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Best Approximating Entire Functions to $|x|^{\alpha}$ in L_2

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Abstract. Let $\alpha > 0$ not be an even integer. We discuss two methods to derive an explicit representation for the entire function H_{α}^{*} of exponential type 1 that minimizes

$$\left\| |x|^{\alpha} - f(x) \right\|_{L_2(\mathbb{R})}$$

amongst all entire functions f of exponential type at most 1. These functions arise in the Bernstein constants problem, of best polynomial approximation of $|x|^{\alpha}$.

1. Introduction

One classical problem in approximation theory is that of the Bernstein constants of polynomial approximation. Let $1 \leq p \leq \infty$ and

$$E_n[|x|^{\alpha}; L_p[-1,1]] = \inf_{\deg(P) \leq n} \| \, |x|^{\alpha} - P(x) \, \|_{L_p[-1,1]}$$

denote the error in best L_p approximation of $|x|^{\alpha}$ by polynomials of degree $\leq n$, in the L_p norm. Starting with Bernstein [**2**], [**3**], a series of authors established the limit

$$\begin{aligned} \Lambda_{p,\alpha}^{*} &= \lim_{n \to \infty} n^{\alpha + \frac{1}{p}} E_n[|x|^{\alpha}; L_p[-1,1]] \\ &= \inf \left\{ \| \, |x|^{\alpha} - f(x) \, \|_{L_p(\mathbb{R})} : f \text{ is entire of exponential type} \leq 1 \right\}, \qquad (1.1) \end{aligned}$$

for $\alpha > 0$, not an even integer.

Only for $p = 1$ and $p = 2$ is $\Lambda_{p,\alpha}^{*}$ known, due largely, respectively, to Nikolskii [**16**] and Raitsin [**17**]:

$$\Lambda_{1,\alpha}^{*} = \frac{\left|\sin \frac{\alpha\pi}{2}\right|}{\pi} 8\Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^{k} (2k+1)^{-\alpha-2};$$

$$\Lambda_{2,\alpha}^{*} = \frac{\left|\sin \frac{\alpha\pi}{2}\right|}{\pi} 2\Gamma(\alpha+1) \sqrt{\pi / (2\alpha+1)}.$$

The exact value of $\Lambda_{\infty,\alpha}^{*}$ is not known for any α, and the search for it has inspired much research. See [**7**], [**12**], [**13**] for references and [**6**] for a survey of the many extensions of this result. For $p = 1$, the unique minimizing entire function of

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exponential type 1 in (1.1) may be expressed as interpolation series at the points $\left\{\left(j-\frac{1}{2}\right)\pi\right\}_{j=1}^{\infty}$, a result established by the first author [**7**]. For $p=\infty$, an analogous interpolation series (at unknown interpolation points) was established in [**14**].

In this paper, we discuss two methods of deriving a representation for the best approximating entire function in the L_2 case. Surprisingly, these are the first published representations in the L_2 case, even though Raitsin's result goes back nearly 40 years. The first method involves elementary facts from distribution theory including Paley-Wiener theorems. The second method is based on the fact that best polynomial approximations in L_2 are partial sums of orthonormal expansions, and that suitably scaled, these best polynomial approximants converge to the best approximating entire function.

Approximation by entire functions of exponential type is a much studied topic. Given $\sigma>0$, and a measurable function g, the error

$$A_\sigma\left[g;L_p\left(\mathbb{R}\right)\right]=\inf\left\{\|g-f\|_{L_p(\mathbb{R})}:f\text{ is entire of exponential type }\leq\sigma\right\}$$

has been estimated especially when g is bounded or has bounded derivatives of some order [**1**], [**4**], [**18**], [**21**], [**22**]. With a view to applications in number theory, there are also explicit representations of the best approximating entire function when $p=1$ and g is one of a special class of functions. For example for $g(x)=\operatorname{sign}(x)$, the best L_1 entire function was determined by Vaaler [**23**]. For other special g, it can be determined using the theory of minimal extrapolations [**18**, Chapter 7], which involve Fourier transforms and Paley-Wiener theory. Quite recently Littman [**11**] has used these ideas, to determine a representation for the best L_1 entire function when $g(x)=x_+^n$, that is $g(x)=x^n$ in $[0,\infty)$ and is 0 on the negative real axis. Then one can deduce from this the extremal entire function for $g(x)=|x|^n=2x_+^n-x^n$.

To the best of our knowledge, this paper contains the first explicit representations for the best approximating entire functions of exponential type to $|x|^\alpha$ in L_2. Our first result is the representation for this function derived using Paley-Wiener theory:

THEOREM 1.1. *Let $\alpha>-1/2$, not an even integer. The unique entire function H_α^* of exponential type 1 satisfying*

$$\left\|\,|x|^\alpha-H_\alpha^*(x)\right\|_{L_2(\mathbb{R})}=A_1\left[|x|^\alpha;L_2(\mathbb{R})\right] \tag{1.2}$$

admits the representation

$$H_\alpha^*(z)=-\frac{2}{\pi}\sin\frac{\alpha\pi}{2}\,\Gamma(\alpha+1)\sum_{k=0}^{\infty}(-1)^k\frac{z^{2k}}{(2k-\alpha)(2k)!}\,. \tag{1.3}$$

Our second representation involves two kernels, the first of which is a Bessel kernel, familiar in universality laws in random matrix theory:

$$\begin{aligned}\mathbb{J}(z,s)&=\frac{1}{2}\left[\frac{\sin(s+z)}{s+z}+\frac{\sin(z-s)}{z-s}\right]\\&=\frac{z\sin z\cos s-s\sin s\cos z}{z^2-s^2}\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}\mathbb{K}(z,s)&=s\mathbb{J}(z,s)-\sin s\cos z\\&=\frac{sz\sin z\cos s-z^2\cos z\sin s}{z^2-s^2}.\end{aligned} \tag{1.5}$$

THEOREM 1.2. *(I) If $-\frac{1}{2} < \alpha < 1$, and $\alpha \neq 0$, then*

$$H_{\alpha}^{*}(z) = \frac{2}{\pi}\int_{0}^{\infty} s^{\alpha}\mathbb{J}(z,s)\,ds. \tag{1.6}$$

If $-\frac{1}{2} < \alpha < 2$, and $\alpha \neq 0$, then

$$H_{\alpha}^{*}(z) = \frac{2}{\pi}\int_{0}^{\infty} s^{\alpha-1}\mathbb{K}(z,s)\,ds + \frac{2^{\alpha}\Gamma\left(\frac{\alpha+1}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\alpha}{2}\right)}\cos z. \tag{1.7}$$

(II) If $\alpha > 2$ and is not an even integer, let ℓ be the even integer in $(\alpha-2,\alpha)$. Then

$$H_{\alpha}^{*}(z) = z^{\ell}H_{\alpha-\ell}^{*}(z) + Q_{1}(z)\cos z - Q_{2}(z)\sin z, \tag{1.8}$$

where

$$Q_{1}(z) = \frac{2^{\alpha}}{\sqrt{\pi}}\sum_{j=0}^{\ell/2-1}\frac{\Gamma\left(\frac{\alpha+1}{2}-j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)}\left(\frac{z}{2}\right)^{2j};$$

$$Q_{2}(z) = \frac{2^{\alpha}}{\sqrt{\pi}}\sum_{j=0}^{\ell/2-1}\frac{\Gamma\left(\frac{\alpha-1}{2}-j\right)}{\Gamma\left(1-\frac{\alpha}{2}+j\right)}\left(\frac{z}{2}\right)^{2j+1}. \tag{1.9}$$

We note that the integral in (1.6) diverges if $\alpha \geq 1$. We shall prove Theorem 1.1 in Section 2, and Theorem 1.2 in Sections 3 and 4. We close off this section with some notation. In the sequel, $C, C_1, C_2, \dots$ denote constants independent of n, x, z. The same symbol does not necessarily denote the same constant, even in successive occurrences. We let $P_{n,\alpha}^{*}$ denote the best L_2 approximant of degree $\leq n$ to $|x|^{\alpha}$ on $[-1,1]$, that is, the unique polynomial of degree $\leq n$ that satisfies

$$\left\||x|^{\alpha} - P_{n,\alpha}^{*}\right\|_{L_2[-1,1]} = \inf_{\deg(P)\leq n}\left\||x|^{\alpha} - P\right\|_{L_2[-1,1]}.$$

The Fourier transform and the inverse Fourier transform of a function or a tempered distribution f is denoted by $\mathcal{F}(f)$ and $\mathcal{F}^{-1}(f)$, respectively. In particular, for $f \in L_1(\mathbb{R})$,

$$\mathcal{F}(f)(y) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(x)e^{-ixy}dx,$$
$$\mathcal{F}^{-1}(f)(y) = (2\pi)^{-1/2}\int_{\mathbb{R}} f(x)e^{ixy}dx,$$

and for $f \in L_2(\mathbb{R})$,

$$\mathcal{F}(f)(y) = \underset{A\to\infty}{\text{l.i.m.}}\,(2\pi)^{-1/2}\int_{-A}^{A} f(x)e^{-ixy}dx,$$
$$\mathcal{F}^{-1}(f)(y) = \underset{A\to\infty}{\text{l.i.m.}}\,(2\pi)^{-1/2}\int_{-A}^{A} f(x)e^{ixy}dx.$$

2. The Paley-Wiener Approach

In this section, we prove Theorem 1.1 in a more general setting (for complex α with $\operatorname{Re}\alpha > -1/2$) using a classical Fourier approach to L_2-approximation of $f_{\alpha}(x) := |x|^{\alpha}$ by entire functions of exponential type ≤ 1. The proof of Theorem 1.1 is based on the following generalized Paley-Wiener theorem (see for example [**19**, Them. 7.2.3, p. 122]):

LEMMA 2.1. *Let h_1 and h_2 be tempered distributions supported in $[-\sigma,\sigma]$, $\sigma > 0$. Then $g_1 := F(h_1)$ and $g_2 := F^{-1}(h_2)$ are entire functions of exponential type σ satisfying for all $z \in C$*

$$|g_1(z)| \le C(1+|z|)^N \exp(\sigma|Im\, z|), \quad |g_2(z)| \le C(1+|z|)^N \exp(\sigma|Im\, z|)$$

for some constants $C > 0$ and $N \ge 0$. Conversely, if entire functions g_1 and g_2 of exponential type $\sigma > 0$ satisfy such growth estimates, then there exist tempered distributions h_1 and h_2 supported in $[-\sigma,\sigma]$ such that $g_1 := F(h_1)$and $g_2 := F^{-1}(h_2)$.

Proof of Theorem 1.1

Step 1. Since $f_\alpha \notin L_2(\mathbb{R})$, we first prove that

$$A_1(|x|^\alpha, L_2(\mathbb{R})) < \infty, \qquad \operatorname{Re}\alpha > -1/2, \quad \alpha \ne 0, 2, \dots \tag{2.1}$$

This fact for real $\alpha > -1/2$ was established in [**17**] by using the limit theorem for L_2-polynomial approximation. Our proof is based on a different idea.

It is known [**8**, eqn. (12), p. 173] that for $y \in \mathbb{R}\backslash\{0\}$ and $\operatorname{Re}\alpha > -1/2$, $\alpha \ne 0, 2, \dots$, the Fourier transform of the tempered distribution f_α is

$$\mathcal{F}(f_\alpha)(y) = \mathcal{F}^{-1}(f_\alpha)(y) = -(2/\pi)^{1/2} \sin\frac{\alpha\pi}{2}\,\Gamma(\alpha+1)\,|y|^{-\alpha-1}. \tag{2.2}$$

Next, we extend $\mathcal{F}(f_\alpha)(y)$ from $\mathbb{R}\backslash(-1,1)$ to $\mathbb{R}$ by the formula:

$$F(y) := \begin{cases} F(f_\alpha)(y), & y \in \mathbb{R}\backslash(-1,1), \\ F(f_\alpha)(1), & y \in (-1,1). \end{cases}$$

Then $F \in L_2(\mathbb{R})$ and $\mathcal{F}^{-1}(F) \in L_2(\mathbb{R})$. Further, it is easy to see that $H := \mathcal{F}(f_\alpha) - F$ is a tempered distribution supported in $[-1,1]$. Hence by Lemma 2.1,

$$g_1 := \mathcal{F}^{-1}(H) = f_\alpha - \mathcal{F}^{-1}(F)$$

is an entire function of exponential type ≤ 1. Therefore, $f_\alpha - g_1 \in L_2(\mathbb{R})$ and (2.1) follows.

Step 2. Next we find a representation for the entire function H^*_α of L_2-best approximation to f_α involving the distributional Fourier transform of f_α.

Let g be an entire function of exponential type 1 such that $f_\alpha - g \in L_2(\mathbb{R})$, where $\operatorname{Re}\alpha > -1/2$, $\alpha \ne 0, 2, \dots$ The existence of such a function follows from (2.1). Then by the Plancherel formula,

$$\begin{aligned} \int_R |f_\alpha - g|^2 dx &= \int_R \left|\mathcal{F}^{-1}(f_\alpha - g)\right|^2 dy \\ &= \int_{|y|\le 1} \left|\mathcal{F}^{-1}(f_\alpha - g)\right|^2 dy + \int_{|y|>1} \left|\mathcal{F}^{-1}(f_\alpha - g)\right|^2 dy. \end{aligned} \tag{2.3}$$

Further, we show that for a.e. $y \in (-\infty,-1) \cup (1,\infty)$,

$$\mathcal{F}^{-1}(f_\alpha - g)(y) = \mathcal{F}^{-1}(f_\alpha)(y), \tag{2.4}$$

where $\mathcal{F}^{-1}(f_\alpha)$ is given in (2.2). Indeed, setting

$$f^*_\alpha(x) := \begin{cases} f_\alpha(x), & x \in \mathbb{R}, \quad \text{Re } \alpha > 0, \quad \alpha \ne 0, 2, \dots \\ f_\alpha(x), & |x| > 1, \quad -1/2 < \text{Re } \alpha < 0 \\ f_\alpha(1), & |x| \le 1, \quad -1/2 < \text{Re } \alpha < 0, \end{cases}$$

we have $f_\alpha^* - g \in L_2(\mathbb{R})$ and $|f_\alpha^*(x)| \le C(1+|x|)^N$ for all $x \in \mathbb{R}$, where $N := \max(\operatorname{Re}\alpha, 0)$. It is known [**5**] (see also [**6**, Lemma 11.4, p. 539]) that these conditions imply the inequality $|g(x)| \le C(1+|x|)^N$, $x \in \mathbb{R}$. Hence [**9**] for any $z \in \mathbb{C}$,

$$|g(z)| \le (1+|z|)^N \exp(|\operatorname{Im} z|).$$

Therefore by Lemma 2.1, $\mathcal{F}^{-1}(g)$ is a tempered distribution supported in $[-1,1]$. In other words, the functional $(\mathcal{F}^{-1}(g),\psi) = 0$ for every rapidly decreasing function ψ from the Schwartz class $S(\mathbb{R})$ with support in $\mathbb{R}-[-1,1]$. Consequently, for every $\psi \in S(\mathbb{R})$ with support in $\mathbb{R}-[-1,1]$ we have

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}^{-1}(f_\alpha - g)(s)\psi(s)ds &= (\mathcal{F}^{-1}(f_\alpha),\psi) - (\mathcal{F}^{-1}(g),\psi)\\ &= \int_{\mathbb{R}} \mathcal{F}^{-1}(f_\alpha)(s)\psi(s)ds. \end{aligned} \tag{2.5}$$

Choosing ψ as a peak delta-like function from $S(\mathbb{R})$ with support in the interval $[y-\varepsilon, y+\varepsilon]$, where $0 < \varepsilon < |y|-1$, and letting $\varepsilon \to 0$, we conclude that (2.4) follows from (2.5).

Combining now (2.3) and (2.4) with (2.2), we have that for every entire function g of exponential type 1 such that $f_\alpha - g \in L_2(\mathbb{R})$, the following inequalities hold:

$$\begin{aligned}\left(\int_R |f_\alpha - g|^2 dx\right)^{1/2} &\ge \left(\int_{|y|\ge 1} |\mathcal{F}^{-1}(f_\alpha - g)|^2 dy\right)^{1/2}\\ &= \left(\int_{|y|\ge 1} |\mathcal{F}^{-1}(f_\alpha)|^2 dy\right)^{1/2}\\ &= (2/\sqrt{\pi})\left|\sin\frac{\alpha\pi}{2}\Gamma(\alpha+1)\right|(2\operatorname{Re}\alpha+1)^{-1/2}. \end{aligned} \tag{2.6}$$

In addition, if there exists an entire function H_α^* of exponential type 1 such that $f_\alpha - H_\alpha^* \in L_2(\mathbb{R})$ and

$$F^{-1}(f_\alpha - H_\alpha^*)(y) = 0 \qquad \text{a.e. on } [-1,1], \tag{2.7}$$

then (2.3) and (2.6) imply the equations

$$\begin{aligned} A_1(|x|^\alpha, L_2(\mathbb{R})) &= \left(\int_R |f_\alpha - H_\alpha^*|^2 dx\right)^{1/2}\\ &= (2/\sqrt{\pi})\left|\sin\frac{\alpha\pi}{2}\Gamma(\alpha+1)\right|(2\operatorname{Re}\alpha+1)^{-1/2}. \end{aligned} \tag{2.8}$$

Therefore, H_α^* is a function of best approximation to f_α in $L_2(\mathbb{R})$, and it is unique since $L_2(\mathbb{R})$ is a strictly convex space.

Step 3. We now show the existence of H_α^* such that (2.7) holds and $f_\alpha - H_\alpha^* \in L_2(\mathbb{R})$. We first note that the function

$$h_\alpha(y) := \mathcal{F}^{-1}(f_\alpha)(y)\chi_{[-1,1]}(y), \tag{2.9}$$

where χ_E is the characteristic function of a set E, is a tempered distribution for $\alpha \ne 0, 2, \ldots$ with support in $[-1,1]$. Indeed, this fact follows from the following representation of the functional $(|y|^{-\alpha-1}\chi_{-1,1]}(y),\psi)$ on $S(\mathbb{R})$ for $\operatorname{Re}\alpha<2m$, $\alpha\neq 0,2,\dots,2m-2$:

$$(|y|^{-\alpha-1}\chi_{[-1,1]}(y),\psi)=\int_0^1 y^{-\alpha-1}\Bigg(\psi(y)+\psi(-y)-2\sum_{k=0}^{m-1}\frac{y^{2k}\psi^{(2k)}(0)}{(2k-\alpha)(2k)!}\Bigg)dy+\sum_{k=0}^{m-1}\frac{\psi^{(2k)}(0)}{(2k-\alpha)(2k)!}$$

(see [**8**, eqn. (3), p. 48]). Therefore by Lemma 2.1, $H^*_\alpha:=\mathcal{F}(h_\alpha)$ is an entire function of exponential type 1. Moreover,

$$\mathcal{F}^{-1}(f_\alpha-H^*_\alpha)(y)=\begin{cases}0, & |y|\le 1\\ \mathcal{F}^{-1}(f_\alpha)(y), & |y|>1.\end{cases} \tag{2.10}$$

Thus (2.7) holds and by (2.2) and (2.10), $\mathcal{F}^{-1}(f_\alpha-H^*_\alpha)\in L_2(\mathbb{R})$, which implies that $f_\alpha-H^*_\alpha\in L_2(\mathbb{R})$. Then it follows from Step 2 that H^*_α is the unique entire function of best approximation in $L_2(\mathbb{R})$ to f_α.

Step 4. It remains to prove representation (1.3). We first assume that $-1/2<\operatorname{Re}\alpha<0$. Then the function h_α given in (2.9) is integrable on $\mathbb{R}$ whence it follows that

$$\begin{aligned}H^*_\alpha(x)&=\mathcal{F}(-(2/\pi)^{1/2}\sin\frac{\alpha\pi}{2}\Gamma(\alpha+1)|y|^{-\alpha-1}\chi_{[-1,1]})(x)\\&=-\frac{1}{\pi}\sin\frac{\alpha\pi}{2}\,\Gamma(\alpha+1)\int_{-1}^1\frac{\cos(xy)}{|y|^{\alpha+1}}dy\\&=-\frac{2}{\pi}\sin\frac{\alpha\pi}{2}\,\Gamma(\alpha+1)\sum_{k=0}^\infty(-1)^k\frac{x^{2k}}{(2k-\alpha)(2k)!}.\end{aligned} \tag{2.11}$$

Therefore, (1.3) holds for $-1/2<\operatorname{Re}\alpha<0$.

Next we use an idea of analytic extension of the distributional Fourier transform developed in [**8**, p. 171]. The function

$$H^*_\alpha(x):=-\frac{2}{\pi}\sin\frac{\alpha\pi}{2}\,\Gamma(\alpha+1)\sum_{k=0}^\infty(-1)^k\frac{x^{2k}}{(2k-\alpha)(2k)!}$$

and the distribution h_α depend analytically on α in the sense that for every $\psi\in S(\mathbb{R})$, the functionals (H^*_α,ψ) and (h_α,ψ) are analytic functions of α in the domain $D:=\{\alpha\in\mathbb{C}:\alpha\neq 0,2,\dots,\operatorname{Re}\alpha>-1/2\}$. Since by (2.11), $\mathcal{F}(h_\alpha)=H^*_\alpha$ for $-1/2<\operatorname{Re}\alpha<0$, the uniqueness of the analytic extension implies that this identity is valid for all $\alpha\in D$ (see [**8**] for more details).

Therefore, (1.3) is established and this completes the proof of Theorem 1.1. □

Remark. The exact value of $A_1(|x|^\alpha,L_2(\mathbb{R}))$, Re $\alpha>-1/2$, $\alpha\neq 0,2,\dots$, is given in (2.8). In case of real α, this provides a new and shorter proof of Raitsin's result.

Moreover, our approach allows us to find $A_1(f,L_2(\mathbb{R}))$ and elements $H^*(f,x)$ of L_2-best approximation to f for some other functions f. For example, we can prove similarly that for $\alpha \neq 1, 3, \ldots$, Re $\alpha > -1/2$,

$$A_1(|x|^\alpha \operatorname{sign} x, L_2(\mathbb{R})) = (2/\sqrt{\pi}) \left| \cos \frac{\alpha\pi}{2} \Gamma(\alpha+1) \right| (2\mathrm{Re}\, \alpha + 1)^{-1/2},$$

$$H^*(|x|^\alpha \operatorname{sign} x, x) = \frac{2}{\pi} \cos \frac{\alpha\pi}{2} \Gamma(\alpha+1) \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1-\alpha)(2k+1)!}.$$

In particular,

$$A_1(\operatorname{sign} x, L_2(\mathbb{R})) = 2/\sqrt{\pi},$$

$$H^*(\operatorname{sign} x, x) = \frac{2}{\pi} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)(2k+1)!}.$$

In addition, the similar relations can be obtained for the functions $|x|^\alpha \log |x|$ and $|x|^\alpha \log |x| \operatorname{sign} x$, Re $\alpha > -1/2$, $\alpha \neq 0, 1, \ldots$

3. The Orthonormal Expansions Approach for $\alpha < 2$

In this section, we analyze the L_2 case for $\alpha < 2$, using the fact that best approximations in L_2 are partial sums of orthonormal expansions. We denote by $\{p_n\}_{n=0}^\infty$ the orthonormal polynomials for the Legendre weight 1 on $[-1, 1]$, so that

$$\int_{-1}^{1} p_n(t)\, p_m(t)\, dt = \delta_{mn}.$$

Moreover, we let γ_n denote the leading coefficient of p_n, and $K_n(x,t)$ denote the reproducing kernel, so that

$$K_n(x,t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_{n-1}(x)\, p_n(t)}{x-t},$$

by the Christoffel-Darboux formula. The classical Legendre polynomials are denoted by $\{P_n\}_{n=0}^\infty$, normalized by the condition $P_n(1) = 1$. Their relation to the orthonormal polynomials is given by

$$p_n(x) = \sqrt{n + \frac{1}{2}}\, P_n(x). \tag{3.1}$$

We let $P_{n,\alpha}^*$ denote the best approximation to $|x|^\alpha$ from the polynomials of degree $\leq n$ in the $L_2[-1,1]$ norm. In the sequel, for $m = n, n+1$, and $x > 0$, let

$$I_n(m, \beta, x) = n^\beta \int_0^1 \frac{t^{\beta+1} p_m(t)}{\left(\frac{x}{n}\right)^2 - t^2} dt. \tag{3.2}$$

The integral is taken in a principal value sense if $x \in (0, n)$. We also set

$$J_n(\beta) = (n-1)^{\beta+1} \int_0^1 t^\beta p_n(t)\, dt. \tag{3.3}$$

The basic idea is to combine the scaled limit in Lemma 3.1(a) below, with the asymptotics in (3.7) and (3.8):

Lemma 3.1. *(a) Let $\alpha > -\frac{1}{2}$, not an even integer. Then uniformly in compact subsets of $\mathbb{C}$,*

$$\lim_{n\to\infty} n^\alpha P_{n,\alpha}^*(z/n) = H_\alpha^*(z). \tag{3.4}$$

(b) Assume that n is even. Then

$$n^{\alpha}P^{*}_{n,\alpha}(x/n) = 2\frac{\gamma_n}{\gamma_{n+1}}\left[xp_{n+1}\left(\frac{x}{n}\right)I_n(n,\alpha-1,x)-p_n\left(\frac{x}{n}\right)I_n(n+1,\alpha,x)\right]. \tag{3.5}$$

If $\alpha>0$, we may also write

$$n^{\alpha}P^{*}_{n,\alpha}(x/n) = 2\frac{\gamma_n}{\gamma_{n+1}}\begin{bmatrix} xp_{n+1}\left(\frac{x}{n}\right)I_n(n,\alpha-1,x) \\ +p_n\left(\frac{x}{n}\right)J_{n+1}(\alpha-1)-x^2p_n\left(\frac{x}{n}\right)I_n(n+1,\alpha-2,x)\end{bmatrix}. \tag{3.6}$$

(c) As $n\to\infty$ through even integers,

$$J_n(\beta)=(-1)^{\frac{n}{2}}2^{\beta+\frac{1}{2}}\frac{\Gamma\left(\frac{1+\beta}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)}+o(1); \tag{3.7}$$

$$J_{n+1}(\beta)=(-1)^{\frac{n}{2}}2^{\beta+\frac{1}{2}}\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\beta}{2}\right)}+o(1). \tag{3.8}$$

Proof. (a) This is part of Theorem 1.1 in [**13**]. (b) As $P^{*}_{n,\alpha}$ is the $(n+1)$st partial sum of the orthonormal expansion of t^{α} in $\{p_j\}_{j=0}^{\infty}$, and as p_n is even, while p_{n+1} is odd,

$$\begin{aligned} P^{*}_{n,\alpha}(x) &= \int_{-1}^{1}|t|^{\alpha}K_{n+1}(x,t)\,dt \\ &= \int_0^1 t^{\alpha}\left[K_{n+1}(x,t)+K_{n+1}(x,-t)\right]dt \\ &= 2\frac{\gamma_n}{\gamma_{n+1}}\begin{bmatrix} xp_{n+1}(x)\int_0^1\frac{t^{\alpha}p_n(t)}{x^2-t^2}dt \\ -p_n(x)\int_0^1\frac{t^{\alpha+1}p_{n+1}(t)}{x^2-t^2}dt\end{bmatrix}. \end{aligned}$$

The first identity (3.5) now follows by a substitution $x\to\frac{x}{n}$ in this last equation. For the second, we write

$$\begin{aligned} I_n(n+1,\alpha,x) &= n^{\alpha}\int_0^1\frac{t^{\alpha+1}p_{n+1}(t)}{\left(\frac{x}{n}\right)^2-t^2}dt \\ &= n^{\alpha}\int_0^1\left[-1+\frac{\left(\frac{x}{n}\right)^2}{\left(\frac{x}{n}\right)^2-t^2}\right]t^{\alpha-1}p_{n+1}(t)\,dt \\ &= -n^{\alpha}\int_0^1 t^{\alpha-1}p_{n+1}(t)\,dt+x^2n^{\alpha-2}\int_0^1\frac{t^{\alpha-1}p_{n+1}(t)}{\left(\frac{x}{n}\right)^2-t^2}dt \\ &= -J_{n+1}(\alpha-1)+x^2I_n(n+1,\alpha-2,x). \end{aligned}$$

Substitute this into (3.5) to get (3.6).
(c) If $n = 2k$, then [**10**, p. 822, (7.231.1)]

$$J_n(\beta) = (n-1)^{\beta+1}\sqrt{n+\frac{1}{2}}\,(-1)^k\,\frac{\Gamma\left(k-\frac{\beta}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\beta}{2}\right)}{2\Gamma\left(-\frac{\beta}{2}\right)\Gamma\left(k+\frac{3}{2}+\frac{\beta}{2}\right)}$$

$$= (-1)^{n/2}\,2^{\beta+\frac{1}{2}}\,\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(-\frac{\beta}{2}\right)} + o(1),$$

by Stirling's formula. Similarly, [**10**, p. 822, (7.231.2)]

$$J_{n+1}(\beta) = n^{\beta+1}\sqrt{n+\frac{3}{2}}\,(-1)^k\,\frac{\Gamma\left(k+\frac{1}{2}-\frac{\beta}{2}\right)\Gamma\left(1+\frac{\beta}{2}\right)}{2\Gamma\left(\frac{1}{2}-\frac{\beta}{2}\right)\Gamma\left(k+2+\frac{\beta}{2}\right)}$$

$$= (-1)^{n/2}\,2^{\beta+\frac{1}{2}}\,\frac{\Gamma\left(1+\frac{\beta}{2}\right)}{\Gamma\left(\frac{1}{2}-\frac{\beta}{2}\right)} + o(1). \qquad \square$$

Our main task will be to estimate $I_n(m, \beta, x)$. We start by recording asymptotics of Legendre polynomials:

Lemma 3.2. *(a) Uniformly for $b \in [0, 1]$,*

$$\int_0^b p_m = O\left(m^{-1}\right). \tag{3.9}$$

(b) As $n \to \infty$ through even integers, we have uniformly for $0 \le s \le n^{1/2}$ and $m = n, n+1$,

$$p_m\left(\frac{s}{n}\right) = \sqrt{\frac{2}{\pi}}\,(-1)^{n/2}\left[\phi_m(s) + O\left(\frac{s+1}{n}\right)\right], \tag{3.10}$$

where

$$\phi_m(t) = \cos t \text{ if } m = n \quad \text{and} \quad \phi_m(t) = \sin t \quad \text{if } m = n+1. \tag{3.11}$$

Proof. (a) Let, as in Szegö, [**20**, p. 194],

$$g_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdots(2n)} = \frac{(2n)!}{(2^n n!)^2} = \frac{1}{\sqrt{\pi n}}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{3.12}$$

Let $\varepsilon \in \left(0, \frac{\pi}{2}\right)$. Then using asymptotics for P_m, [**20**, Theorem 8.21.4, p. 195], and (3.1), we have uniformly for $\theta \in [\varepsilon, \pi - \varepsilon]$,

$$p_m(\cos\theta) = \sqrt{2m+1}\,g_m\frac{\cos\left\{\left(m+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right\}}{(\sin\theta)^{1/2}} + O\left(m^{-1}\right). \tag{3.13}$$

Integrating

$$\int_0^b p_m = \int_{\arccos(b)}^{\frac{\pi}{2}} p_m(\cos\theta)\sin\theta\,d\theta$$

by parts gives (3.9) for $b \in \left[0, \frac{1}{2}\right]$. Since (3.7) and (3.8) with $\beta = 0$ show that

$$\int_0^1 p_m = (m-1)^{-1} J_m(0) = O\left(m^{-1}\right),$$

we then obtain (3.9) for all $b \in [0,1]$.
(b) Let $\theta = \arccos\left(\frac{s}{n}\right)$. Then

$$\theta = \arccos\left(\frac{s}{n}\right) = \frac{\pi}{2} - \frac{s}{n} + O\left(\frac{s}{n}\right)^3,$$

so

$$\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4} = \frac{n}{2}\pi - s + O\left(\frac{s}{n}\right),$$

recall that $s \le \sqrt{n}$. From this, as n is even, we deduce that

$$\cos\left(\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right) = (-1)^{\frac{n}{2}}\cos(s) + O\left(\frac{s}{n}\right);$$

$$\cos\left(\left(n+\frac{3}{2}\right)\theta - \frac{\pi}{4}\right) = (-1)^{\frac{n}{2}}\sin(s) + O\left(\frac{s}{n}\right).$$

We substitute these into the asymptotic (3.13) and use $(\sin\theta)^{-\frac{1}{2}} = 1 + O\left(\frac{s^2}{n^2}\right)$, as well as (3.12) to get the result. □

The most difficult calculation is contained in

LEMMA 3.3. *Let $-2 < \beta < 1$, $x > 0$ be fixed, and $m = n$ or $n+1$, and ϕ_m be given by* (3.11). *Then as $n \to \infty$ through even integers,*

$$I_n(m,\beta,x) = (-1)^{n/2}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{s^{\beta+1}\phi_m(s)}{x^2 - s^2}ds + o(1). \tag{3.14}$$

For $m = n+1$, we may also allow $-3 < \beta < 1$.

Proof. We split

$$\begin{aligned} I_n(m,\beta,x) &= n^\beta\left[\int_0^{\frac{2x}{n}} + \int_{\frac{2x}{n}}^{\frac{\log n}{n}} + \int_{\frac{\log n}{n}}^1\right]\frac{t^{\beta+1}p_m(t)}{\left(\frac{x}{n}\right)^2 - t^2}dt \\ &=: I_n^{(1)} + I_n^{(2)} + I_n^{(3)}. \end{aligned} \tag{3.15}$$

Note that $I_n^{(1)}$ is a Cauchy principal value integral.

Step 1. We establish that

$$I_n^{(3)} = n^\beta\int_{\frac{\log n}{n}}^1 \frac{t^{\beta+1}p_m(t)}{\left(\frac{x}{n}\right)^2 - t^2}dt = o(1). \tag{3.16}$$

Let

$$f(t) = \frac{t^{\beta+1}}{\left(\frac{x}{n}\right)^2 - t^2}.$$

For large n, we see that uniformly for $t \in \left[\frac{\log n}{n}, 1\right]$,

$$|f(t)| = O\left(t^{\beta-1}\right).$$

We integrate by parts:

$$\begin{aligned}
I_n^{(3)} &= n^\beta \int_{\frac{\log n}{n}}^1 f(t)\, p_m(t)\, dt \\
&= n^\beta \left[f(1) \int_0^1 p_m - f\left(\frac{\log n}{n}\right) \int_0^{\frac{\log n}{n}} p_m - \int_{\frac{\log n}{n}}^1 f'(t) \left(\int_0^t p_m \right) dt \right] \\
&= O\left(n^{\beta-1}\right) + O\left((\log n)^{\beta-1}\right) + O\left(n^{\beta-1}\right) \int_{\frac{\log n}{n}}^1 |f'|,
\end{aligned}$$

by Lemma 3.2(a). Here

$$f'(t) \left[\left(\frac{x}{n}\right)^2 - t^2 \right]^2 = t^\beta \left[(\beta+1) \left(\frac{x}{n}\right)^2 + (1-\beta)\, t^2 \right].$$

For large n, we see that $f' > 0$ in $\left[\frac{\log n}{n}, 1\right]$, and hence

$$\int_{\frac{\log n}{n}}^1 |f'| = \int_{\frac{\log n}{n}}^1 f' = O\left(\left| f\left(\frac{\log n}{n}\right) \right| \right) = O\left(\left(\frac{\log n}{n}\right)^{\beta-1} \right).$$

So as $\beta < 1$,

$$\left| I_n^{(3)} \right| = O\left(n^{\beta-1}\right) + O\left((\log n)^{\beta-1}\right) = o(1).$$

Step 2. We establish that

$$I_n^{(2)} = n^\beta \int_{\frac{2x}{n}}^{\frac{\log n}{n}} \frac{t^{\beta+1} p_m(t)}{\left(\frac{x}{n}\right)^2 - t^2} dt = (-1)^{n/2} \sqrt{\frac{2}{\pi}} \int_{2x}^\infty \frac{s^{\beta+1} \phi_m(s)}{x^2 - s^2} ds + o(1). \tag{3.17}$$

By the substitution $t = s/n$, and then Lemma 3.2(b),

$$\begin{aligned}
I_n^{(2)} &= \int_{2x}^{\log n} \frac{s^{\beta+1} p_m(s/n)}{x^2 - s^2} ds \\
&= (-1)^{n/2} \sqrt{\frac{2}{\pi}} \int_{2x}^{\log n} \frac{s^{\beta+1} \phi_m(s)}{x^2 - s^2} ds \\
&\quad + O\left(\frac{1}{n} \left\{ \begin{array}{ll} (\log n)^{\max\{\beta+1,0\}}, & \beta \neq -1 \\ \log(\log n), & \beta = -1 \end{array} \right\} \right).
\end{aligned}$$

The integral over $(0, \infty)$ is conditionally convergent as $\beta+1 < 2$, and (3.17) follows.

Step 3. Finally, we deal with $I_n^{(1)}$, establishing

$$I_n^{(1)} = n^\beta \int_0^{\frac{2x}{n}} \frac{t^{\beta+1} p_m(t)}{\left(\frac{x}{n}\right)^2 - t^2} dt = (-1)^{n/2} \sqrt{\frac{2}{\pi}} \int_0^{2x} \frac{s^{\beta+1} \phi_m(s)}{x^2 - s^2} ds + o(1). \tag{3.18}$$

We emphasize that $x > 0$ is fixed. We see that

$$I_n^{(1)} = \int_0^{2x} \frac{s^{\beta+1} \left[p_m\left(\frac{s}{n}\right) - p_m\left(\frac{x}{n}\right) \right]}{x^2 - s^2} ds + p_m\left(\frac{x}{n}\right) \int_0^{2x} \frac{s^{\beta+1}}{x^2 - s^2} ds. \tag{3.19}$$

Here if $0 < \varepsilon < \frac{x}{2}$, and n is large enough,

$$\left| \int_{x-\varepsilon}^{x+\varepsilon} \frac{s^{\beta+1} \left[p_m\left(\frac{s}{n}\right) - p_m\left(\frac{x}{n}\right) \right]}{x^2 - s^2} ds \right| \leq C\varepsilon \max_{\left[0, \frac{1}{4}\right]} |p_m'| \, \frac{1}{n},$$

with C depending on x, but independent of m, n, ε. Now as p_m is bounded in $\left[-\frac{1}{2}, \frac{1}{2}\right]$, Bernstein's inequality implies that $\max_{\left[0,\frac{1}{4}\right]} |p'_m| = O(n)$. Hence uniformly in ε and n,

$$\left| \int_{x-\varepsilon}^{x+\varepsilon} \frac{s^{\beta+1}\left[p_m\left(\frac{s}{n}\right) - p_m\left(\frac{x}{n}\right)\right]}{x^2 - s^2} ds \right| = O(\varepsilon). \tag{3.20}$$

Next, the asymptotic in Lemma 3.2(b) shows that

$$\begin{aligned}
&\int_{[0,2x]\setminus[x-\varepsilon,x+\varepsilon]} \frac{s^{\beta+1}\left[p_m\left(\frac{s}{n}\right) - p_m\left(\frac{x}{n}\right)\right]}{x^2 - s^2} ds \\
&= (-1)^{n/2}\sqrt{\frac{2}{\pi}} \int_{[0,2x]\setminus[x-\varepsilon,x+\varepsilon]} \frac{s^{\beta+1}\left[\phi_m(s) - \phi_m(x)\right]}{x^2 - s^2} ds \\
&\quad + O\left(\frac{1}{n}\int_{[0,2x]\setminus[x-\varepsilon,x+\varepsilon]} \frac{s^{\beta+2} + s^{\beta+1}}{|x^2 - s^2|} ds\right) \\
&= (-1)^{n/2}\sqrt{\frac{2}{\pi}} \left[\int_0^{2x} \frac{s^{\beta+1}\left[\phi_m(s) - \phi_m(x)\right]}{x^2 - s^2} ds + O(\varepsilon)\right] + O\left(\frac{1}{n\varepsilon}\right).
\end{aligned}$$

We now choose $\varepsilon = \frac{1}{\sqrt{n}}$. Combining this, (3.19), (3.20) and Lemma 3.2(b) again gives

$$\begin{aligned}
&I_n^{(1)} \\
&= (-1)^{n/2}\sqrt{\frac{2}{\pi}} \left[\int_0^{2x} \frac{s^{\beta+1}\left[\phi_m(s) - \phi_m(x)\right]}{x^2 - s^2} ds + \phi_m(x) \int_0^{2x} \frac{s^{\beta+1}}{x^2 - s^2} ds\right] + o(1).
\end{aligned}$$

So we have established (3.18). Finally, combining (3.15) to (3.18) gives the result. When $-3 < \beta < 2$ and $m = n + 1$, one splits off part of the integral in $I_n^{(1)}$ near 0, say over $[0, \varepsilon]$ and estimates it separately. We leave this case to the reader. □

Finally we deduce a special case of Theorem 1.2:

Proof of (1.6) of Theorem 1.2 for $-\frac{1}{2} < \alpha < 1$. Recall that [**15**], [**20**, eqn. (4.21), p. 63]

$$\frac{\gamma_n}{\gamma_{n+1}} = \frac{1}{2} + o(1), n \to \infty.$$

Then (3.4), (3.5) and (3.10) give for $x > 0$,

$$\begin{aligned}
H_\alpha^*(x) &= \lim_{n\to\infty, n \text{ even}} \left[x p_{n+1}\left(\frac{x}{n}\right) I_n(n, \alpha - 1, x) - p_n\left(\frac{x}{n}\right) I_n(n+1, \alpha, x)\right] \\
&= \frac{2}{\pi}\left[x \sin x \int_0^\infty \frac{s^\alpha \cos(s)}{x^2 - s^2} ds - \cos x \int_0^\infty \frac{s^{\alpha+1}\sin(s)}{x^2 - s^2} ds\right] \\
&= \frac{2}{\pi}\int_0^\infty s^\alpha \mathbb{J}(x, s)\, ds.
\end{aligned}$$

Note that in all the applications of Lemma 3.3, $\beta = \alpha - 1$ or $\alpha < 1$. Since both sides are entire, this identity remains valid in the entire plane. □

Proof of (1.7) of Theorem 1.2 for $-\frac{1}{2} < \alpha < 2$. Here (3.4) and (3.6) give

$$H_\alpha^*(x) = \lim_{n\to\infty, n \text{ even}} \begin{bmatrix} xp_{n+1}\left(\frac{x}{n}\right) I_n(n, \alpha-1, x) \\ +p_n\left(\frac{x}{n}\right) J_{n+1}(\alpha-1) - x^2 p_n\left(\frac{x}{n}\right) I_n(n+1, \alpha-2, x) \end{bmatrix}$$

$$= \frac{2}{\pi}\left[x\sin x \int_0^\infty \frac{s^\alpha \cos(s)}{x^2-s^2} ds + \frac{\sqrt{\pi}2^{\alpha-1}\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(1-\frac{\alpha}{2}\right)}\cos x \right.$$

$$\left. - x^2\cos x \int_0^\infty \frac{s^{\alpha-1}\sin s}{x^2-s^2} ds\right],$$

by (3.8), (3.10) and (3.14). Then (1.7) follows. □

4. The L_2 Case for $\alpha > 2$

Recall that ℓ is the even integer in $(\alpha-2, \alpha)$.

Lemma 4.1. *Let n be even. Then*

$$P_{n,\alpha}^*(x) = x^\ell P_{n,\alpha-\ell}^*(x) + 2\frac{\gamma_n}{\gamma_{n+1}} \sum_{j=0}^{\ell/2-1} x^{2j}\left[p_n(x)\int_0^1 t^{\alpha-2j-1} p_{n+1}(t)\,dt - xp_{n+1}(x)\int_0^1 t^{\alpha-2-2j} p_n(t)\,dt\right].$$

Proof. We substitute the identity

$$t^\ell = x^\ell + \left(t^2-x^2\right)\sum_{j=0}^{\ell/2-1} x^{2j} t^{\ell-2-2j}$$

in

$$P_{n,\alpha}^*(x) = \int_{-1}^1 |t|^{\alpha-\ell}\, t^\ell K_{n+1}(x,t)\,dt$$

to deduce

$$P_{n,\alpha}^*(x) = x^\ell P_{n,\alpha-\ell}^*(x) + \sum_{j=0}^{\ell/2-1} x^{2j}\int_{-1}^1 |t|^{\alpha-2-2j}\left(t^2-x^2\right) K_{n+1}(x,t)\,dt. \tag{4.1}$$

Here

$$\int_{-1}^1 |t|^{\alpha-2-2j}\left(t^2-x^2\right) K_{n+1}(x,t)\,dt$$

$$= \int_0^1 t^{\alpha-2-2j}\left(t^2-x^2\right)\left[K_{n+1}(x,t) + K_{n+1}(x,-t)\right] dt$$

$$= 2\frac{\gamma_n}{\gamma_{n+1}}\left[p_n(x)\int_0^1 t^{\alpha-1-2j} p_{n+1}(t)\,dt - xp_{n+1}(x)\int_0^1 t^{\alpha-2-2j} p_n(t)\,dt\right]$$

by the Christoffel-Darboux formula. Now substitute in (4.1). □

Proof of Theorem 1.2(II) for $\alpha > 2$. From Lemma 4.1 we deduce

$$n^{\alpha} P^{*}_{n,\alpha}\left(\frac{x}{n}\right) = x^{\ell} n^{\alpha-\ell} P^{*}_{n,\alpha-\ell}\left(\frac{x}{n}\right) + 2\frac{\gamma_n}{\gamma_{n+1}} \sum_{j=0}^{\ell/2-1} x^{2j} \left[\begin{array}{r} p_n\left(\frac{x}{n}\right) J_{n+1}\left(\alpha-2j-1\right) \\ -\left(\frac{n}{n-1}\right)^{\alpha-2j-1} x p_{n+1}\left(\frac{x}{n}\right) J_n\left(\alpha-2j-2\right) \end{array} \right].$$

From Lemma 3.1(a)

$$\lim_{n\to\infty} n^{\alpha-\ell} P^{*}_{n,\alpha}\left(\frac{x}{n}\right) = H^{*}_{\alpha-\ell}\left(x\right).$$

Now just substitute in this and the limits (3.7), (3.8), (3.10). □

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