EXPLICIT ORTHOGONAL POLYNOMIALS FOR RECIPROCAL POLYNOMIAL WEIGHTS ON $(-\infty, \infty)$

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ABSTRACT. Let S be a polynomial of degree $2n+2$, that is positive on the real axis, and let $w = 1/S$ on $(-\infty, \infty)$. We present an explicit formula for the nth orthogonal polynomial and related quantities for the weight w. This is an analogue for the real line of the classical Bernstein-Szegő formula for $(-1, 1)$.

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1. The Result[1]

The Bernstein-Szegő formula provides an explicit formula for orthogonal polynomials for a weight of the form $\sqrt{1-x^2}/S(x)$, $x \in (-1,1)$, where S is a polynomial positive in $(-1,1)$, possibly with at most simple zeros at ± 1. It plays a key role in asymptotic analysis of orthogonal polynomials.

In this paper, we present an explicit formula for the nth degree orthogonal polynomial for weights w on the whole real line of the form

$$w = 1/S, \tag{1.1}$$

where S is a polynomial of degree $2n+2$, positive on $\mathbb{R}$. In addition, we give representations for the $(n+1)$st reproducing kernel and Christoffel function. We present elementary proofs, although they follow partly from the theory of de Branges spaces [1]. The formulae do not seem to be recorded in de Branges' book, nor in the orthogonal polynomial literature [2], [3], [7], [8], [9]. We believe they will be useful in analyzing orthogonal polynomials for weights on $\mathbb{R}$.

Recall that we may define orthonormal polynomials $\{p_m\}_{m=0}^{n}$, where

$$p_m(x) = \gamma_m x^m + ..., \gamma_m > 0, \tag{1.2}$$

satisfying

$$\int_{-\infty}^{\infty} p_j p_k w = \delta_{jk}.$$

Because the denominator S in w has degree $2n+2$, orthogonal polynomials of degree higher than n are not defined. The $(n+1)$ st reproducing kernel for w is

$$K_{n+1}(x,y) = \sum_{j=0}^{n} p_j(x)\, p_j(y). \tag{1.3}$$

Inasmuch as S is a positive polynomial, we can write

$$S(z) = E(z)\, \overline{E(\bar{z})}, \tag{1.4}$$

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where E is a polynomial of degree $n+1$, with all zeros in the lower-half plane $\{z : \operatorname{Im} z < 0\}$. We ensure E is unique by normalizing E so that

$$E(i) \text{ is real and positive.} \tag{1.5}$$

Write

$$E(z) = \sum_{j=0}^{n+1} e_j z^j, \; S(z) = \sum_{j=0}^{2n+2} s_j z^j \tag{1.6}$$

and

$$E^*(z) = \overline{E(\bar{z})}. \tag{1.7}$$

Denote the first difference of a function f by

$$[f, t, x] = \frac{f(t) - f(x)}{t - x}. \tag{1.8}$$

We shall need various Cauchy principal value integrals: for real x, and suitable functions h,

$$\begin{aligned} PV_x \int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt &= \lim_{\varepsilon \to 0+} \int_{|t-x| \geq \varepsilon} \frac{h(t)}{t-x} dt; \\ PV_\infty \int_{-\infty}^{\infty} h(t) \; dt &= \lim_{R \to \infty} \int_{-R}^{R} h(t) \; dt; \\ PV_{x,\infty} \int_{-\infty}^{\infty} \frac{h(t)}{t-x} dt &= \lim_{\varepsilon \to 0+, R \to \infty} \int_{|t| \leq R, |t-x| \geq \varepsilon} \frac{h(t)}{t-x} dt. \end{aligned}$$

With the above assumptions on w, we prove:

Theorem 1 *(a) For* $\operatorname{Im} z > 0$,

$$E(z) = \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log w(t)}{1+t^2} dt \right), \tag{1.9}$$

and

$$e_{n+1} = s_{2n+2}^{1/2} (-i)^{n+1} \exp\left(\frac{1}{2\pi i} PV_\infty \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} t \; dt \right). \tag{1.10}$$

(b) For $z \neq v$,

$$K_{n+1}(z, v) = \frac{i}{2\pi} \frac{E(z) E^*(v) - E^*(z) E(v)}{z - v}; \tag{1.11}$$

$$K_{n+1}(z, z) = \frac{i}{2\pi} \left(E'(z) E^*(z) - E(z) E^{*\prime}(z) \right). \tag{1.12}$$

(c)

$$\gamma_n = \left\{ \frac{1}{\pi} \operatorname{Im} \left(\overline{e_{n+1}} e_n \right) \right\}^{1/2} \tag{1.13}$$

and

$$p_n(z) = -\frac{1}{\gamma_n} \frac{i}{2\pi} \left(\overline{e_{n+1}} E(z) - e_{n+1} E^*(z) \right). \tag{1.14}$$

Theorem 2 *For $x \in \mathbb{R}$,*
(a)

$$p_n(x)\, w(x)^{1/2} = \frac{s_{2n+2}^{1/2}}{\pi \gamma_n} \cos\left(\frac{n\pi}{2} + \frac{1}{2\pi} PV_{x,\infty} \int_{-\infty}^{\infty} \frac{\log w(t)}{t-x} dt \right). \tag{1.15}$$

(b)

$$\pi K_{n+1}(x,x)\, w(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} [\log w, t, x] \frac{t}{1+t^2} dt$$
$$-\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} [\log w, t, x] \frac{1+tx}{1+t^2} dt. \tag{1.16}$$

(c) If $s_{2n+1} = 0$,

$$\gamma_n = \frac{1}{\pi} \left\{ \frac{s_{2n+2}}{2} \int_{-\infty}^{\infty} \log \left[\frac{S(t)}{s_{2n+2} t^{2n+2}} \right] dt \right\}^{1/2}. \tag{1.17}$$

Remarks (a) The function E is a Szegő/ outer function associated with w for the upper-half plane. It has been used in the relative asymptotics of G. Lopez [6] and in the orthogonal rational functions of Bultheel et al [2].
(b) It is easily seen that for $\operatorname{Im} z > 0$,

$$E^*(z) = C E(z) \prod_{a: E(a)=0} \frac{z-\bar{a}}{z-a}, \tag{1.18}$$

where

$$C = \frac{\overline{e_{n+1}}}{e_{n+1}} = (-1)^{n+1} \exp\left(-\frac{1}{\pi i} PV_{\infty} \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} t\, dt \right).$$

(c) Of course if S is even, then s_{2n+1} is 0. The latter condition ensures that the integral in (1.17) converges.
(d) Explicit formulae for the Christoffel function $K_n(x,x)^{-1}$ for Bernstein-Szegő weights appear in [3], [5], [7], [8], [9], [10]. We will present one application of (1.11-12) in Section 3.

2. Proofs

As we noted above, our original proofs arose from de Branges spaces, but we present elementary proofs. Let us choose E satisfying (1.4) and (1.5).

Proof of (1.9) of Theorem 1(a) Let H denote the right side of (1.9), so that

$$H(z) = \exp\left(-\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log w(t)}{1+t^2} dt \right).$$

Then for $z = x + iy$,

$$\begin{aligned} \log|H(z)| &= -\operatorname{Re}\left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1+tz}{t-z} \frac{\log w(t)}{1+t^2} dt \right] \\ &= \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\log|E(t)|}{(t-x)^2 + y^2} dt \\ &= \log|E(z)|, \end{aligned} \tag{2.1}$$

by a Theorem in [4, p. 47]. This may be applied as $E(z)$ is analytic and non-zero in the closed upper-half plane, and $\log|E(z)|$ is $O(\log|z|)$ as $|z| \to \infty$. Since H/E

is analytic there, we deduce that for some C with $|C| = 1$, $E = CH$. Now by hypothesis, $E(i)$ is real and positive, while

$$H(i) = \exp\left(-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}dt\right) > 0$$

so $C = 1$.

Proof of (1.10) of Theorem 1(a) We first show that

$$1 - iz = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left(1+t^2\right)}{1+t^2}\frac{1+tz}{t-z}dt\right),\ \operatorname{Im} z > 0. \tag{2.2}$$

Indeed, $1 - iz$ serves as the Szegő function for the weight $1/\left(1+t^2\right)$, so (1.9) of Theorem 1 applied to the weight $1/\left(1+t^2\right)$ gives this identity. Then for $\operatorname{Im} z > 0$,

$$E(z)/(1-iz)^{n+1} = \exp(I_1 + I_2), \tag{2.3}$$

where

$$\begin{aligned} I_1 &= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right]}{1+t^2}\frac{1+tz}{t-z}dt; \\ I_2 &= \frac{\log s_{2n+2}}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{1+t^2}\frac{1+tz}{t-z}dt. \end{aligned}$$

The integrand in I_2 has simple poles in the upper-half plane at i and z, and is $O\left(t^{-2}\right)$ as $|t| \to \infty$, so the residue calculus gives

$$I_2 = \frac{\log s_{2n+2}}{2}. \tag{2.4}$$

Next, $\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right] = O\left(\frac{1}{t}\right)$ as $|t| \to \infty$. Thus the integrand in I_1 is bounded in absolute value for $z = iy, y \geq 1$ and all t by

$$C\frac{1}{(1+t^2)(1+|t|)}\frac{1+|t|\,y}{|t|+y} \leq \frac{C}{1+t^2}.$$

Here C is independent of t and z. We may then apply Lebesgue's Dominated Convergence Theorem to I_1, with $z = iy$, $y \to \infty$, to deduce that

$$\begin{aligned} I_1 &\to \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\log\left[w(t)\,s_{2n+2}\left(1+t^2\right)^{n+1}\right]}{1+t^2}t\ dt \\ &= \frac{1}{2\pi i}PV_\infty\int_{-\infty}^{\infty}\frac{\log w(t)}{1+t^2}t\ dt, \end{aligned} \tag{2.5}$$

as

$$PV_\infty\int_{-\infty}^{\infty}\frac{t}{1+t^2}\ dt = 0 = PV_\infty\int_{-\infty}^{\infty}\frac{\log\left(1+t^2\right)}{1+t^2}t\ dt,$$

the integrands being odd. Substituting (2.5) and (2.4) into (2.3) and letting also $z = iy$, $y \to \infty$, in the left-hand side there, gives (1.10). ■

Proof of Theorem 1(b) We need prove only (1.11), for (1.12) then follows by l'Hospital's rule. Set

$$G(u,v) = \frac{i}{2\pi}\frac{E(u)\,E^*(v) - E^*(u)\,E(v)}{u-v}.$$

Observe that for fixed v, $G(u,v)$ is a polynomial of degree at most n in u. Assume that P is a polynomial of degree $\leq n$ and that $\operatorname{Im} u > 0$. Now for real t, $w(t) = 1/(E(t)E^*(t))$, so

$$\begin{aligned} & \int_{-\infty}^{\infty} P(t)\, G(u,t)\, w(t)\, dt \\ (2.6) \quad = \; & \frac{i}{2\pi}\left(E^*(u) \int_{-\infty}^{\infty} \frac{P(t)}{E^*(t)(t-u)} dt - E(u) \int_{-\infty}^{\infty} \frac{P(t)}{E(t)(t-u)} dt \right). \end{aligned}$$

Recall that E has all its zeros in the lower-half plane, so E^* has all its zeros in the upper-half plane. Then the integrand $\frac{P(t)}{E^*(t)(t-u)}$ in the first integral is analytic in the closed lower-half plane, and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. By Cauchy's integral theorem, the first integral is 0. Next, the integrand $\frac{P(t)}{E(t)(t-u)}$ in the second integral is analytic in the closed upper-half plane, except for a simple pole at u (unless $P(u) = 0$) and is $O\left(|t|^{-2}\right)$ as $|t| \to \infty$. The residue theorem shows that

$$\int_{-\infty}^{\infty} \frac{P(t)}{E(t)(t-u)} dt = 2\pi i \frac{P(u)}{E(u)}.$$

Substituting this into (2.6) gives

$$\int_{-\infty}^{\infty} P(t)\, G(u,t)\, w(t)\, dt = P(u)$$

for $\operatorname{Im} u > 0$. As both sides are polynomials in u, analytic continuation gives it for all u. Finally, (1.11) follows from uniqueness of reproducing kernels:

$$K_{n+1}(u,v) = \int_{-\infty}^{\infty} K_{n+1}(t,v)\, G(u,t)\, w(t)\, dt = G(u,v).$$

Proof of Theorem 1(c) We note that since p_{n+1} is not defined, we cannot use the Christoffel-Darboux formula for K_{n+1}. However, we can use it for K_n :

$$K_{n+1}(u,v) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(u)\, p_{n-1}(v) - p_n(v)\, p_{n-1}(u)}{u-v} + p_n(u)\, p_n(v).$$

Multiplying by $u - v$ leads to

$$\begin{aligned} & \frac{\gamma_{n-1}}{\gamma_n}\left(p_n(u)\, p_{n-1}(v) - p_n(v)\, p_{n-1}(u)\right) + (u-v)\, p_n(u)\, p_n(v) \\ = \; & (u-v)\, K_{n+1}(u,v) = \frac{i}{2\pi}\left(E(u)\, E^*(v) - E^*(u)\, E(v)\right), \end{aligned}$$

by (1.11). Now we compare coefficients of u^{n+1} on both sides above:

$$(2.7) \qquad \gamma_n p_n(v) = \frac{i}{2\pi}\left(e_{n+1} E^*(v) - \overline{e_{n+1}} E(v)\right),$$

giving (1.14). For (1.13), we compare the coefficients of v^n on both sides above:

$$\gamma_n^2 = \frac{i}{2\pi}\left(e_{n+1}\overline{e_n} - \overline{e_{n+1}} e_n\right).$$

(Note that the coefficient of v^{n+1} on the right-hand side in (2.7) is zero). ■

Proof of Theorem 2(a) From (1.14), for real x,

$$\pi \gamma_n p_n(x) = \operatorname{Im}\left(\overline{e_{n+1}} E(x)\right).$$

We take non-tangential boundary values $z \to x$ from the upper-half plane in (1.9). The Sokhotsky-Plemelj formulae give

$$E(x) = \exp\left(-\frac{1}{2\pi i} PV_x \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} \frac{1+tx}{t-x} dt - \frac{1}{2}\log w(x)\right), \tag{2.8}$$

and this and (1.10) give

$$\begin{aligned} & \pi\gamma_n p_n(x) w(x)^{1/2} \\ = \;& s_{2n+2}^{1/2} \operatorname{Im}[i^{n+1} \exp\left(-\frac{1}{2\pi i} PV_\infty \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} t\, dt - \frac{1}{2\pi i} PV_x \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} \frac{1+tx}{t-x} dt\right)] \\ = \;& s_{2n+2}^{1/2} \operatorname{Im}[i^{n+1} \exp\left(-\frac{1}{2\pi i} PV_{x,\infty} \int_{-\infty}^{\infty} \frac{\log w(t)}{t-x} dt\right)]. \end{aligned}$$

Proof of Theorem 2(b) For real x, and E as above, we define a phase function φ (cf. [1, p. 54]) by

$$E(x) = |E(x)| e^{-i\varphi(x)}. \tag{2.9}$$

Here, as in [1, p. 54], φ is an increasing differentiable function. We have, as there

$$K_{n+1}(x,x) = \frac{1}{\pi} |E(x)|^2 \varphi'(x) = \frac{1}{\pi} w(x)^{-1} \varphi'(x). \tag{2.10}$$

Indeed, for real x,

$$E^*(x) = |E(x)| e^{i\varphi(x)},$$

so for real $t \neq x$, (1.11) gives

$$K_{n+1}(x,t) = \frac{|E(x)|\,|E(t)|}{\pi} \frac{\sin(\varphi(x) - \varphi(t))}{x-t}.$$

L'Hospital's rule gives the first equality in (2.10). Next, from (2.8) and the definition of φ, we have for some constant C independent of x,

$$\varphi(x) = -\frac{1}{2\pi} PV_x \int_{-\infty}^{\infty} \frac{\log w(t)}{1+t^2} \frac{1+tx}{t-x} dt + C. \tag{2.11}$$

The residue theorem shows that for $\operatorname{Im} z > 0$,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{1+tz}{t-z} dt = \frac{1}{2}, \tag{2.12}$$

so also for real x, the Sokhotsky-Plemelj formulae give

$$\frac{1}{2\pi i} PV_x \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{1+tx}{t-x} dt + \frac{1}{2} = \frac{1}{2},$$

thus

$$\frac{1}{2\pi i} PV_x \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{1+tx}{t-x} dt = 0. \tag{2.13}$$

Hence we may write

$$\begin{aligned} \varphi(x) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log w(t) - \log w(x)}{t-x} \frac{1+tx}{1+t^2} dt + C \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} [\log w, t, x] \frac{1+tx}{1+t^2} dt + C, \end{aligned}$$

where the integral is now a Lebesgue integral. Then

$$\varphi'(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}[\log w, t, x]\frac{t}{1+t^2}dt - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\partial}{\partial x}[\log w, t, x]\frac{1+tx}{1+t^2}dt.$$

The interchange of derivative and integral is justified by uniform in x (and absolute) convergence of the differentiated integrals. Finally, apply (2.10). ■

Proof of Theorem 2(c) We compute γ_n by comparing both sides of (2.10) as $x \to \infty$. First observe that if $a > 0$, and

$$w_a(x) = \left(x^2 + a^2\right)^{-(n+1)},$$

then the Szegő/ outer function E_a for the weight w_a is given by

$$E_a(z) = (a - iz)^{n+1} \text{ and } E_a^*(z) = (a + iz)^{n+1}.$$

If $K_{n+1}(w_a, \cdot, \cdot)$ denotes the kernel for w_a, (1.11) leads to

$$K_{n+1}(w_a, x + iy, x - iy) = \frac{\left(x^2 + (a+y)^2\right)^{n+1} - \left(x^2 + (a-y)^2\right)^{n+1}}{4\pi y}.$$

Letting $y \to 0+$, and using l'Hospital's rule gives

$$K_{n+1}(w_a, x, x) = \frac{n+1}{\pi}a\left(x^2 + a^2\right)^n$$

and

$$K_{n+1}(w_a, x, x)\, w_a(x) = \frac{(n+1)\,a}{\pi\left(x^2 + a^2\right)}. \tag{2.14}$$

Next, if we write

$$E_a(x) = |E_a(x)|\, e^{-i\varphi_a(x)},$$

then, as at (2.11),

$$\varphi_a(x) = -\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{\log w_a(t)}{1+t^2}\frac{1+tx}{t-x}dt + C_a. \tag{2.15}$$

Let

$$g_a(t) = \log\left[w(t)\, s_{2n+2}/w_a(t)\right] = \log\left[\frac{s_{2n+2}\left(t^2 + a^2\right)^{n+1}}{S(t)}\right].$$

In view of (2.11), (2.13) and (2.15), we may then write

$$\varphi(x) - \varphi_a(x) = -\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{1+t^2}\frac{1+tx}{t-x}dt + C - C_a \tag{2.16}$$

and then (2.14), followed by (2.10) and (2.16) give

$$\begin{aligned} & \pi K_{n+1}(x,x)\, w(x) - \frac{(n+1)\,a}{x^2 + a^2} \\ = {} & \pi K_{n+1}(x,x)\, w(x) - \pi K_{n+1}(w_a, x, x)\, w_a(x) \\ = {} & \varphi'(x) - \varphi_a'(x) \\ = {} & \frac{d}{dx}\left[-\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{1+t^2}\frac{1+tx}{t-x}dt\right]. \end{aligned} \tag{2.17}$$

Since $s_{2n+1} = 0$, it is easily seen that for each $j \geq 0$,

$$g_a^{(j)}(t) = O(|t|^{-j-2}) \text{ as } |t| \to \infty. \tag{2.18}$$

As
$$\frac{1}{1+t^2}\frac{1+tx}{t-x}=\frac{1}{t-x}-\frac{t}{1+t^2},$$
the decay of g_a at ∞ enables us to deduce that
$$\begin{aligned}&\pi K_{n+1}(x,x)\,w(x)-\frac{(n+1)\,a}{x^2+a^2}\\ (2.19)\qquad &=\frac{d}{dx}\left[-\frac{1}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt\right].\end{aligned}$$
It is well known that the derivative of a Cauchy principal value is a Hadamard finite part integral, but we sketch what we need here. Fix x, let $R>|x|$, and split
$$PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt=PV_x\left(\int_{-R}^{R}+\int_{\mathbb{R}\setminus[-R,R]}\right)\frac{g_a(t)}{t-x}dt=:F_R(x)+G_R(x).$$
Here, because the differentiated integrand has uniformly convergent integral,
$$G_R'(x)=\int_{\mathbb{R}\setminus[-R,R]}\frac{g_a(t)}{(t-x)^2}dt.$$
Note too that $G_R'(x)\to 0$ as $R\to\infty$. Next, adding and subtracting a principal value integral gives
$$F_R(x)=\int_{-R}^{R}\frac{g_a(t)-g_a(x)}{t-x}dt+g_a(x)\ln\left|\frac{R-x}{R+x}\right|,$$
so, again, as the differentiated integrand has uniformly convergent integral,
$$\begin{aligned}F_R'(x)&=\int_{-R}^{R}\frac{g_a(t)-g_a(x)-g_a'(x)(t-x)}{(t-x)^2}dt+g_a'(x)\ln\left|\frac{R-x}{R+x}\right|+g_a(x)\left(\frac{1}{x-R}-\frac{1}{x+R}\right)\\&=PV_x\int_{-R}^{R}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt+g_a(x)\left(\frac{1}{x-R}-\frac{1}{x+R}\right).\end{aligned}$$
As $x\to\infty$, the decay of g_a at ∞ ensures that
$$F_R'(x)\to PV_x\int_{-\infty}^{\infty}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt.$$
We deduce that
$$\frac{d}{dx}\left[PV_x\int_{-\infty}^{\infty}\frac{g_a(t)}{t-x}dt\right]=PV_x\int_{-\infty}^{\infty}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt.$$
Thus, from (2.19),
$$\begin{aligned}\pi x^2K_{n+1}(x,x)\,w(x)-\frac{(n+1)\,ax^2}{x^2+a^2}&=-\frac{x^2}{2\pi}PV_x\int_{-\infty}^{\infty}\frac{g_a(t)-g_a(x)}{(t-x)^2}dt\\ (2.20)\qquad &=-\frac{1}{2\pi}\int_{-\infty}^{\infty}h_a(t,x)\,dt,\end{aligned}$$
where
$$h_a(t,x)=\begin{cases}\frac{x^2[g_a(t)-g_a(x)]}{(t-x)^2} & ,t\notin\left[\frac{x}{2},\frac{3x}{2}\right]\\ \frac{x^2\left[g_a(t)-g_a(x)-g_a'(x)(t-x)\right]}{(t-x)^2} & ,t\in\left[\frac{x}{2},\frac{3x}{2}\right]\end{cases}.$$
Observe that for each fixed t,
$$\lim_{x\to\infty}h_a(t,x)=g_a(t).$$

(We use (2.18) for this). We next obtain an integrable bound on $h_a(t,x)$ that is independent of large x. If $t \in \left(-\infty, \frac{x}{2}\right)$,

$$|h_a(t,x)| \leq C |g_a(t)| + \frac{C}{1+t^2}.$$

Similarly if $t \in \left(\frac{3x}{2}, \infty\right)$, this bound holds. If $t \in \left[\frac{x}{2}, \frac{3x}{2}\right]$, then for some ξ in this interval, (2.18) shows that

$$|h_a(t,x)| = \frac{x^2}{2} |g_a''(\xi)| \leq \frac{C}{1+t^2}.$$

In all occurrences, C is independent of x and t. It follows that we may apply Lebesgue's Dominated Convergence Theorem to the integral in the right-hand side of (2.20) and let $x \to \infty$ on both sides to deduce that

$$\frac{\pi \gamma_n^2}{s_{2n+2}} - (n+1) a = -\frac{1}{2\pi} \int_{-\infty}^{\infty} g_a(t)\, dt.$$

Now we let $a \to 0+$, and use the definition of g_a (and an easier Dominated Convergence) to deduce that

$$\frac{\pi \gamma_n^2}{s_{2n+2}} = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \log\left[\frac{s_{2n+2} t^{2n+2}}{S(t)}\right] dt. \quad \blacksquare$$

3. An Application to Reciprocal Entire Weights

Suppose $z_j = x_j + iy_j$, $j \geq 1$, with all $y_j < 0$ and

$$\sum_{j=1}^{\infty} \left(\frac{x_j}{|z_j|}\right)^2 < \infty. \tag{3.1}$$

Let

$$E(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right) \text{ and } E_n(z) = \prod_{j=1}^{n+1} \left(1 - \frac{z}{z_j}\right),\ n \geq 1.$$

Assume that E is entire, and let

$$W = \frac{1}{|E|^2} \text{ and } w_n = \frac{1}{|E_n|^2},\ n \geq 1.$$

For real x, it is easily seen that

$$\frac{w_n}{W}(x) \geq \prod_{j=n+2}^{\infty} \left(1 - \left(\frac{x_j}{|z_j|}\right)^2\right) =: \rho_n.$$

Let $K_{n+1}(W,\cdot,\cdot)$ and $K_{n+1}(w_n,\cdot,\cdot)$ denote the nth reproducing kernels for W and w_n respectively. This last inequality and extremal properties of K_{n+1} yield

$$K_{n+1}(W,z,\bar{z}) \geq \rho_n^{-1} K_{n+1}(w_n,z,\bar{z}) \text{ for all } z \in \mathbb{C}.$$

In view of (3.1), $\rho_n \to 1$ as $n \to \infty$. Then the explicit formula (1.11) for $K_{n+1}(w_n,z,\bar{z})$ and the fact that $E_n \to E$ as $n \to \infty$ give, for non-real z,

$$\liminf_{n\to\infty} K_{n+1}(W,z,\bar{z}) \geq \frac{i}{2\pi} \frac{E(z)E^*(\bar{z}) - E^*(z)E(\bar{z})}{z - \bar{z}}. \tag{3.2}$$

For real z, we instead use (1.12). Now let $\mathcal{H}(E)$ be the de Branges space corresponding to E. This consists [1, p. 50 ff.] of all entire functions g for which both g/E and g^*/E belong to the Hardy 2 space of the upper-half plane $H^2(\mathbb{C}^+)$, with

$$\int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2 < \infty.$$

The reproducing kernel for this space is [1, p. 51]

$$K(z,v) = \frac{i}{2\pi} \frac{E(z)E^*(v) - E^*(z)E(v)}{z-v}, \; z \neq v,$$

with a confluent form when $z = v$. Moreover, for such g, we have [1, p. 53]

$$|g(z)|^2 \leq K(z,\bar{z}) \int_{-\infty}^{\infty} \left|\frac{g}{E}\right|^2, \; z \in \mathbb{C}.$$

Since $\mathcal{H}(E)$ contains all polynomials, we may apply this last inequality to $g(t) = K_{n+1}(W,t,\bar{z})$ for fixed z, and deduce that

$$K_{n+1}(W,z,\bar{z})^2 \leq K(z,\bar{z}) \int_{-\infty}^{\infty} |K_{n+1}(W,t,\bar{z})|^2 W(t)\,dt = K(z,\bar{z}) K_{n+1}(W,z,\bar{z}),$$

so

$$K_{n+1}(W,z,\bar{z}) \leq K(z,\bar{z}).$$

Together with (3.2), this yields, for non-real z,

$$\lim_{n\to\infty} K_n(W,z,\bar{z}) = K(z,\bar{z}) = \frac{i}{2\pi} \frac{E(z)E^*(\bar{z}) - E^*(z)E(\bar{z})}{z-\bar{z}}.$$

Similarly, for x real,

$$\lim_{n\to\infty} K_n(W,x,x) = K(x,x) = \frac{i}{2\pi}\left(E'(x)E^*(x) - E(x)E^{*\prime}(x)\right).$$

In particular, as this is finite, the moment problem corresponding to W is indeterminate (cf. [3]).

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