ORTHOGONAL POLYNOMIALS AND PADÉ APPROXIMANTS FOR RECIPROCAL POLYNOMIAL WEIGHTS

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Abstract. Let Γ be a closed oriented contour on the Riemann sphere. Let E and F be polynomials of degree $n+1$, with zeros respectively on the positive and negative sides of Γ. We compute the $[n/n]$ and $[n-1/n]$ Padé denominator at ∞ to

$$f(z)=\int_\Gamma \frac{1}{z-t}\frac{dt}{E(t)F(t)}.$$

As a consequence, we compute the nth orthogonal polynomial for the weight $1/(EF)$. In particular, when Γ is the unit circle, this leads to an explicit formula for the Hermitian orthogonal polynomial of degree n for the weight $1/|F|^2$. This complements the classical Bernstein-Szegő formula for the orthogonal polynomials of degree $\geq n+1$.

Padé approximant, de Branges space, reproducing kernel, Orthogonal Polynomials, Bernstein-Szegő formula. AMS Classification: 41A21, 42C99
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1. The Result

Let f be a formal power series at ∞ of the form

$$f(z)=\sum_{j=0}^{\infty} c_j z^{-j}.$$

The n/n Padé approximant to f at ∞ is a rational function $[n/n](z)=p_n(z)/q_n(z)$, where p_n and q_n are polynomials of degree $\leq n$, with, as $z\to\infty$,

$$f(z)q_n(z)-p_n(z)=O\left(z^{-n-1}\right). \tag{1.1}$$

While the numerator p_n and denominator q_n are non-unique, the approximant $[n/n]$ is unique. Explicit Padé approximants are known for classical special functions, and convergence as $n\to\infty$, has been established in many senses, and for a great variety of functions. There are close connections to orthogonal polynomials. See [1], [2], [4], [12], [13].

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In this note, we use ideas from the theory of de Branges spaces to give an explicit representation for the denominator polynomial q_n, when

$$f(z) = \int_\Gamma \frac{1}{z-t} \frac{dt}{E(t) F(t)}, \ z \text{ on the negative side of } \Gamma. \tag{1.2}$$

Here Γ is a simple closed positively oriented rectifiable contour on the Riemann sphere, and E and F are polynomials of degree $n+1$ with zeros respectively on the positive and negative sides of Γ. We write

$$E(z) = e_{n+1} z^{n+1} + ... \text{ and } F(z) = f_{n+1} z^{n+1} + ... \ , \ e_{n+1} \neq 0, f_{n+1} \neq 0. \tag{1.3}$$

In the case, where Γ is an unbounded contour (so that it passes through ∞ on the Riemann sphere), we assume that Γ admits a suitable form of the residue theorem on its positive and negative sides. More precisely, if a function g is defined on Γ and its negative side, and analytic there except for poles, while $g(t) = O(t^{-2})$ as $t \to \infty$, then $\frac{1}{2\pi i} \int_\Gamma g$ is the sum of the residues on the negative side. An analogous statement is assumed for functions g defined on the positive side of Γ. In particular, both are true if Γ is the real line, or a smooth curve from $-\infty$ to ∞. In this case, too, the approach of z to ∞ in (1.1) must be suitably restricted, and we assume the approach of z to ∞ is from the negative side of Γ.

The result of this note extends the rather limited class of functions for which explicit Padé approximants are available. In addition to classical special functions, there are explicit representations of Padé denominators in terms of orthogonal polynomials for Markov-Stieltjes series, and some representations derived from continued fractions for a broader array of special functions [2]. In some respects, however, our result is closer in spirit to the Bernstein-Szegő formula for orthogonal polynomials, since it would most likely be applied for varying E and F.

We prove:

Theorem 1 *Assume that E and F are polynomials of degree $n+1$, with zeros respectively on the negative and positive sides of Γ. Assume that f is defined by (1.2), for z on the negative side of Γ.*
(a) We may take as the Padé denominator in $[n/n]$,

$$q_n(z) = e_{n+1} F(z) - f_{n+1} E(z); \tag{1.4}$$

$$p_n(z) = \int_\Gamma \frac{q_n(z) - q_n(t)}{z-t} \frac{dt}{E(t) F(t)}. \tag{1.5}$$

(b) For all polynomials S of degree $\leq n-1$,

$$\int_\Gamma q_n(t) S(t) \frac{dt}{E(t) F(t)} = 0. \tag{1.6}$$

(c) Let

$$K_{n+1}(t,z) = \frac{i}{2\pi} \frac{F(t)E(z) - F(z)E(t)}{t-z}. \tag{1.7}$$

Then for every polynomial P of degree $\leq n$, and for all complex z,

$$\int_\Gamma P(t) K_{n+1}(t,z) \frac{dt}{E(t)F(t)} = P(z). \tag{1.8}$$

Remarks (a) The function K_{n+1} is of course a reproducing kernel for the weight $\frac{1}{EF}$ on Γ. The idea for the form of K_{n+1} comes from the theory of de Branges spaces [3].
(b) The polynomial q_n cannot be identically zero. If it were, E and F would be multiples of one another, contradicting that their zeros lie on opposite sides of Γ. However, q_n need not have degree n. For example, if $\Gamma = \{t : |t| = 1\}$ is positively oriented, and $E(z) = (3z)^{n+1} - 1$ and $F(z) = \left(\frac{z}{3}\right)^{n+1} - 1$, then q_n is the constant polynomial $q_n(z) = 3^{-n-1} - 3^{n+1}$.
(c) In general, p_n above is actually of degree $\leq n-1$, so that p_n/q_n can also serve as $[n-1/n]$.
(d) The function $f(z)$ is a rational function of type $n/(n+1)$. Indeed, if E has all simple zeros, the residue theorem shows that for z on the negative side of Γ,

$$f(z) = 2\pi i \sum_{t:E(t)=0} \frac{1}{F(t)E'(t)} \frac{1}{z-t}.$$

There is a substantial convergence theory of Padé approximation for functions that are the sum of a rational function and a Stieltjes transform - two of the earlier major references are [6], [10].
(e) Although the formulation above refers to non-Hermitian orthogonality, the two most interesting cases involve classical orthogonality. In [8], we explored the special case where Γ is the whole real line, and $E(z) = \overline{F(\bar{z})}$, so that on the real line $\frac{1}{EF} = \frac{1}{|F|^2}$. This leads to new Bernstein-Szegő type formulas for the orthogonal polynomial of degree n for the weight $\frac{1}{|F^2|}$ on the whole real line. The classical Bernstein-Szegő formulas deal with weights on $[-1,1]$ [5], [7], [11], [14].
(f) Another interesting case is where Γ is the unit circle, and $E(z) = z^{n+1}\overline{F(1/\bar{z})}$. Then the orthogonality relation (1.6) becomes

$$\int_\Gamma q_n(t) S(t) \frac{dt}{t^{n+1}|F(t)|^2} = 0.$$

Setting $q_n^*(z) = z^n \overline{q_n(1/\bar{z})}$, we obtain the classical orthogonality relation

$$\int_0^{2\pi} \overline{q_n^*(e^{i\theta})} S\left(e^{i\theta}\right) \frac{d\theta}{|F(e^{i\theta})|^2} = 0.$$

Thus, q_n^* is an orthogonal polynomial of degree n for the weight $\frac{1}{|F|^2}$. Using (1.4), we obtain

$$q_n^*(z) = z^n F(0) \overline{F\left(\frac{1}{\bar{z}}\right)} - \overline{f_{n+1}} \frac{F(z)}{z}.$$

Note that the term in $\frac{1}{z}$ cancels, so q_n^* is indeed a polynomial. The leading coefficient is $|F(0)|^2 - |f_{n+1}|^2$. It is positive, since

$$|F(0)| = |f_{n+1}| \prod_{z:F(z)=0} |z| > |f_{n+1}|.$$

This formula seems to be new. Indeed, the classical Bernstein-Szegő formulas [11, p. 111], [14, p. 289] give explicit representations for the orthogonal polynomials of degree $m \geq n+1$, but not for $m = n$. We summarize this as:

Corollary 2 *Let F be a polynomial of degree $n+1$, with leading coefficient f_{n+1} and all zeros outside the unit circle. Then the monic orthogonal polynomial Φ_n of degree n, satisfying*

$$\int_0^{2\pi} \Phi_n\left(e^{i\theta}\right) \overline{S\left(e^{i\theta}\right)} \frac{d\theta}{\left|F\left(e^{i\theta}\right)\right|^2} = 0,$$

for polynomials S of degree $\leq n-1$, is

$$\Phi_n(z) = \frac{z^n F(0) \overline{F\left(\frac{1}{\bar{z}}\right)} - \overline{f_{n+1}} \frac{F(z)}{z}}{|F(0)|^2 - |f_{n+1}|^2}.$$

Proof of Theorem 1(c)

Let P be a polynomial of degree at most n and K_{n+1} be given by (1.7). Let z lie on the positive side of Γ. Then

$$\int_\Gamma P(t) K_{n+1}(t,z) \frac{dt}{E(t) F(t)} = I_1 E(z) - I_2 F(z), \tag{1.9}$$

where

$$I_1 = \frac{i}{2\pi} \int_\Gamma \frac{P(t)}{E(t)} \frac{dt}{t-z} \text{ and } I_2 = \frac{i}{2\pi} \int_\Gamma \frac{P(t)}{F(t)} \frac{dt}{t-z}.$$

Now note that $\frac{P(t)}{E(t)} \frac{1}{t-z}$ and $\frac{P(t)}{F(t)} \frac{1}{t-z}$ are $O\left(t^{-2}\right)$ at ∞. Also, in I_1, $\frac{P(t)}{E(t)} \frac{1}{t-z}$ has all its poles on the positive side of Γ, so is analytic on the negative side of Γ. Cauchy's integral theorem (or the residue theorem) gives $I_1 = 0$. Next, $\frac{P(t)}{F(t)} \frac{1}{t-z}$ is analytic on the positive side of Γ except for a simple pole at $t = z$, so the residue theorem gives

$$I_2 = \frac{i}{2\pi} 2\pi i \frac{P(z)}{F(z)} = -\frac{P(z)}{F(z)}.$$

The result (1.8) for z on the positive side of Γ, then follows from (1.9). As both sides of (1.8) are polynomials, analytic continuation gives it for all z.

Proof of Theorem 1(b)

We apply (c) to the polynomial $P(t) = (t-z)\,S(t)$, where S is a polynomial of degree $\leq n-1$. We obtain

$$\int_\Gamma (t-z)\,S(t)\,K_n(t,z)\,\frac{dt}{E(t)F(t)} = 0$$

$$\Rightarrow \int_\Gamma (F(t)\,E(z) - F(z)\,E(t))\,S(t)\,\frac{dt}{E(t)F(t)} = 0.$$

Comparing the coefficients of z^{n+1} on both sides gives

$$\int_\Gamma (F(t)\,e_{n+1} - f_{n+1}E(t))\,S(t)\,\frac{dt}{E(t)\,F(t)} = 0.$$

This is just the orthogonality relation (1.6), taking account of our choice (1.4) of q_n. Note that the coefficients of t^{n+1} in the definition of q_n cancel, so that q_n is a polynomial of degree at most n. Moreover, as we noted before, q_n is not identically zero since F and E have zeros on opposite sides of Γ.

Proof of Theorem 1(a)

Let p_n be given by (1.5), so that it is a polynomial of degree $\leq n-1$. Recall that f is given by (1.2). Then

$$\Delta(z) := q_n(z)\,f(z) - p_n(z) = \int_\Gamma \frac{q_n(t)}{z-t}\frac{dt}{E(t)F(t)}.$$

Using the orthogonality properties of q_n, we continue this as

$$\begin{aligned}\Delta(z) &= \int_\Gamma q_n(t)\left[\frac{1}{z-t} - \frac{1}{z}\sum_{j=0}^{n-1}\left(\frac{t}{z}\right)^j\right]\frac{dt}{E(t)F(t)} \\ &= \frac{1}{z^n}\int_\Gamma \frac{q_n(t)\,t^n}{z-t}\frac{dt}{E(t)F(t)}.\end{aligned}$$

If Γ is closed and bounded, we deduce that as $z \to \infty$,

$$\Delta(z) = O\left(z^{-n-1}\right).$$

If Γ is unbounded, this remains valid if we let $z \to \infty$ in such a way that $\mathrm{dist}(z,\Gamma) \geq C\,|z|$. Thus $[n/n] = p_n/q_n$ and also, $[n-1/n] = p_n/q_n$. ∎

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