

ON LOWER BOUNDS FOR ERDŐS SZEKERES PRODUCTS

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ABSTRACT. Let $\{s_j\}_{j=1}^n$ be positive integers. We show that for any $1 \leq L \leq n$,

$$\left\| \prod_{j=1}^n (1 - z^{s_j}) \right\|_{L_\infty(|z|=1)} \geq \exp \left(\frac{1}{2e} \frac{L}{(s_1 s_2 \dots s_L)^{1/L}} \right).$$

In particular, this gives geometric growth if a positive proportion of the $\{s_j\}$ are bounded. We also show that when the $\{s_j\}$ grow regularly and faster than $j(\log j)^{2+\varepsilon}$, some $\varepsilon > 0$, then the norms grow faster than $\exp((\log n)^{1+\delta})$ for some $\delta > 0$.

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1. INTRODUCTION

A celebrated short 1959 paper of Erdős and Szekeres [14] posed a number of problems about the growth or decay of "pure power products"

$$(1.1) \quad P_n(z) = \prod_{j=1}^n (1 - z^{s_j})$$

and their norms

$$\|P_n\| = \|P_n\|_{L_\infty(|z|=1)}.$$

Here $\{s_j\}_{j=1}^n$ are positive integers. Perhaps the most well known is the following:

Problem

Let

$$M(s_1, s_2, \dots, s_n) = \left\| \prod_{j=1}^n (1 - z^{s_j}) \right\|$$

and

$$f(n) := \inf \{M(s_1, s_2, \dots, s_n) : s_1, s_2, \dots, s_n \geq 1\}.$$

Determine the growth of $f(n)$ as $n \rightarrow \infty$.

Erdős and Szekeres proved that

$$\lim_{n \rightarrow \infty} f(n)^{1/n} = 1.$$

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This provided a contrast to a 1964 paper of C. Sudler [26] where it was shown that

$$\lim_{n \rightarrow \infty} M(1, 2, \dots, n)^{1/n} = 1.219\dots > 1.$$

Perhaps the first major advance is due to Atkinson in 1961 [4], showing that

$$f(n) = \exp\left(O\left(n^{1/2} \log n\right)\right),$$

while in 1982, Odlyzko [22] proved that

$$f(n) = \exp\left(O\left(n^{1/3} (\log n)^{4/3}\right)\right).$$

As far as we are aware, the best current result is the 1996 estimate of Belov and Konyagin [7]

$$f(n) = \exp\left(O\left((\log n)^4\right)\right),$$

a consequence of their work on nonnegative trigonometric polynomials. We note that in many of these upper bounds, the $\{s_j\}$ are not necessarily distinct. Erdős and Szekeres asserted that $f(n) \geq \sqrt{2n}$. This is still the best general lower bound, though for $n = 7, 9, 10, 11$, Maltby established a larger lower bound [20], [21]. Erdős [13, p. 55] later conjectured that $f(n)$ should grow faster than any power of n .

There are several important related results: for example, Bell, Borwein, and Richmond [6] showed in 1998, that if L is a positive integer,

$$\liminf_{n \rightarrow \infty} M(1, 2^L, 3^L, \dots, n^L)^{1/n} > 1.$$

Borwein [9] showed that if none of the $\{s_j\}$ are divisible by a given prime p , $M(s_1, s_2, \dots, s_n)$ grows at least as fast as p^n with strict inequality if $p \geq 15$. Bourgain and Chang [11] showed that we can choose $\{s_1, s_2, \dots, s_n\} \subset \{1, 2, \dots, N\}$ with $n/N \rightarrow 1/2$ such that

$$M(s_1, s_2, \dots, s_n) \leq \exp\left(O\left(\sqrt{n} \sqrt{\log n} \log \log n\right)\right)$$

but if $\tau > 0$ is small enough and $n > (1 - \tau)N$, then for all $\{s_1, s_2, \dots, s_n\} \subset \{1, 2, \dots, N\}$,

$$M(s_1, s_2, \dots, s_n) > \exp(\tau n).$$

There are several other pointwise problems in [14] that we do not have space to review here. Some relevant references for these are [1], [2], [3], [5], [15], [16], [17], [19], [27].

This paper is organized as follows. We state our new results in Section 2. Theorem 2.1 is proved in Section 3. Theorem 2.2 and Corollaries 2.3, 2.4 are proved in Section 4. We close this section with some notation. We use $[x]$ to denote the largest integer $\leq x$ to denote the fractional part. P_n is always defined by (1.1).

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2. NEW RESULTS

Our first result, proved using Poisson integral representation, provides a lower bound for $M(s_1, s_2, \dots, s_n)$ that is useful, when s_j grows somewhat slower than j .

Theorem 2.1

Let $n \geq 1$ and $1 \leq s_1 \leq s_2 \leq \dots \leq s_n$. Let $1 \leq L \leq n$. Let P_n be defined by (1.1). Then

(a)

$$(2.1) \quad \|P_n\| \geq \exp\left(\frac{1}{2e} \frac{L}{(s_1 s_2 \dots s_L)^{1/L}}\right).$$

(b) Moreover, for any $p > 0$,

$$(2.2) \quad \|P_n\| \geq \left(\frac{\sqrt{\pi}\Gamma(\frac{1}{2}p+1)}{\Gamma(\frac{1}{2}p+\frac{1}{2})}\right)^{1/p} \left(\exp\left(\frac{pL}{2e(s_1 s_2 \dots s_L)^{1/L}}\right) - 1\right)^{1/p}.$$

Remarks

(i) Let $r \in (0, 1)$. Observe that if, for example, $s_j \leq A$ for $1 \leq j \leq [rn]$, then (a) gives

$$\|P_n\| \geq \exp\left(\frac{[rn]}{2eA}\right).$$

Thus we obtain geometric growth. As a second example, if $s_j \leq \frac{j}{1+(\log j)^2}$, for $1 \leq j \leq [rn]$, estimation of the product in (2.1) shows that

$$\|P_n\| \geq \exp\left(\frac{1}{2}(\log[rn])^2(1+o(1))\right).$$

However, if all $\{s_j\}$ are distinct, so that $s_j \geq j$, the estimate is not useful.

(ii) When $p = 1$, (b) gives

$$\|P_n\| \geq \frac{\pi}{2} \left(\exp\left(\frac{L}{2e(s_1 s_2 \dots s_L)^{1/L}}\right) - 1\right).$$

This is better than the estimate in (a) except when the exponential term is close to 1.

While Theorem 2.1 works well when the $\{s_j\}$ do not grow rapidly, our second result, proved using Kellogg's extension of the Hausdorff-Young inequalities, works well for rapidly growing or separated $\{s_j\}$:

Theorem 2.2

Let $I_k = \{2^{k-1}, 2^{k-1} + 1, \dots, 2^k - 1\}$ for $k \geq 1$. Let $1 \leq s_1 \leq s_2 \leq \dots \leq s_n$. Assume that I_k contains $\ell_k \geq 0$ of the $\{s_j\}_{j=1}^n$ for $k \geq 1$, so that $\sum_{k=1}^{\infty} \ell_k = n$. Let $1 < p \leq 2$ and $\varepsilon = \frac{2}{p}(p-1)$. Then for $n \geq 2$,

$$(2.3) \quad \|P_n\| \geq \exp\left(C \left\{\frac{\sum_{k=1}^{\infty} \ell_k \varepsilon}{(n \log n)^\varepsilon}\right\}^{p/2}\right).$$

Here C depends on p but is independent of n and the $\{s_j\}$.

Corollary 2.3

If $C_1 > 0, B > 2$ and for all k

$$(2.4) \quad \ell_k \leq \frac{C_1 n}{(\log n)^B},$$

then for some $\delta > 0, C_2 > 0$,

$$(2.5) \quad \|P_n\| \geq \exp\left(C_2 (\log n)^{1+\delta}\right).$$

Here C_2, δ are independent of n and the $\{s_j\}$, but depend on B, C_1 .

Corollary 2.4

If for some $\tau > 0$, there are at least n^τ of the $\{I_k\}$ containing at least one s_j , then for some $\delta > 0$,

$$(2.6) \quad \|P_n\| \geq \exp(Cn^\delta).$$

Example

Let $B > 2$ and

$$s_j = \lceil j (\log j)^B \rceil, \quad j \geq 1.$$

Given $k \geq 1$, the largest j for which

$$s_j \leq 2^k - 1$$

satisfies

$$j \leq 2^k (\log 2^k)^{-B} (1 + o(1))$$

It follows that given large $n \geq 1$, there are at most $O\left(n (\log n)^{-B}\right)$ of $\{s_j\}_{j=1}^n$ lying in any I_k . Then the hypotheses of Corollary 2.3 are satisfied, and we have the lower bound (2.5).

3. PROOF OF THEOREM 2.1

Erdős and Szekeres asserted that

$$f(n) := \inf \{M(s_1, s_2, \dots, s_n) : s_1, s_2, \dots, s_n \geq 1\} \geq \sqrt{2n}.$$

Briefly, their proof [14, p. 34 ff.] runs as follows: assume that for some increasing integers, $\{a_j\}_{j=1}^r$ and another distinct set of increasing integers $\{b_j\}_{j=1}^r$, (so that there is no intersection between the $\{a_j\}, \{b_j\}$)

$$(3.1) \quad \sum_{j=1}^r z^{a_j} - \sum_{j=1}^r z^{b_j} = \prod_{j=1}^n (1 - z^{s_j}).$$

In particular all coefficients of powers of z are ± 1 . Then as the right-hand side has a zero of multiplicity n at 1, we can differentiate the left-hand side k times and set $z = 1$ to obtain

$$\sum_{j=1}^r a_j^k = \sum_{j=1}^r b_j^k, \quad k = 0, 1, \dots, n - 1.$$

They then deduce that this solution of the Prouhet-Tarry-Escott problem [10] requires $r \geq n$, so that there are at least $2n$ non-zero coefficients. The latter can directly be justified by showing that if $r < n$,

$$\prod_{j=1}^n (x - a_j) = \prod_{j=1}^n (x - b_j)$$

and hence the $\{a_j\}, \{b_j\}$ are identical, a contradiction.

There are problems with this proof that do not seem to have been addressed in the literature. The identity (3.1) is simply not true for all n , as very often there are coefficients other than ± 1 . For example,

$$\prod_{j=1}^4 (1 - z^j) = 1 - z - z^2 + 2z^5 - z^8 - z^9 + z^{10}.$$

For the moment, let us ignore this problem and continue: write

$$P_n(z) = \prod_{j=1}^n (1 - z^{s_j}) = \sum_{k=0}^N a_k z^k.$$

If as asserted by Erdős-Szekeres, there are at least $2n$ non-zero coefficients, then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{it})|^2 dt = \sum_{k=0}^N |a_k|^2 \geq 2n,$$

so that

$$(3.2) \quad \|P_n\| \geq \sqrt{2n}.$$

This bound can, however, be properly proved using a well known method: since P_n has a zero of order n at 1, we have $P_n^{(j)}(1) = 0$, for $j = 0, 1, \dots, n-1$, leading to

$$\sum_{k=0}^N a_k k(k-1)(k-2) \dots (k-j+1) = 0, \quad 0 \leq j \leq n-1.$$

Since every polynomial $S(x)$ of degree at most $n-1$ can be expressed as a linear combination of the polynomials $\omega_0(x) = 1$ and $\omega_j(x) = x(x-1) \dots (x-j+1)$, $1 \leq j \leq n-1$, we obtain

$$\sum_{k=0}^N a_k S(k) = 0$$

for every such S . As P_n is not identically 0, this forces at least n coefficients in P_n not to be 0. So

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{it})|^2 dt = \sum_{k=0}^N |a_k|^2 \geq n.$$

Since all zeros are on the unit circle, a result of O'Hara and Rodriguez [23, Corollary 1, p. 333] shows that

$$\|P_n\|_{L^\infty(|z|=1)}^2 \geq 2 \sum_{k=0}^N |a_k|^2 \geq 2n$$

so that indeed

$$f(n) \geq \sqrt{2n}.$$

If the original Erdős-Szekeres proof could be fixed, the O'Hara Rodriguez bound would give $f(n) \geq 2\sqrt{n}$. We note that the O'Hara Rodriguez bound is a special case of a bound of Saff and Sheil-Small [25].

We turn to the

Proof of Theorem 2.1

(a) We use the Poisson integral representation. Since $\log |P_n|$ is harmonic in the unit disc, and integrable on the unit circle, we have for $r < 1$, $\theta \in [-\pi, \pi]$,

$$(3.3) \quad \log |P_n(re^{i\theta})| = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log |P_n(e^{it})|) \mathcal{P}(r, \theta - t) dt.$$

Here $\mathcal{P}(r, t)$ is the Poisson kernel, satisfying for $0 < r < 1$, $t \in [-\pi, \pi]$,

$$(3.4) \quad 0 \leq \mathcal{P}(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2} \leq \frac{2}{1 - r}.$$

Next, with $\log^+ x = \max\{0, \log x\}$ and $\log^- x = -\min\{0, \log x\}$, we have $\log x = \log^+ x - \log^- x$, so

$$(3.5) \quad \begin{aligned} \log \frac{1}{|P_n(re^{i\theta})|} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^- |P_n(e^{it})| - \log^+ |P_n(e^{it})|) \mathcal{P}(r, \theta - t) dt \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^- |P_n(e^{it})|) \mathcal{P}(r, \theta - t) dt. \end{aligned}$$

Next, from the identity

$$\int_{-\pi}^{\pi} \log |1 - e^{it}| dt = 0,$$

we obtain

$$(3.6) \quad \begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |P_n(e^{it})| dt &= 0 \\ \Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |P_n(e^{it})| dt &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt. \end{aligned}$$

Then (3.4-3.6) give for any $0 < r < 1$,

$$(3.7) \quad \log \frac{1}{|P_n(r)|} \leq \frac{2}{1 - r} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt.$$

Now we choose r . First recall that for $0 < r < 1$,

$$\begin{aligned} \frac{1 - r^{s_j}}{1 - r} &= \sum_{k=0}^{s_j-1} r^k \leq s_j \\ \Rightarrow \log (1 - r^{s_j})^{-1} &\geq -\log s_j - \log (1 - r). \end{aligned}$$

Since also all $\log(1 - r^{s_j})^{-1} \geq 0$, we can drop terms to obtain for $1 \leq L \leq n$,

$$\begin{aligned} \log \frac{1}{|P_n(r)|} &\geq \sum_{j=1}^L \log(1 - r^{s_j})^{-1} \\ &\geq - \left(\sum_{j=1}^L \log s_j \right) - L \log(1 - r) \\ &= -L \{ \log M_L + \log(1 - r) \}, \end{aligned}$$

where

$$M_L = (s_1 s_2 \dots s_L)^{1/L}.$$

Then (3.7) gives

$$-L(1 - r) \{ \log M_L + \log(1 - r) \} \leq 2 \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt.$$

By differentiation with respect to r , we find that the best choice of r is given by $1 - r = \frac{1}{eM_L}$. Then this last inequality gives

$$(3.8) \quad \frac{L}{2eM_L} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt \leq \log \|P_n\|.$$

This yields (2.1).

(b) Here we use the arithmetic-geometric inequality/ Jensen's inequality and a result of Saff and Sheil-Small [25]:

$$\begin{aligned} &1 + \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{it})|^p dt \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \max \{ 1, |P_n(e^{it})|^p \} dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(p \log^+ |P_n(e^{it})|) dt \\ &\geq \exp \left(p \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt \right) \\ &\geq \exp \left(\frac{pL}{2eM_L} \right), \end{aligned}$$

by (3.8). So

$$(3.9) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{it})|^p dt \geq \exp \left(\frac{pL}{2eM_L} \right) - 1.$$

Next, a result of Ed Saff and T. Sheil-Small [25, Theorem 1, p. 17] shows that

$$\|P_n\|^p \geq \frac{\sqrt{\pi} \Gamma(\frac{1}{2}p + 1)}{\Gamma(\frac{1}{2}p + \frac{1}{2})} \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(e^{it})|^p dt.$$

Combining this with (3.9) and taking p th roots gives (2.2). ■

4. PROOF OF THEOREM 2.2 AND COROLLARIES 2.3, 2.4

For $t > 0$, let

$$(4.1) \quad \mu_n(t) = \text{meas} \{ \theta \in [-\pi, \pi] : \log^- |P_n(e^{i\theta})| > t \},$$

denote the distribution function of $\log^- |P_n|$. Its usefulness is apparent from the formula [24, p. 172]

$$(4.2) \quad \int_0^\infty pt^{p-1} \mu_n(t) dt = \int_{-\pi}^\pi (\log^- |P_n(e^{i\theta})|)^p d\theta, \quad p > 0.$$

We can obtain estimates of μ_n from Cartan's lemma on small values of polynomials or Nazarov's estimate for exponential sums, but these involve the size of $\{s_n\}$. Instead we use:

Lemma 4.1

$$(4.3) \quad \mu_n(t) \leq \pi n e^{-t/n}, \quad t > 0.$$

Proof

Let $\delta \in (0, 2n)$ and

$$\mathcal{F}_j = \{ \theta \in [-\pi, \pi] : |1 - e^{is_j\theta}| \leq \delta/n \} = \left\{ \theta \in [-\pi, \pi] : \left| \sin \frac{s_j\theta}{2} \right| \leq \frac{\delta}{2n} \right\}.$$

Here if k is the integer closest to $\frac{s_j\theta}{2\pi}$,

$$\begin{aligned} \left| \sin \frac{s_j\theta}{2} \right| &= \left| \sin \pi \left(\frac{s_j\theta}{2\pi} - k \right) \right| \leq \frac{\delta}{2n} \\ \Leftrightarrow \pi \left| \frac{s_j\theta}{2\pi} - k \right| &\leq \arcsin \left(\frac{\delta}{2n} \right) \\ \Leftrightarrow \theta &\in \left[\frac{2k\pi}{s_j} - \frac{2}{s_j} \arcsin \left(\frac{\delta}{2n} \right), \frac{2k\pi}{s_j} + \frac{2}{s_j} \arcsin \left(\frac{\delta}{2n} \right) \right]. \end{aligned}$$

So

$$\mathcal{F}_j \subset [-\pi, \pi] \cap \bigcup_{|k| \leq s_j/2} \left[\frac{2k\pi}{s_j} - \frac{2}{s_j} \arcsin \left(\frac{\delta}{2n} \right), \frac{2k\pi}{s_j} + \frac{2}{s_j} \arcsin \left(\frac{\delta}{2n} \right) \right]$$

$$\Rightarrow \text{meas}(\mathcal{F}_j) \leq (s_j) \frac{4}{s_j} \arcsin \left(\frac{\delta}{2n} \right) \leq \pi \frac{\delta}{n},$$

by the inequality $|\arcsin v| \leq \frac{\pi}{2} |v|$, $v \in (-1, 1)$. Let

$$\mathcal{F} = \bigcup_{j=1}^n \mathcal{F}_j$$

so that $\text{meas}(\mathcal{F}) \leq \pi\delta$. Also if $\theta \notin \mathcal{F}$,

$$|P_n(e^{i\theta})| > \left(\frac{\delta}{n} \right)^n.$$

Then if $\delta \in (0, n)$,

$$\text{meas} \left\{ \theta \in [-\pi, \pi] : \log |P_n(e^{i\theta})|^{-1} > \log \left(\frac{\delta}{n} \right)^{-n} \right\} \leq \text{meas}(\mathcal{F}) \leq \pi\delta.$$

Note that necessarily $|P_n(e^{i\theta})| < 1$ for such θ . Setting $t = \log\left(\frac{\delta}{n}\right)^{-n} \Leftrightarrow \delta = ne^{-t/n}$, and noting that $\delta \in (0, n) \Leftrightarrow t \in (0, \infty)$, we obtain (4.3). ■

The classical Hausdorff-Young Inequality [24, p. 261], [28, p. 101, Thm. 12.2.3] asserts that if $1 < p \leq 2$ and $f \in L_p[-\pi, \pi]$, and has Fourier coefficients

$$f_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} dt, \quad j \in \mathbb{Z},$$

then

$$\left\{ \sum_{j=-\infty}^{\infty} |f_j|^{p'} \right\}^{1/p'} \leq \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right\}^{1/p},$$

where $p' = \frac{p}{p-1}$. Kellogg's extension of the Hausdorff-Young inequality states that [18, p.125, Theorem 3]

$$(4.4) \quad \left\{ \sum_{k=-\infty}^{\infty} \left(\sum_{j \in I_k} |f_j|^{p'} \right)^{2/p'} \right\}^{1/2} \leq A_p \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right\}^{1/p},$$

where if $k > 0$, $I_k = \{j \in \mathbb{Z} : 2^{k-1} \leq j < 2^k\}$ while $I_{-k} = -I_k$ if $k < 0$, and $I_0 = \{0\}$. The constant A_p depends only on p . We let

$$(4.5) \quad C_0 = \frac{1}{2} \left(2^{p/2} A_p^p \right)^{-1}.$$

We apply this to $f = \log |P_n|$. Recall that ℓ_k denotes the number of s_j in I_k for $k \geq 1$.

Lemma 4.2

(a)

$$(4.6) \quad \log |P_n(e^{i\theta})| = - \sum_{\ell=1}^{\infty} \frac{\Lambda_{\ell}}{\ell} \cos \ell\theta,$$

where

$$(4.7) \quad \Lambda_{\ell} = \sum_{j: s_j | \ell} s_j.$$

(b) If $1 < p \leq 2$, and $p' = \frac{p}{p-1}$,

$$(4.8) \quad n^{p-1} \leq \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2} \leq (2C_0)^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |P_n(e^{it})||^p dt.$$

(c) Either

$$(4.9) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^+ |P_n(e^{it})|)^p dt \geq C_0 \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2}$$

or for $n \geq n_0(p)$,

$$(4.10) \quad \int_{-\pi}^{\pi} \log^- |P_n(e^{it})| dt \geq \frac{C_1}{(n \log n)^{p-1}} \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2}.$$

The threshold n_0 and the constant C_1 depend only on p and not on $n, \{s_j\}$.

Proof

(a) The Taylor series

$$\log(1-z) = -\sum_{k=1}^{\infty} \frac{z^k}{k}, |z| < 1,$$

gives for such z ,

$$\log P_n(z) = -\sum_{j=1}^n \sum_{k=1}^{\infty} \frac{z^{ks_j}}{k} = -\sum_{\ell=1}^{\infty} \frac{\Lambda_\ell}{\ell} z^\ell,$$

where Λ_ℓ is given by (4.7). Also then, by taking real parts, we obtain the Fourier series expansion (4.6). This converges uniformly in closed subarcs of the unit circle omitting zeros of P_n as $\log |P_n|$ is differentiable in such arcs.

(b) Now (4.4) with $f_{\pm\ell} = \frac{1}{2} \frac{\Lambda_\ell}{\ell}$, $\ell \geq 1$ and $f_0 = 0$, gives

$$\left\{ 2 \sum_{k=1}^{\infty} \left(\sum_{j \in I_k} \left(\frac{\Lambda_j}{2j} \right)^{p'} \right)^{2/p'} \right\}^{1/2} \leq A_p \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |P_n(e^{it})||^p dt \right\}^{1/p}.$$

(4.11)

Here as $s_j |s_j$,

$$\Lambda_{s_j} \geq s_j.$$

Then (4.11) gives

$$(4.12) \quad 2C_0 \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |P_n(e^{it})||^p dt.$$

This gives the rightmost inequality in (4.8). Recall we defined C_0 by (4.5). Next,

$$\sum_{k=1}^{\infty} \ell_k = n.$$

Since $2/p' < 1$, repeated use of $(x+y)^{2/p'} \leq x^{2/p'} + y^{2/p'}$ for $x, y \geq 0$, gives

$$(4.13) \quad \sum_{k=1}^{\infty} \ell_k^{2/p'} \geq n^{2/p'}.$$

Finally, $p/p' = p-1$, so we obtain the leftmost inequality in (4.8).

(c) As the functions \log^\pm have disjoint support, so

$$(4.14) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |P_n(e^{it})||^p dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^+ |P_n(e^{it})|)^p dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} (\log^- |P_n(e^{it})|)^p dt.$$

If (4.9) fails, then (4.12) shows that

$$(4.15) \quad \geq C_0 \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2}$$

We now turn this into an estimate for $\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |P_n(e^{it})| dt$. By Lemma 4.1,

$$\begin{aligned} & \int_{(p+1)n \log n}^{\infty} pt^{p-1} \mu_n(t) dt \\ & \leq \pi n \int_{(p+1)n \log n}^{\infty} pt^{p-1} e^{-t/n} dt \\ & = \pi n^{p+1} \int_{(p+1) \log n}^{\infty} py^{p-1} e^{-y} dy. \end{aligned}$$

Integrating by parts, and using that $p \leq 2$, we continue this as

$$\begin{aligned} & = \pi n^{p+1} \left\{ -py^{p-1} e^{-y} \Big|_{y=(p+1) \log n}^{y=\infty} + \int_{(p+1) \log n}^{\infty} p(p-1) y^{p-2} e^{-y} dy \right\} \\ & \leq \pi p ((p+1) \log n)^{p-1} + \pi p (p-1), \end{aligned}$$

provided $(p+1) \log n \geq 1$. (This is true for $n \geq 2$). Then from (4.2),

$$\begin{aligned} & \int_{-\pi}^{\pi} \log^- |P_n(e^{it})| dt \\ & = \int_0^{\infty} \mu_n(t) dt \\ & \geq \int_0^{(p+1)n \log n} \frac{pt^{p-1}}{p((p+1)n \log n)^{p-1}} \mu_n(t) dt \\ & = \frac{1}{p((p+1)n \log n)^{p-1}} \left[\int_0^{\infty} - \int_{(p+1)n \log n}^{\infty} \right] pt^{p-1} \mu_n(t) dt \\ & \geq \frac{1}{p((p+1)n \log n)^{p-1}} \left(2\pi C_0 \left\{ \sum_{k=1}^{\infty} \ell_k^{2/p'} \right\}^{p/2} - \pi p ((p+1) \log n)^{p-1} - \pi p (p-1) \right) \end{aligned}$$

by (4.15). In view of the leftmost inequality in (4.8), we obtain (4.10) for $n \geq n_0(p)$.

■

Proof of Theorem 2.2

If (4.9) is true, then recalling $\varepsilon = \frac{2}{p}(p-1) = \frac{2}{p'}$,

$$(\log \|P_n\|)^p \geq C_0 \left\{ \sum_{k=1}^{\infty} \ell_k^{\varepsilon} \right\}^{p/2}.$$

Then as

$$(\log \|P_n\|)^{p-1} \leq (\log 2^n)^{p-1} = (n \log 2)^{\varepsilon p/2},$$

so

$$(4.16) \quad \log \|P_n\| \geq C_0 \left\{ \frac{\sum_{k=1}^{\infty} \ell_k^{\varepsilon}}{(n \log 2)^{\varepsilon}} \right\}^{p/2}.$$

This is a stronger estimate than (2.3). Now suppose (4.9) fails. From (3.6),

$$\begin{aligned} \log \|P_n\|_{L_\infty(|z|=1)} &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |P_n(e^{it})| dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |P_n(e^{it})| dt \\ &\geq \frac{C_1}{2\pi} \frac{1}{(n \log n)^{p-1}} \left\{ \sum_{k=1}^{\infty} \ell_k^\varepsilon \right\}^{p/2} \\ &= \frac{C_1}{2\pi} \left\{ \frac{\sum_{k=1}^{\infty} \ell_k^\varepsilon}{(n \log n)^\varepsilon} \right\}^{p/2}, \end{aligned}$$

by (4.10) and provided $n \geq n_0(p)$. This is easily reformulated as (2.3) for $n \geq n_0(p)$. For the integers $n = 2, 3, \dots, n_0(p) - 1$,

$$\frac{\sum_{k=1}^{\infty} \ell_k^\varepsilon}{(n \log n)^\varepsilon} \leq \frac{n}{(\log 2)^\varepsilon} \leq \frac{n_0}{(\log 2)^\varepsilon}$$

while the left-hand side exceeds $\log \sqrt{2n} \geq \log \sqrt{2}$, so increasing the size of C gives the inequality (2.3) for all $n \geq 2$. ■

Proof of Corollary 2.3

Here if all $\ell_k \leq \frac{C_1 n}{(\log n)^B}$,

$$n = \sum_{k=1}^{\infty} \ell_k \leq \left(\frac{C_1 n}{(\log n)^B} \right)^{1-\varepsilon} \sum_{k=1}^{\infty} \ell_k^\varepsilon.$$

So

$$\left\{ \frac{\sum_{k=1}^{\infty} \ell_k^\varepsilon}{(n \log n)^\varepsilon} \right\}^{p/2} \geq \left(C_1^{-(1-\varepsilon)} (\log n)^{B(1-\varepsilon)-\varepsilon} \right)^{p/2}.$$

Here,

$$[B(1-\varepsilon) - \varepsilon] \frac{p}{2} > 1 \Leftrightarrow B > \frac{2p}{2-p}.$$

If $B > 2$, then we can choose p close enough to 1 so that this last inequality is satisfied. Together with (2.3), this gives for some $\delta > 0$,

$$\|P_n\| \geq \exp \left(C (\log n)^{1+\delta} \right).$$

■

Remark

The same conclusion holds if we can find m different $\{I_{k_i}\}$ such that I_{k_i} contains ℓ_{k_i} of the $\{s_j\}$, where each $\ell_{k_i} \leq Cn/(\log n)^B$ and for some fixed $\rho \in (0, 1)$,

$$\sum_{i=1}^m \ell_{k_i} \geq \rho n.$$

Proof of Corollary 2.4

If at least n^τ of the $\{I_k\}$ contain some s_j , so that at least n^τ of the $\ell_k \geq 1$, we then have

$$\frac{\sum_{k=1}^{\infty} \ell_k^\varepsilon}{(n \log n)^\varepsilon} \geq n^{\tau-\varepsilon} (\log n)^{-\varepsilon}$$

which will grow like a power of n if ε is small enough. ■

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