AVERAGE GROWTH OF L_p NORMS OF ERDŐS-SZEKERES POLYNOMIALS

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Abstract. We study the average growth of pth powers of L_p noms on the unit circle of Erdős-Szekeres polynomials

$$P_n(\{s_j\}, z) = \prod_{j=1}^{n} (1 - z^{s_j})$$

where $1 \leq s_1, s_2, ..., s_n \leq M$ and $M, n \to \infty$. In particular, we show the average growth is geometric and determine the precise geometric growth. We also analyze the variance.

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1. Introduction

In a 1959 paper, Erdős and Szekeres [12] posed the problem of determining the behavior, especially as $n \to \infty$, of

$$M_n = \inf_{s_1, s_2, ..., s_n \geq 1} M(s_1, s_2, ..., s_n) = \inf_{s_1, s_2, ..., s_n \geq 1} \left\| \prod_{j=1}^{n} (1 - z^{s_j}) \right\|_{L_\infty(|z|=1)}$$

over all $n-$tuples of positive integers $s_2, s_2, ..., s_n$. The best current upper bound is the 1996 estimate of Belov and Konyagin [6]

$$M_n = \exp\left(O\left((\log n)^4\right)\right).$$

The best lower bound is still that of Erdős and Szekeres:

$$M_n \geq \sqrt{2n}.$$

Erdős later conjectured that M_n grows faster than any power of n [11]. The complexity of the problem is perhaps best illustrated by the contrast in the results of Bourgain and Chang [10]. They proved that there exist $\{s_1, s_2, ..., s_n\} \subset \{1, 2, ..., N\}$ with $n/N \to 1/2$ as $N \to \infty$ such that

$$M(s_1, s_2, ..., s_n) \leq \exp\left(O\left(\sqrt{n}\sqrt{\log n} \log\log n\right)\right)$$

but if $\tau > 0$ is small enough and $n > (1 - \tau) N$, then for all $\{s_1, s_2, ..., s_n\} \subset \{1, 2, ..., N\}$,

$$M(s_1, s_2, ..., s_n) > \exp(\tau n).$$

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There is an extensive literature - see for example, [5], [7], [8], [9], [10], [18], [19]. There is also an extensive literature on the closely related pointwise growth of Sudler products $\prod_{j=1}^{n}(1-z^j)$, where all $s_j = j$, and also some other special $\{s_j\}$, are considered. See [1], [2], [3], [4], [13], [14], [15], [17], [21].

The primary focus of this paper is the average behavior of L_p norms of Erdős-Szekeres polynomials, motivated by the contrast mentioned above in the results of Bourgain and Chang. For $0 < p < \infty$, we set

$$\|P\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p}.$$

Given $s_1, s_2, ..., s_n \geq 1$, we set

$$P_n\left(\{s_j\}, z\right) = \prod_{j=1}^{n}(1-z^{s_j}).$$

For $M \geq 1$, and $p > 0$, form the average of the pth powers of the L_p norms over all $1 \leq s_j \leq M$:

$$A_p(M,n) = \frac{1}{M^n}\sum_{1\leq s_1,s_2,\ldots,s_n\leq M}\left\|P_n\left(\{s_j\},\cdot\right)\right\|_p^p. \tag{1.1}$$

The corresponding variance is

$$V_p(M,n) = \left\{\frac{1}{M^n}\sum_{1\leq s_1,s_2,\ldots,s_n\leq M}\left\{\left\|P_n\left(\{s_j\},\cdot\right)\right\|_p^p - A_p(M,n)\right\}^2\right\}^{1/2}. \tag{1.2}$$

The following simple expressions facilitate analysis:

Proposition 1.1
(a)

$$A_p(M,n) = 2^{np}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\frac{1}{M}\sum_{k=1}^{M}|\sin kt|^p\right)^n dt. \tag{1.3}$$

(b)
(1.4)

$$V_p(M,n)^2 = \left(2^{np}\frac{2}{\pi}\right)^2\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(\frac{1}{M}\sum_{k=1}^{M}(|\sin ks|\,|\sin kt|)^p\right)^n ds\ dt - A_p(M,n)^2.$$

Perhaps surprisingly, the growth of M relative to n is a factor only when M grows much faster than n. The formulation of our results is particularly simple for $p = 2$:

Theorem 1.2
Let $\{M_k\}, \{n_k\}$ be sequences of positive integers with limit ∞ such that for some $\rho \in [1, \infty]$,

$$\lim_{k\to\infty} M_k^{1/n_k} = \rho. \tag{1.5}$$

(a) Let $s_0 \in \left(\pi, \frac{3}{2}\pi\right)$ *be the unique root of the equation* $\tan s = s$ *in the interval* $\left(\pi, \frac{3}{2}\pi\right)$*. Then*

$$\lim_{k\to\infty} A_2\left(M_k, n_k\right)^{1/n_k} = 2\max\left\{1, \frac{1}{\rho}\left(1 - \frac{\sin s_0}{s_0}\right)\right\}. \tag{1.6}$$

(b) If $\rho = 1$,

$$\lim_{k\to\infty} V_2\left(M_k, n_k\right)^{1/n_k} = \sqrt{8}. \tag{1.7}$$

Remarks
(a) If for some $L > 0$, we have $M_k = O\left(\left(n_k\right)^L\right)$, then $\rho = 1$, and

$$\lim_{k\to\infty} A_2\left(M_k, n_k\right)^{1/n_k} = 2\left\{1 - \frac{\sin s_0}{s_0}\right\} = 2.434...$$

while

$$\lim_{k\to\infty} V_2\left(M_k, n_k\right)^{1/n_k} = \sqrt{8}.$$

Recalling that we squared the norm before averaging, this indicates the average L_2 norm of these polynomials grows roughly like $\left(\sqrt{2\left\{1 - \frac{\sin s_0}{s_0}\right\}}\right)^n = (1.56...)^n$. Note that when all $s_j = j$ and we take the sup norm, Sudler showed [20] that the norm grows geometrically, but smaller, namely,

$$\lim_{n\to\infty}\left\|\prod_{j=1}^{n}\left(1 - z^j\right)\right\|_{L_\infty(|z|=1)}^{1/n} = 1.219... \,.$$

(b) It is possible to analyze the variance for $\rho \in \left(1, \frac{3}{2}\right)$, for then the first term in the right-hand side of (1.4) dominates the second term. However, this is quite technical, and there are other factors that arise, for example, from the diagonal $s = t, t \in \left[0, \frac{\pi}{2}\right]$ in the first term in (1.4), so is omitted.

The case of general p is more complicated. When n is fixed, however, the situation is rather simple:

Theorem 1.3
Fix $n \geq 1$*. Then for* $p > 0$,

$$\lim_{M\to\infty} A_p\left(M, n\right) = 2^{np}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left(\sin t\right)^p dt\right)^n \tag{1.8}$$

and

$$\lim_{M\to\infty} V_p\left(M, n\right) = 0. \tag{1.9}$$

For general p, we let

$$g_p\left(t\right) = \left|\sin t\right|^p, t \in \left[-\pi, \pi\right]. \tag{1.10}$$

Its Fourier series has the form

$$g_p\left(t\right) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_{2j}\cos 2jt, \tag{1.11}$$

where

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} g_p(t) \cos jt \, dt, j \geq 0.$$

(As g_p is even, the sine coefficients are 0, while the identity $g_p(\pi - t) = g_p(t)$ shows that the odd order cosine coefficients $a_{2j+1} = 0$). We also need for $k \geq 1$,

$$F_k(s) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_{2jk} \frac{\sin kjs}{kjs}. \tag{1.12}$$

Theorem 1.4
Let $p \geq 1$. Let $\{M_k\}, \{n_k\}$ be sequences of positive integers with limit ∞ such that for some $\rho \in [1, \infty]$, (1.5) holds.
(a) Then

$$\lim_{k \to \infty} A_p(M_k, n_k)^{1/n_k} = 2^p \max \left\{ \frac{1}{2} a_0, \frac{1}{\rho} \|F_{k_0}\|_{L_\infty[0,\infty)} \right\}, \tag{1.13}$$

where k_0 is a positive integer such that

$$\|F_{k_0}\|_{L_\infty[0,\infty)} = \sup_{k \geq 1} \|F_k\|_{L_\infty[0,\infty)} \geq \frac{1}{2} > \frac{1}{2} a_0. \tag{1.14}$$

(b) When $p \geq 4$, this simplifies to

$$\lim_{k \to \infty} A_p(M_k, n_k)^{1/n_k} = 2^p \max \left\{ \frac{1}{2} a_0, \frac{1}{2\rho} \right\}. \tag{1.15}$$

(c) If $\rho = 1$ and $p \geq 2$, then

$$\lim_{k \to \infty} V_p(M_k, n_k)^{1/n_k} = 2^{p-1/2}. \tag{1.16}$$

Remarks
From Hölder's inequality, the average without pth powers, namely

$$A_p^*(M, n) = \frac{1}{M^n} \sum_{1 \leq s_1, s_2, \ldots, s_n \leq M} \|P_n(\{s_j\}, \cdot)\|_p$$

satisfies for $p \geq 1$,

$$A_1(M, n) \leq A_p^*(M, n) \leq A_p(M, n)^{1/p},$$

so under the hypotheses of the above theorem,

$$\limsup_{k \to \infty} A_p^*(M_k, n_k)^{1/n_k} \leq 2 \max \left\{ \frac{2}{\pi} \int_0^{\pi/2} |\sin t|^p \, dt, \frac{1}{\rho} \|F_{k_0}\|_{L_\infty[0,\infty)} \right\}^{1/p}$$

where F_{k_0} arises from the $\{F_k\}$ for p. In the other direction, we have from our results for $A_1(M, n)$,

$$\liminf_{k \to \infty} A_p^*(M_k, n_k)^{1/n_k} \geq 2 \left(\frac{2}{\pi} \int_0^{\pi/2} |\sin t| \, dt \right) = \frac{4}{\pi}.$$

In particular,

$$\liminf_{k \to \infty} A_\infty^*(M_k, n_k)^{1/n_k} \geq \frac{4}{\pi}.$$

This paper is organized as follows: we prove Proposition 1.1 and Theorem 1.3 in Section 2. We prove Theorem 1.2(a) and 1.4(a), (b) in Section 3 and Theorems 1.2(b), 1.4(b) in Section 4. We present some further results in Section 5.

2. Proof of Proposition 1.1 and Theorem 1.3

Proof of Proposition 1.1

(a) We have

$$\|P_n(\{s_j\},\cdot)\|_p^p = \frac{1}{2\pi}\int_{-\pi}^{\pi}\prod_{j=1}^{n}\left(2\left|\sin\frac{s_j\theta}{2}\right|\right)^p d\theta = 2^{np}\frac{2}{\pi}\int_0^{\pi/2}\prod_{j=1}^{n}|\sin s_j\theta|^p\, d\theta.$$

So

$$\begin{aligned} A_p(M,n) &= \frac{1}{M^n}\sum_{s_1=1}^{M}\sum_{s_2=1}^{M}\cdots\sum_{s_n=1}^{M}\left(2^{np}\frac{2}{\pi}\int_0^{\pi/2}\prod_{j=1}^{n}|\sin s_j\theta|^p\, d\theta\right) \\ &= 2^{np}\frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1}{M}\sum_{k=1}^{M}|\sin k\theta|^p\right)^n d\theta. \end{aligned}$$

(b) We have

$$\begin{aligned} V_p(M,n)^2 &= \frac{1}{M^n}\sum_{1\le s_1,s_2,\ldots,s_n\le M}\|P_n(\{s_j\},\cdot)\|_p^{2p} - A_p(M,n)^2 \\ &= B_p(M,n) - A_p(M,n)^2, \end{aligned}$$

say. Here as above,

$$\begin{aligned} B_p(M,n) &= \frac{1}{M^n}\sum_{s_1=1}^{M}\sum_{s_2=1}^{M}\cdots\sum_{s_n=1}^{M}\left(2^{np}\frac{2}{\pi}\int_0^{\pi/2}\prod_{j=1}^{n}|\sin s_j\theta|^p\, d\theta\right)^2 \\ &= \left(2^{np}\frac{2}{\pi}\right)^2\int_0^{\pi/2}\int_0^{\pi/2}\left(\frac{1}{M}\sum_{k=1}^{M}|\sin k\theta\sin k\phi|^p\right)^n d\phi\, d\theta. \quad (2.1) \end{aligned}$$

■

Proof of Theorem 1.3

Recall that if $f:[0,1]\to\mathbb{R}$ is continuous, and α is irrational, while $\{k\alpha\}$ denotes the fractional part of $k\alpha$, the theory of uniform distribution [16] gives

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}f(\{k\alpha\}) = \int_0^1 f(t)\,dt.$$

Applying this to $f(t)=|\sin\pi t|^p$, we see that for t/π irrational, and hence for a.e. $t\in[0,\pi]$,

$$\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}|\sin kt|^p = \lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\left|\sin\pi\left\{k\frac{t}{\pi}\right\}\right|^p = \int_0^1|\sin\pi t|^p\,dt. \quad (2.2)$$

In addition,

$$\frac{1}{M}\sum_{k=1}^{M}|\sin kt|^p \le 1.$$

Lebesgue's Dominated Convergence Theorem shows that

$$\lim_{M\to\infty} A_p(M,n) = 2^{np}\frac{2}{\pi}\int_0^{\pi/2}\left(\int_0^1 |\sin \pi t|^p\, dt\right)^n d\theta.$$

(b) Let $B_p(M,n)$ be given by (2.1). The theory of uniform distribution [16, Chapter 6] shows that for a.e. $(\theta,\phi)\in\left[0,\frac{\pi}{2}\right]^2$, we have

$$\begin{aligned}
&\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}|\sin k\theta \sin k\phi|^p \\
=\ &\lim_{M\to\infty}\frac{1}{M}\sum_{k=1}^{M}\left(\left|\sin\pi\left\{k\frac{\theta}{\pi}\right\}\right|\left|\sin\pi\left\{k\frac{\phi}{\pi}\right\}\right|\right)^p \\
=\ &\int_0^1\int_0^1 \left(|\sin \pi t|\,|\sin \pi s|\right)^p ds\ dt \\
=\ &\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}|\sin s|^p\, ds\right)^2.
\end{aligned}$$

Then

$$\lim_{M\to\infty} B_p(M,n) = 2^{2np}\left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}}|\sin s|^p\, ds\right)^{2n} = \lim_{M\to\infty} A_p(M,n)^2,$$

so we obtain (1.9). ■

3. Proof of Theorems 1.2(a), 1.4(a) and 1.4(b)

Let

$$h_{M,p}(t) = \frac{1}{M}\sum_{k=1}^{M}|\sin kt|^p.$$

Lemma 3.1
Let $p\geq 1$.
(a) There exists $C_p>0$ such that for $M\geq 1$ and $s,t\in\mathbb{R}$,

$$|h_{M,p}(t)-h_{M,p}(s)|\leq C_p M\,|t-s|.$$

(b) Given $\varepsilon>0$, there exists M_0 and δ_0 such that for $M\geq M_0$ and $\left|t-\frac{\pi}{2}\right|\leq \delta_0/M$,

$$\left|h_{M,p}(t)-\frac{1}{2}\right|\leq\varepsilon.$$

Proof
(a) We use the fact that there exists $C_p>0$ such that for $u,v\in\mathbb{R}$,

$$\left||\sin u|^p-|\sin v|^p\right|\leq C_p\,|u-v|.$$

Then

$$|h_{M,p}(t)-h_{M,p}(s)|\leq\frac{C_p}{M}\sum_{k=1}^{M}|k(t-s)| = \frac{C_p}{M}|t-s|\frac{M(M+1)}{2}.$$

(b) Now

$$h_{M,p}\left(\frac{\pi}{2}\right) = \frac{1}{M}\sum_{1\leq k\leq M, k \text{ odd}} 1 = \frac{1}{2}+O\left(\frac{1}{M}\right).$$

The result then follows from (a). ■

We can now prove a preliminary lower bound:

Lemma 3.2
Let $p > 0$ *and* $\{M_k\}, \{n_k\}$ *be sequences of positive integers with* $M_k \to \infty$ *as* $k \to \infty$. *Then*

$$\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} \geq 2^p \max\left\{\frac{2}{\pi}\int_0^{\frac{\pi}{2}} |\sin t|^p \, dt, \frac{1}{2\limsup_{k\to\infty} M_k^{1/n_k}}\right\}.$$

Proof
First, from (1.3) and Hölder's inequality,

$$A_p(M_k, n_k)^{1/n_k} \geq 2^p \frac{2}{\pi}\int_0^{\frac{\pi}{2}} h_{M_k,p}(t)\, dt.$$

Using Fatou's Lemma, and uniform distribution as in (2.2),

$$\begin{aligned}\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} &\geq 2^p \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \liminf_{k\to\infty} h_{M_k,p}(t)\, dt \\ &= 2^p \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(\int_0^1 |\sin \pi\theta|^p \, d\theta\right) dt \\ &= 2^p \int_0^1 |\sin \pi\theta|^p \, d\theta. \qquad (3.1)\end{aligned}$$

Next, let $\varepsilon \in \left(0, \frac{1}{2}\right)$. From Lemma 3.1(b), there exists K_0 and δ_0 such that for $k \geq K_0$,

$$\int_{\frac{\pi}{2}-\frac{\delta_0}{M}}^{\frac{\pi}{2}} h_{M_k,p}(t)^{n_k}\, dt \geq \int_{\frac{\pi}{2}-\frac{\delta_0}{M}}^{\frac{\pi}{2}} \left(\frac{1}{2}-\varepsilon\right)^{n_k} dt = \frac{\delta_0}{M}\left(\frac{1}{2}-\varepsilon\right)^{n_k},$$

so that

$$A_p(M_k, n_k)^{1/n_k} \geq \left(\frac{2}{\pi}\frac{\delta_0}{M}\right)^{1/n_k} 2^p \left(\frac{1}{2}-\varepsilon\right).$$

Letting $k \to \infty$,

$$\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} \geq 2^p \left(\frac{1}{2}-\varepsilon\right) \liminf_{k\to\infty} \frac{1}{M_k^{1/n_k}}.$$

Here as $\varepsilon > 0$ is arbitrary, we obtain

$$\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} \geq 2^{p-1} \frac{1}{\limsup_{k\to\infty} M_k^{1/n_k}}.$$

Combining this and (3.1) gives the result. ■

We now consider the special case $p = 2$, where there is a simple formula for $h_{M,p}$.

Lemma 3.3
(a)

$$h_{M,2}(t) = \frac{1}{2}\left(1 + \frac{1}{2M} - \frac{\sin(2M+1)t}{2M\sin t}\right). \qquad (3.2)$$

(b)

$$\|h_{M,2}\|_{L_\infty\left[0,\frac{\pi}{2}\right]} = \frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)+o\left(1\right), \tag{3.3}$$

where $s_0 \in \left(\pi, \frac{3}{2}\pi\right)$ *is the unique root of the equation* $\tan s = s$ *in that interval. The sup norm of* $h_{M,2}$ *is attained at a point of the form* $t_M = \frac{s_0}{2M+1}\left(1+o\left(1\right)\right)$*.*

Proof

(a) This uses the standard trick from Fourier series:

$$\begin{aligned} h_{M,2}\left(t\right) &= \frac{1}{2M}\sum_{k=1}^{M}\left(1-\cos 2kt\right) \\ &= \frac{1}{2}-\frac{1}{2M}\sum_{k=1}^{M}\frac{\sin\left(2k+1\right)t-\sin\left(2k-1\right)t}{2\sin t} \\ &= \frac{1}{2}-\frac{\sin\left(2M+1\right)t}{4M\sin t}+\frac{1}{4M}. \end{aligned}$$

(b) If first $t \in [0, \frac{\pi}{2M+1}]$, then $\sin\left(2m+1\right)t \geq 0$, so

$$0 \leq h_{M,2}\left(t\right) \leq \frac{1}{2}+\frac{1}{4M}.$$

If $t \in \left[\frac{3}{2}\frac{\pi}{2M+1}, \frac{\pi}{2}\right]$, then

$$\begin{aligned} 0 &\leq h_{M,2}\left(t\right) \leq \frac{1}{2}+\frac{1}{4M\sin t}+\frac{1}{4M} \\ &\leq \frac{1}{2}+\frac{1}{4M\sin\frac{3}{2}\frac{\pi}{2M+1}}+\frac{1}{4M} = h_{M,2}\left(\frac{3}{2}\frac{\pi}{2M+1}\right). \end{aligned}$$

So $\|h_{M,2}\|_{L_\infty\left[0,\frac{\pi}{2}\right]}$ is attained in the interval $\left[\frac{\pi}{2M+1}, \frac{3}{2}\frac{\pi}{2M+1}\right]$. As $M \to \infty$, uniformly for $s \in \left[\pi, \frac{3}{2}\pi\right]$, we have

$$\begin{aligned} h_{M,2}\left(\frac{s}{2M+1}\right) &= \frac{1}{2}\left(1+\frac{1}{2M}-\frac{\sin s}{2M\sin\frac{s}{2M+1}}\right) \\ &= \frac{1}{2}\left(1-\frac{\sin s}{s}\right)+O\left(\frac{1}{M}\right). \end{aligned}$$

The function $\frac{\sin s}{s}$ has a unique minimum in $\left(\pi, \frac{3}{2}\pi\right)$, at the point s_0, where $\tan s_0 = s_0$. Then we have the result. ■

Proof of Theorem 1.2(a)

We first establish the asymptotic lower bound. Let $\varepsilon \in \left(0, \frac{1}{4}\right)$. From Lemma 3.1(a) and Lemma 3.3(b), there exists $\delta_0 > 0$ such that for large enough M,

$$\int_{t_M-\frac{\delta_0}{2M+1}}^{t_M-\frac{\delta_0}{2M+1}} h_{M,2}\left(t\right)^n dt \geq \frac{2\delta_0}{2M+1}\left(\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)-\varepsilon\right)^n,$$

so that

$$\begin{aligned}\liminf_{k\to\infty} A_2(M_k,n_k)^{1/n_k} &\geq \liminf_{k\to\infty} 2^2\left(\frac{2}{\pi}\frac{2\delta_0}{2M_k+1}\right)^{1/n_k}\left(\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)-\varepsilon\right)\\ &= 2^2\frac{1}{\rho}\left(\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)-\varepsilon\right).\end{aligned}$$

As $\varepsilon > 0$ is arbitrary,

$$\liminf_{k\to\infty} A_2(M_k,n_k)^{1/n_k} \geq 2^2\frac{1}{\rho}\left(\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)\right).$$

Together with Lemma 3.2 and the fact that $\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right) > \frac{1}{2}$, this gives

$$\begin{aligned}\liminf_{k\to\infty} A_2(M_k,n_k)^{1/n_k} &\geq 2^2\max\left\{\frac{2}{\pi}\int_0^{\frac{\pi}{2}}|\sin t|^2\,dt, \frac{1}{\rho}\left(\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right)\right)\right\}\\ &= 2\max\left\{1,\frac{1}{\rho}\left(1-\frac{\sin s_0}{s_0}\right)\right\}. \qquad (3.4)\end{aligned}$$

We now turn to the matching upper bound. Let $R > 0$. We have

$$\int_0^{\frac{R}{2M+1}} h_{M,2}(t)^n\,dt \leq \frac{R}{2M+1}\|h_{M,2}\|^n_{L_\infty[0,\frac{\pi}{2}]}.$$

Next, for $t \in \left[\frac{R}{2M+1},\frac{\pi}{2}\right]$, we have from (3.2), for large enough M,

$$h_{M,2}(t) \leq \frac{1}{2}\left(1+\frac{1}{2M}+\frac{1}{2M\sin\frac{R}{2M+1}}\right) \leq \frac{1}{2}\left(1+\frac{2}{R}\right).$$

Combining the above estimates, gives for large enough k,

$$\begin{aligned}A_2(M_k,n_k) &= 2^{2n_k}\frac{2}{\pi}\left[\int_0^{\frac{R}{2M+1}}+\int_{\frac{R}{2M+1}}^{\frac{\pi}{2}}\right]h_{M_k,2}(t)^n\,dt\\ &\leq 2^{2n_k}\frac{2}{\pi}\left[\frac{R}{2M_k+1}\|h_{M_k,2}\|^{n_k}_{L_\infty[0,\frac{\pi}{2}]}+\frac{\pi}{2}\left(\frac{1}{2}\left(1+\frac{2}{R}\right)\right)^{n_k}\right].\end{aligned}$$

Then using Lemma 3.3(b),

$$A_2(M_k,n_k)^{1/n_k} \leq 2^2(1+o(1))\max\left\{\frac{1}{\rho}\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right),\frac{1}{2}\left(1+\frac{2}{R}\right)\right\}.$$

Since R may be made arbitrarily large, we obtain

$$\limsup_{k\to\infty} A_2(M_k,n_k)^{1/n_k} \leq 2\max\left\{1,\frac{1}{\rho}\left(1-\frac{\sin s_0}{s_0}\right)\right\}.$$

This and (3.4) give the result. ■

We turn to the more difficult case of general p. Recall that we expanded $g_p(t) = |\sin t|^p$ as a Fourier series in (1.11) and defined F_k by (1.12). Recall too that

$$h_{M,p}(t) = \frac{1}{M}\sum_{k=1}^{M}|\sin kt|^p.$$

Lemma 3.4
Let $p \geq 1$, $R > 1$, and $\varepsilon \in (0,1)$.
(a)

$$h_{M,p}(t) = \frac{1}{2}a_0\left(1+\frac{1}{2M}\right)+\sum_{j=1}^{\infty}a_{2j}\frac{\sin\left(j\left(2M+1\right)t\right)}{2M\sin jt}. \tag{3.5}$$

(b) There exists N such that if

$$h_{M,p,N}(t) = \frac{1}{2}a_0\left(1+\frac{1}{2M}\right)+\sum_{j=1}^{N}a_{2j}\frac{\sin\left(j\left(2M+1\right)t\right)}{2M\sin jt}, \tag{3.6}$$

then for $M \geq 1$ and $t \in \mathbb{R}$,

$$\left|h_{M,p}(t) - h_{M,n,p}(t)\right| \leq \varepsilon \tag{3.7}$$

and

$$\sum_{j=N+1}^{\infty}\left|a_{2j}\right| < \varepsilon. \tag{3.8}$$

(c) Let $M \geq R$. With N as in (b), let

$$\mathcal{I} = \left\{t \in \left[0,\frac{\pi}{2}\right] : \left|\sin jt\right| \geq \frac{R}{M} \text{ for } 1 \leq j \leq N\right\}. \tag{3.9}$$

Then for $t \in \mathcal{I}$, we have

$$h_{M,p,N}(t) \leq \frac{1}{2}a_0 + \frac{C}{R}, \tag{3.10}$$

where C is independent of M, R, N, t.
(d) Let $\mathcal{J} = \left[0,\frac{\pi}{2}\right]\setminus\mathcal{I}$. Then for $t \in \mathcal{J}$, and $M \geq M_0(\varepsilon)$, we have

$$h_{M,p}(t) \leq \sup_{k\geq 1}\left\|F_k\right\|_{L_\infty[0,\infty)} + 3\varepsilon. \tag{3.11}$$

(e) Given $1 \leq j_0 \leq N$, there exists for large enough M, $t_M \in \left[0,\frac{\pi}{2}\right]$ and $\eta > 0$ such that for $|t - t_M| \leq \frac{\eta}{M}$,

$$h_{M,p}(t) \geq \left\|F_{j_0}\right\|_{L_\infty[0,\infty)} - \varepsilon. \tag{3.12}$$

Remark
The sets $\mathcal{I}$ and $\mathcal{J}$ depend on M, N and R, but we do not explicitly display this dependence.
Proof
(a) We have

$$\begin{aligned} h_{M,p}(t) &= \frac{1}{M}\sum_{k=1}^{M}\left(\frac{a_0}{2}+\sum_{j=1}^{\infty}a_{2j}\cos 2jkt\right) \\ &= \frac{a_0}{2}+\sum_{j=1}^{\infty}a_{2j}\frac{1}{M}\sum_{k=1}^{M}\cos 2jkt \\ &= \frac{a_0}{2}+\sum_{j=1}^{\infty}a_{2j}\left[\frac{\sin j\left(2M+1\right)t}{2M\sin jt}-\frac{1}{2M}\right], \end{aligned}$$

by the usual sums of Fourier series. Here as g_p has left and right derivatives at each point of $[-\pi,\pi]$, it equals its Fourier series there. In particular at $t=0$,

$$0=\frac{a_0}{2}+\sum_{j=1}^{\infty}a_{2j}, \tag{3.13}$$

so that (3.5) follows.
(b) A direct computation shows that if $p=1$,

$$a_{2j}=-\frac{4}{\pi}\frac{1}{4j^2-1}, j\geq 1.$$

If $p>1$, integrating by parts twice shows that

$$a_{2j}=-\frac{p\,(p-1)}{2\pi j^2}\int_0^{\pi}(\sin t)^{p-2}\cos(2jt)\,dt.$$

Consequently if $p\geq 1$, there exists $C>0$ such that for $j\geq 1$,

$$|a_j|\leq\frac{C}{j^2}, j\geq 1.$$

Then if N is large enough,

$$|h_{M,p}(t)-h_{M,p,N}(t)|=\left|\sum_{j=N+1}^{\infty}a_{2j}\frac{\sin(j\,(2M+1)\,t)}{2M\sin jt}\right|\leq 2\sum_{j=N+1}^{\infty}\frac{C}{j^2}<\varepsilon.$$

Thus we obtain (3.7) and (3.8).
(c) Here

$$\begin{aligned}h_{M,p,N}(t) &\leq \frac{1}{2}a_0\left(1+\frac{1}{2M}\right)+\frac{1}{2R}\sum_{j=1}^{N}|a_{2j}|\\ &\leq \frac{1}{2}a_0+\frac{C}{R},\end{aligned}$$

where C is independent of $M\geq R$ and N,t.
(d) We assume that $M>>N^2R$. Let $t\in\mathcal{J}$. Then for some $1\leq j\leq N$, we have $|\sin jt|<\frac{R}{M}$. For the given t, let

$$S_t=\left\{j:1\leq j\leq N\text{ and }|\sin jt|<\frac{R}{M}\right\}.$$

Let j_0 be the smallest integer in S_t. Then necessarily j_0t is close to a multiple of π. Let us make this more precise. Since $0\leq j_0t\leq j_0\frac{\pi}{2}$, there exists an integer $0\leq m_0\leq\frac{j_0}{2}$ such that $|j_0t-m_0\pi|\leq\frac{\pi}{2}$ and $m_0\pi$ is the closest multiple of π to j_0t. Then

$$\begin{aligned}\frac{R}{M} &\geq |\sin(j_0t-m_0\pi)|\geq\frac{2}{\pi}|j_0t-m_0\pi|\\ &\Rightarrow \left|t-\frac{m_0}{j_0}\pi\right|\leq\frac{\pi R}{2j_0M}\leq\frac{\pi R}{2M}.\end{aligned} \tag{3.14}$$

We claim that we can assume either $m_0=0$ or j_0,m_0 are coprime. For suppose $m_0\neq 0$ but j_0,m_0 are not coprime. Then $j_0=j_1k$ and $m_0=m_1k$ for some $k\geq 2$, and we have

$$\begin{aligned}|\sin j_1 t| &= |\sin(j_1 t - m_1\pi)| = \left|\sin\left(\frac{1}{k}(j_0 t - m_0\pi)\right)\right| \\ &\leq \frac{j_0}{k}\left|t - \frac{m_0}{j_0}\pi\right| \leq \frac{\pi R}{2kM} < \frac{R}{M}\end{aligned}$$

as $k \geq 2$. This contradicts our choice of j_0 as the smallest element of S_t. We next claim that

$$S_t \subseteq \{kj_0 : 1 \leq k \leq N/j_0\}. \tag{3.15}$$

If first $m_0 = 0$, then $|\sin j_0 t| \leq \frac{R}{M}$, and since j_0 is the smallest member of S_t, so necessarily $j_0 = 1$. So all this last statement asserts is $S_t \subseteq \{1, 2, ..., N\}$, which follows from the definition. Next suppose $m_0 > 0$ so that j_0 and m_0 are coprime. If j_1 is not a multiple of j_0 and $j_1 \in S_t$, we have for some $m_1 \leq j_1/2$ that

$$\left|t - \frac{m_1}{j_1}\pi\right| \leq \frac{\pi R}{2M}$$

as at (3.14). Then

$$\left|\frac{m_0}{j_0} - \frac{m_1}{j_1}\right| \leq \frac{R}{M}$$

$$\Rightarrow |m_0 j_1 - m_1 j_0| \leq \frac{R}{M}N^2 < 1.$$

Then $m_0 j_1 - m_1 j_0 = 0$, and so $j_0 | j_1$, a contradiction. Thus we have (3.15) in all cases. Next, we can write

$$t = \frac{m_0}{j_0}\pi + \frac{s}{2M+1}, \text{ where } |s| \leq \frac{\pi R}{2}\frac{2M+1}{2M}. \tag{3.16}$$

Then from (3.6),

$$h_{M,p,N}(t) = \frac{1}{2}a_0\left(1 + \frac{1}{2M}\right) + \sum_{j=1}^{N} a_{2j}\frac{\sin\left(j(2M+1)\frac{m_0}{j_0}\pi + js\right)}{2M\sin j\left(\frac{m_0}{j_0}\pi + \frac{s}{2M+1}\right)}. \tag{3.17}$$

If first $m_0 = 0$, this yields uniformly in s,

$$h_{M,p,N}(t) = \frac{1}{2}a_0\left(1 + \frac{1}{2M}\right) + \sum_{j=1}^{N} a_{2j}\frac{\sin js}{js} + O\left(\frac{1}{M}\right). \tag{3.18}$$

Next suppose $m_0 \neq 0$ but j_0, m_0 are coprime. The main contributions to the sum in (3.17) come from those $j \leq N$ that are multiples of j_0, say $j = j_0\ell$, where $\ell \leq N/j_0$. Then

$$\begin{aligned}\frac{\sin\left(j(2M+1)\frac{m}{j_0}\pi + js\right)}{2M\sin j\left(\frac{m}{j_0}\pi + \frac{s}{2M+1}\right)} &= \frac{\sin((2M+1)\ell m\pi + j_0\ell s)}{2M\sin\left(\ell m\pi + j_0\ell\frac{s}{2M+1}\right)} \\ &= \frac{\sin(j_0\ell s)}{2M\sin\left(j_0\ell\frac{s}{2M+1}\right)} \\ &= \frac{\sin(j_0\ell s)}{j_0\ell s} + O\left(\frac{1}{M}\right),\end{aligned}$$

uniformly for $|s| \leq \frac{\pi R}{2}\frac{2M+1}{2M}$. Note that this holds even if we do not know that $j = j_0\ell \in S_t$. For the remaining terms, we have as $j_0 \nmid jm$ that $j_0 \geq 2$, so

$$\begin{aligned} \left|\sin j\left(\frac{m}{j_0}\pi + \frac{s}{2M+1}\right)\right| &\geq \left|\sin\frac{\pi}{j_0}\right| - \left|\frac{js}{2M+1}\right| \\ &\geq \left|\sin\frac{\pi}{N}\right| - O\left(\frac{1}{M}\right). \end{aligned}$$

Then no matter whether $m = 0$ or j_0, m are coprime,

$$h_{M,p,N}(t) = \frac{1}{2}a_0 + \sum_{1\leq \ell \leq N/j_0} a_{2j_0\ell}\frac{\sin(j_0\ell s)}{j_0\ell s} + O\left(\frac{1}{M}\right). \tag{3.19}$$

Hence

$$|h_{M,p,N}(t) - F_{j_0}(s)| \leq \sum_{k=N+1}^{\infty}|a_{2k}| + O\left(\frac{1}{M}\right) < \varepsilon + O\left(\frac{1}{M}\right), \tag{3.20}$$

by (3.8). Together with (3.7), this gives

$$\begin{aligned} h_{M,p}(t) &\leq F_{j_0}(s) + 2\varepsilon + O\left(\frac{1}{M}\right) \\ &\leq \sup_{k\geq 1}\|F_k\|_{L_\infty[0,\infty)} + 2\varepsilon + O\left(\frac{1}{M}\right). \end{aligned}$$

For large enough M, we obtain (3.11).
(e) With t given by (3.16), we have from (3.7), (3.19), (3.20),

$$h_{M,p}(t) \geq F_{j_0}(s) - 2\varepsilon + O\left(\frac{1}{M}\right).$$

Here we can choose any $1 \leq j_0 \leq N$ and any s with $|s| \leq \frac{\pi R}{2}\frac{2M+1}{2M}$. As R can be as large as we please, we can choose a suitable t and then a suitable j_0 with

$$h_{M,p}(t) \geq \|F_{j_0}\|_{L_\infty[0,\infty)} - 4\varepsilon$$

for large enough M. The Hölder estimate in Lemma 3.1(a) yields the result. ■

Next we establish further properties of the $\{F_k\}$ defined by (1.12):

Lemma 3.5
Let $p \geq 1$.
(a) There is an integer $k_0 \geq 1$ such that

$$\|F_{k_0}\|_{L_\infty[0,\infty)} = \sup_{k\geq 1}\|F_k\|_{L_\infty[0,\infty)} \geq \|F_1\|_{L_\infty[0,\infty)} > \frac{1}{2}a_0$$

and for $k > k_0$,

$$\|F_k\|_{L_\infty[0,\infty)} < \|F_{k_0}\|_{L_\infty[0,\infty)}.$$

(b) In addition,

$$\|F_{k_0}\|_{L_\infty[0,\infty)} \geq F_2(0) = \frac{1}{2}.$$

(c) Each F_k is nonnegative in $[0,\infty)$. Moreover, if $p \geq 2$, then with s_0 as above,

$$\|F_{k_0}\|_{L_\infty[0,\infty)} = \sup_{k\geq 1}\|F_k\|_{L_\infty[0,\infty)} \leq \frac{1}{2}\left(1 - \frac{\sin s_0}{s_0}\right).$$

Proof

(a) Now

$$\lim_{s\to\infty} F_1(s) = \frac{a_0}{2} = F_1(m\pi),\ m \geq 1.$$

If $\|F_1\|_{L_\infty[0,\infty)} = \frac{a_0}{2}$, then for all $m \geq 1$, $F_1'(m\pi) = 0$. Here

$$F_1'(s) = \sum_{j=1}^{\infty} a_{2\ell} \frac{(j \cos js)\, s - \sin js}{js^2}.$$

$$\Rightarrow 0 = F_1'(2\pi) = \frac{1}{2\pi} \sum_{j=1}^{\infty} a_{2j}.$$

But then from (3.13), $a_0 = 0$, which is false. So

$$\sup_{k\geq 1} \|F_k\|_{L_\infty[0,\infty)} \geq \|F_1\|_{L_\infty[0,\infty)} > \frac{1}{2}a_0.$$

Next, for each k,

$$\|F_k\|_{L_\infty[0,\infty)} \leq \frac{1}{2}a_0 + \sum_{j=2k}^{\infty} |a_j| \to \frac{1}{2}a_0$$

as $k \to \infty$, so for sufficiently large k, we obtain

$$\|F_k\|_{L_\infty[0,\infty)} < \|F_1\|_{L_\infty[0,\infty)}.$$

Thus there is a k_0 as described above.

(b) Now

$$F_2(0) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_{4j}. \tag{3.21}$$

Here

$$1 = g_p\left(\frac{\pi}{2}\right) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_{2j}(-1)^j;$$

$$0 = g_p(0) = \frac{1}{2}a_0 + \sum_{j=1}^{\infty} a_{2j},$$

so adding,

$$1 = a_0 + 2\sum_{j=1}^{\infty} a_{4j}.$$

Substituting in (3.21), gives

$$F_2(0) = \frac{1}{2}.$$

(c) Suppose that $p \geq 2$. This essentially follows from the inequality $h_{M,p}(t) \leq h_{M,2}(t)$. By Lemma 3.3(b), for all t,

$$0 \leq h_{M,p}(t) \leq \frac{1}{2}\left(1 - \frac{\sin s_0}{s_0}\right) + o(1).$$

Given $\varepsilon > 0$, we can then choose N, M_0 so large that for $M \geq M_0$ and all t,

$$-\varepsilon \leq h_{M,p,N}(t) \leq \frac{1}{2}\left(1 - \frac{\sin s_0}{s_0}\right) + \varepsilon,$$

as at (3.7). By taking scaling limits of the left-hand side, much as in the proof of Lemma 3.4, we will obtain the result. Let us make this precise. Let $j_0 \geq 1$ and $s \in \mathbb{R}$. From (3.20), with t given by (3.16), we obtain

$$-2\varepsilon \leq F_{j_0}(s) \leq \frac{1}{2}\left(1 - \frac{\sin s_0}{s_0}\right) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary,

$$0 \leq F_{j_0}(s) \leq \frac{1}{2}\left(1 - \frac{\sin s_0}{s_0}\right).$$

Here $s \in (0, \infty]$ is arbitrary, so we obtain the result. The nonnegativity clearly also follows for $p \leq 2$. ■

Proof of Theorem 1.4(a)

We first establish the asymptotic lower bound. Let k_0 be as in the lemma above. Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. From Lemma 3.4(e), Lemma 3.5(a), and Lemma 3.1(a), there exists for large enough k, $t_k \in (0, \infty)$ and $\eta > 0$, such that for $|t - t_k| \leq \frac{\eta}{M_k}$, we have

$$h_{M_k,p}(t) \geq \|F_{k_0}\|_{L_\infty[0,\infty)} - \varepsilon.$$

Then

$$\begin{aligned}\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} &\geq \liminf_{k\to\infty}\left(2^{n_k p}\frac{2}{\pi}\int_{t_k - \frac{\eta}{M_k}}^{t_k + \frac{\eta}{M_k}}\left(\|F_{k_0}\|_{L_\infty[0,\infty)} - \varepsilon\right)^{n_k} dt\right)^{1/n_k} \\ &= 2^p \frac{1}{\rho}\left(\|F_{k_0}\|_{L_\infty[0,\infty)} - \varepsilon\right).\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this last lower bound and Lemma 3.2, give

$$\begin{aligned}\liminf_{k\to\infty} A_p(M_k, n_k)^{1/n_k} &\geq 2^p \max\left\{\frac{1}{2}a_0, \frac{1}{2\rho}, \frac{1}{\rho}\|F_{k_0}\|_{L_\infty[0,\infty)}\right\} \\ &= 2^p \max\left\{\frac{1}{2}a_0, \frac{1}{\rho}\|F_{k_0}\|_{L_\infty[0,\infty)}\right\}, \qquad (3.22)\end{aligned}$$

recall Lemma 3.5(b). Now let us establish the corresponding uper bound. We split $\left[0, \frac{\pi}{2}\right] = \mathcal{I} \cup \mathcal{J}$, where the latter are as in Lemma 3.4. From Lemma 3.4(c), (d),

$$\begin{aligned}A_p(M, n) &= 2^{np}\frac{2}{\pi}\left(\int_{\mathcal{I}} + \int_{\mathcal{J}}\right) h_{M,p}(t)^n\, dt \\ &\leq 2^{np}\frac{2}{\pi}\left(\frac{\pi}{2}\left[\frac{1}{2}a_0 + \frac{C}{R}\right]^n + meas(\mathcal{J})\left[\|F_{k_0}\|_{L_\infty[0,\infty)} + 3\varepsilon\right]^n\right).\end{aligned}$$

Here $meas(\mathcal{J}) \leq \frac{C}{M}$, (as is clear from (3.14) and the fact that there are $O\left(N^2\right)$ pairs (j_0, m_0)) so

$$A_p(M_k, n_k) \leq C2^{n_k p}\max\left\{\left[\frac{1}{2}a_0 + \frac{C}{R}\right]^{n_k}, \frac{1}{M_k}\left[\|F_{k_0}\|_{L_\infty[0,\infty)} + 3\varepsilon\right]^{n_k}\right\},$$

$$\Rightarrow \limsup_{k\to\infty} A_p(M_k, n_k)^{1/n_k} \leq 2^p \max\left\{\frac{1}{2}a_0 + \frac{C}{R}, \frac{1}{\rho}\left[\|F_{k_0}\|_{L_\infty[0,\infty)} + 3\varepsilon\right]\right\}.$$

As R may be as large as we please while ε may be as small as we please,

$$\limsup_{k\to\infty} A_p(M_k, n_k)^{1/n_k} \leq 2^p \max\left\{\frac{1}{2}a_0, \frac{1}{\rho}\|F_{k_0}\|_{L_\infty[0,\infty)}\right\}.$$

This and our lower bound (3.22) give the result. ■

We next look at $p = 4$ in some detail:

Lemma 3.6
Let $p \geq 4$. Then

$$\lim_{M\to\infty} \|h_{M,p}\|_{L_\infty[0,\frac{\pi}{2}]} = \frac{1}{2} = \sup_{k\geq 1} \|F_k\|_{L_\infty[0,\infty)}. \tag{3.23}$$

Proof
The Fourier series of $(\sin t)^4$ can be deduced from trigonometric identities:

$$(\sin t)^4 = \frac{3}{8} - \frac{1}{2}\cos 2t + \frac{1}{8}\cos 4t.$$

Then we see from Lemma 3.4(a) that

$$h_{M,4}(t) = \frac{3}{8}\left(1 + \frac{1}{2M}\right) - \frac{1}{2}\frac{\sin((2M+1)t)}{2M\sin t} + \frac{1}{8}\frac{\sin(2(2M+1)t)}{2M\sin 2t}.$$

Here there are really only 2 of the "F" functions:

$$F_1(s) = \frac{3}{8} - \frac{1}{2}\frac{\sin s}{s} + \frac{1}{8}\frac{\sin 2s}{2s};$$

$$F_2(s) = \frac{3}{8} + \frac{1}{8}\frac{\sin 2s}{2s}.$$

For $k \geq 3$, $F_k = \frac{3}{8}$. Recall from Lemma 3.5(c) that these are nonnegative functions. We see that

$$0 \leq F_2(s) \leq \frac{1}{2} = F_2(0).$$

Next if $s \in [0, \pi)$, we have $\sin s \geq 0$, so

$$0 \leq F_1(s) \leq \frac{3}{8} + \frac{1}{8} = \frac{1}{2}.$$

If $s \geq \frac{3}{2}\pi$, then

$$0 \leq F_1(s) \leq \frac{3}{8} + \frac{1}{3\pi} + \frac{1}{24\pi} = 0.375 + 0.106 + 0.0132 < \frac{1}{2}.$$

It remains to deal with $s \in \left[\pi, \frac{3}{2}\pi\right]$. Here a plot of the function $F_2(s), s \in \left[\pi, \frac{3}{2}\pi\right]$ shows that its maximum is 0.4922... . Combining the above estimates for F_1 and F_2, we see that

$$\sup_{k\geq 1} \|F_k\|_{L_\infty[0,\infty)} = \frac{1}{2} = F_2(0),$$

so that from Lemma 3.4(c), (d), (e),

$$\|h_{M,4}\|_{L_\infty[0,\infty)} = \frac{1}{2} + o(1).$$

Finally for $p \geq 4$, $h_{M,p} \leq h_{M,4}$, which together with Lemma 3.1(b), gives the result. ■

Proof of Theorem 1.4(b)
For $p \geq 4$, this follows from the lemma above and (1.13). ■

4. The Variance

Recall from (2.1) that

$$B_p(M,n)=\left(2^{np}\frac{2}{\pi}\right)^2\int_0^{\pi/2}\int_0^{\pi/2}\left(H_{M,p}(\theta,\phi)\right)^n d\phi\ d\theta,$$

where

$$H_{M,p}(\theta,\phi)=\frac{1}{M}\sum_{k=1}^{M}\left(|\sin k\theta|\,|\sin k\phi|\right)^p.$$

Lemma 4.1
Let $p\geq 1$.
(a)

$$H_{M,p}(\theta,\phi)\leq\sqrt{h_{M,2p}(\theta)\,h_{M,2p}(\phi)},$$

(b) There exists $C_p>0$ *such that for* $M\geq 1$ and $s,t,u,v\in\mathbb{R}$,

$$|H_{M,p}(s,t)-H_{M,p}(u,v)|\leq C_p\left(M|s-u|+M|t-v|\right). \tag{4.1}$$

(c) For $p\geq 2$,

$$\|H_{M,p}\|_{L_\infty\left(\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]\right)}=H_{m,p}\left(\frac{\pi}{2},\frac{\pi}{2}\right)+o(1)=\frac{1}{2}+o(1). \tag{4.2}$$

Proof
(a) This follows directly from Cauchy-Schwarz's inequality and the fact that $H_{M,p}(\theta,\theta)=h_{M,2p}(\theta)$.
(b) This follows much as in Lemma 3.1(a) .
(c) From (a),

$$\|H_{M,p}\|_{L_\infty\left(\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]\right)}=\|h_{M,2p}\|_{L_\infty\left[0,\frac{\pi}{2}\right]}.$$

Also from Lemma 3.6,

$$\|h_{M,2p}\|_{L_\infty\left[0,\frac{\pi}{2}\right]}=\frac{1}{2}+o(1)=h_{M,2p}\left(\frac{\pi}{2}\right)+o(1).$$

■

Lemma 4.2
If $p\geq 2$ *and* $\rho=1$,

$$\lim_{k\to\infty}B_p(M_k,n_k)^{1/n_k}=2^{2p-1}.$$

Proof
Firstly,

$$\begin{aligned}B_p(M_k,n_k)^{1/n_k} &\leq \left\{\left(2^{n_kp}\frac{2}{\pi}\right)^2\int_0^{\pi/2}\int_0^{\pi/2}\left(\|H_{M_k,p}\|_{L_\infty\left(\left[0,\frac{\pi}{2}\right]\times\left[0,\frac{\pi}{2}\right]\right)}\right)^{n_k}d\phi\ d\theta\right\}^{1/n_k}\\ &\leq 2^{2p}\left(\frac{1}{2}+o(1)\right),\end{aligned} \tag{4.3}$$

from Lemma 4.1(c). We turn to the corresponding lower bound. Let $\varepsilon\in\left(0,\frac{1}{2}\right)$. It follows from Lemma 4.1(b), that there exists $\eta>0$ such that for $s,t\in\left[0,\frac{\pi}{2}\right]$ with

$\left|s-\frac{\pi}{2}\right|<\frac{\eta}{M}$ and $\left|t-\frac{\pi}{2}\right|<\frac{\eta}{M}$, that

$$H_{M,p}\left(t\right)\geq\frac{1}{2}-\varepsilon$$

so that

$$\begin{aligned} B_p\left(M,n\right) &\geq \left(2^{np}\frac{2}{\pi}\right)^2\int_{\pi/2-\frac{\eta}{M}}^{\pi/2}\int_{\pi/2-\frac{\eta}{M}}^{\pi/2}\left(\frac{1}{2}-\varepsilon\right)^n d\phi\, d\theta \\ &= \left(2^{np}\frac{2}{\pi}\right)^2\left(\frac{\eta}{M}\right)^2\left(\frac{1}{2}-\varepsilon\right)^n. \end{aligned}$$

Letting $M=M_k$ and $n=n_k$, and $k\to\infty$, gives as $\rho=1$,

$$\liminf_{k\to\infty}B_p\left(M_k,n_k\right)^{1/n_k}\geq 2^{2p}\left(\frac{1}{2}-\varepsilon\right).$$

Here $\varepsilon>0$ is arbitrary. Together with (4.3), this gives the result. ■

Proof of Theorem 1.4(c)
Recall from (1.4) and (2.1) that

$$V_p\left(M,n\right)^2=B_p\left(M,n\right)-A_p\left(M,n\right)^2. \tag{4.4}$$

We shall show that the term $B_p(M_k,n_k)$ is geometrically larger than $A_p\left(M_k,n_k\right)^2$. From Theorem 1.4(a), with $\rho=1$,

$$\lim_{k\to\infty}A_p\left(M_k,n_k\right)^{1/n_k}=2^p\max\left\{\frac{1}{2}a_0,\left\|F_{k_0}\right\|_{L_\infty[0,\infty)}\right\}.$$

Here

$$\frac{1}{2}a_0\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sin t\right)^2dt=\frac{1}{2}$$

and from Lemma 3.5(c)

$$\left\|F_{k_0}\right\|_{L_\infty[0,\infty)}\leq\frac{1}{2}\left(1-\frac{\sin s_0}{s_0}\right).$$

This last right-hand side is larger than $\frac{1}{2}$. Then

$$\begin{aligned} \lim_{k\to\infty}A_p\left(M_k,n_k\right)^{2/n_k} &\leq 2^{2p-2}\left(1-\frac{\sin s_0}{s_0}\right)^2 \\ &< 2^{2p-2}\left(1.217...\right)^2 \\ &< 2^{2p-1}=\lim_{k\to\infty}B_p\left(M_k,n_k\right)^{1/n_k}, \end{aligned}$$

by Lemma 4.2. Now (4.4) gives the result. ■

Proof of Theorem 1.2(b)
This is the special case $p=2$ of Theorem 1.4(c). ■

5. Further Results

We can also estimate the average over subsequences of the integers that generate uniformly distributed subsequences, rather than requiring all $1\leq s_j\leq M$:

Proposition 5.1
Let $\{p_j\}_{j\geq 1}$ be an increasing sequence of positive integers such that for each irrational $\alpha \in (0,1)$ and continuous $f : [0,1] \to \mathbb{R}$, we have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{j=1}^{m} f\left(\{p_j \alpha\}\right) = \int_0^1 f(t)\, dt. \tag{5.1}$$

For $M \geq 1$, let $\mathcal{P}_M = \{p_1, p_2, ...p_M\}$. For $n \geq 1$, and $p > 0$, let

$$A_p\left(\mathcal{P}_M, n\right) = \frac{1}{M^n} \sum_{s_1, s_2 ... s_n \in \mathcal{P}_M} \left\| P_n\left(\{s_j\}, \cdot\right)\right\|_p^p.$$

Let $\{M_k\}, \{n_k\}$ be sequences of positive integers with limit ∞. Then

$$\liminf_{k\to\infty} A_p\left(\mathcal{P}_{M_k}, n_k\right)^{1/n_k} \geq 2^p \max\left\{\frac{2}{\pi}\int_0^{\frac{\pi}{2}} |\sin t|^p\, dt, \frac{1}{2\limsup_{k\to\infty} M_k^{1/n_k}}\right\}.$$

Proof
We see that as in Proposition 1.1,

$$A_p\left(\mathcal{P}_M, n\right) = 2^{np} \frac{2}{\pi} \int_0^{\pi/2} \left(\frac{1}{M} \sum_{k=1}^{M} |\sin p_k \theta|^p\right)^n d\theta$$

and can then proceed as in Lemma 3.2. ■

For example, the prime numbers satisfy (5.1), and for any positive integer L, so also do $p_j = j^L, j \geq 1$. Another direction is to replace the uniform bound M on $\{s_j\}$ with varying bounds. When these grow very rapidly, there is a simple explicit formula for the average of the L_2 norm:

Proposition 5.2
Let $\{M_j\}_{j=1}^n$ be positive integers satisfying for $2 \leq m \leq n$,

$$M_m \geq \sum_{j=1}^{m-1} M_j. \tag{5.2}$$

Let

$$A_n = \frac{1}{M_1 M_2 ... M_n} \sum_{1\leq s_j \leq M_j, 1\leq j\leq n} \left\| P_n\left(\{s_j\}, \cdot\right)\right\|_2^2.$$

Then

$$A_n = 2^n \prod_{j=2}^{n} \left(1 + \frac{1}{2M_j}\right). \tag{5.3}$$

Proof
The proof is essentially via induction. Let $\mathcal{P}_m$ denote the set of all polynomials of the form $\prod_{j=1}^{m} (1 - z^{s_j})$ with $1 \leq s_j \leq M_j$, all $1 \leq j \leq m$. We observe that we obtain all polynomials in $\mathcal{P}_m$ from those in $\mathcal{P}_{m-1}$ by multiplying by factors

$(1 - z^{s_m})$ where $1 \leq s_m \leq M_m$. So fix a polynomial P in $\mathcal{P}_{m-1}$. It will have degree at most M_m because of (5.2). We see that for $m \geq 2$,

$$\begin{aligned} & \sum_{s_m=1}^{M_m} \|P(z)(1 - z^{s_m})\|_2^2 \\ = & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 \left(\sum_{s_m=1}^{M_m} \left|1 - e^{is_m\theta}\right|^2 \right) d\theta \\ = & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 2 \sum_{s_m=1}^{M_m} (1 - \cos s_m\theta)\, d\theta \\ = & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 \{(2M_m + 1) - 2D_{M_m}(\theta)\}\, d\theta, \end{aligned}$$

where

$$D_{M_m}(\theta) = \frac{1}{2} + \sum_{k=1}^{M_m} \cos k\theta$$

is the usual Dirichlet kernel of Fourier series. Here $\left|P\left(e^{i\theta}\right)\right|^2 = P\left(e^{i\theta}\right) P\left(e^{-i\theta}\right)$ is a trigonometric polynomial of degree at most $\sum_{j=1}^{m-1} M_j \leq M_m$. By the usual reproducing kernel property of Fourier series, we then have for $m \geq 2$,

$$\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 D_{M_m}(\theta)\, d\theta = \left|P\left(e^{i0}\right)\right|^2 = 0. \tag{5.4}$$

(Note that when $m = 1$, we have $P = 1$, so we instead obtain 1.) Then for $m \geq 2$,

$$\sum_{s_m=1}^{M_m} \|P(z)(1 - z^{s_m})\|_2^2 = (2M_m + 1) \|P\|_2^2.$$

Adding over all P in $\mathcal{P}_{m-1}$ gives the identity

$$\sum_{P \in \mathcal{P}_m} \|P\|_2^2 = (2M_m + 1) \sum_{P \in \mathcal{P}_{m-1}} \|P\|_2^2.$$

Applying this repeatedly gives

$$\sum_{P \in \mathcal{P}_n} \|P\|_2^2 = (2M_1) \prod_{j=2}^{n} (2M_j + 1),$$

where we have used the fact that for $m = 1$, we have 1 rather then 0 in (5.4). Dividing by $M_1 M_2 ... M_n$ gives the result. ■

When we have an infinite sequence $\{M_n\}$ satisfying (5.2), the product in (5.3) converges, and so the average grows like $c2^n$ for some constant c.

One interesting question is the distribution of the norms of the polynomials. Numerical calculations suggest some sort of bell curve for the distribution of the L_2 norms. It would be good to have a theoretical justification of the bell shape. Following is a typical example that was generated using our algorithm, with $M = n$, and $n = 10, 11, ..., 20$. Here are the steps:

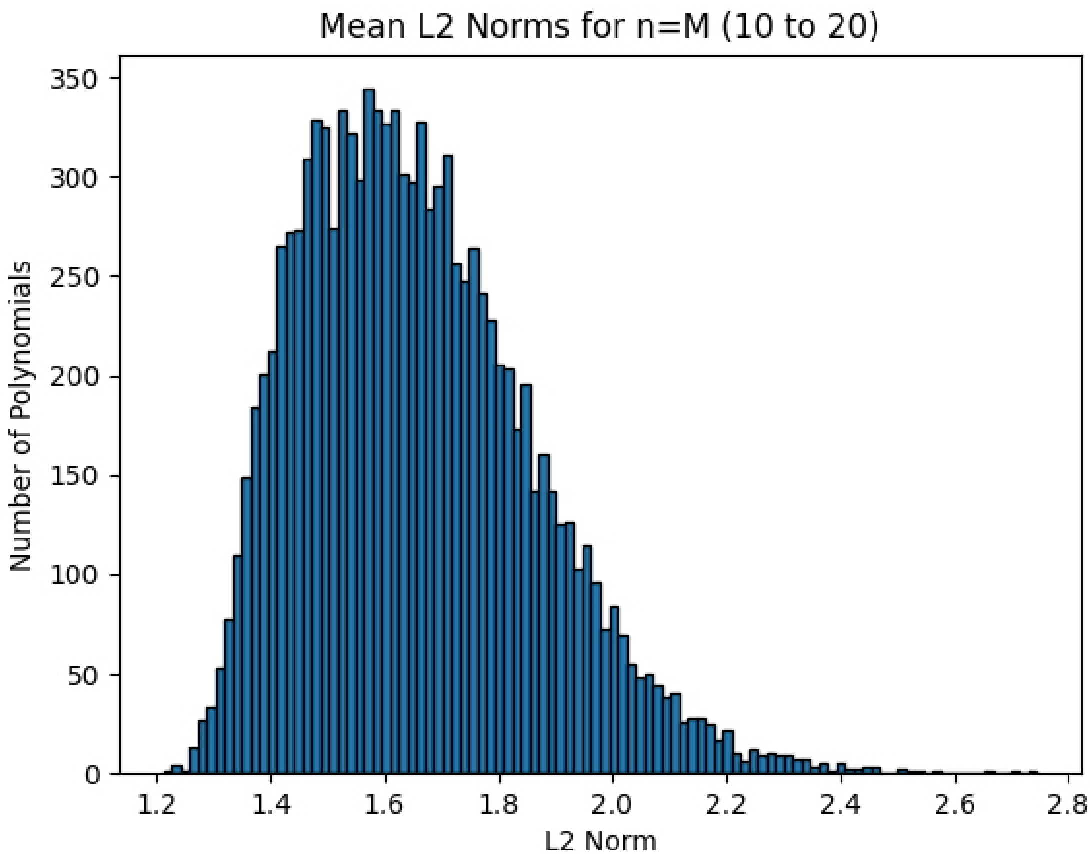
(1) Uniformly sample (with repetition) from the set of all possible n-tuples $(s_1, s_2, ..., s_n)$ with each $1 \leq s_j \leq M$.

(2) Calculate $\left\|\prod_{j=1}^{n}(1-z^{s_j})\right\|_2^2$.

(3) Store the result and return to step (1) until the desired number of polynomials have been sampled.

L2

Norms



$_1.jpg$

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