GAUSSIAN FLUCTUATIONS OF EIGENVALUES OF RANDOM HERMITIAN MATRICES ASSOCIATED WITH FIXED AND VARYING WEIGHTS

D. S. LUBINSKY

Abstract. We investigate fluctuations of eigenvalues in the bulk for random Hermitian matrices associated with (i) a fixed weight on a compact interval (ii) an exponential weight on a possibly unbounded interval (iii) varying exponential weights. In particular, we show that the normalized difference between the $k-$th eigenvalue in the bulk, and the kth zero of an appropriate orthogonal polynomial, is normally distributed in the bulk. This generalizes earlier work of Jonas Gustavsson and Deng Zhang.

1. Introduction

In the theory of random Hermitian matrices [7, p. 102 ff.], one considers a probability distribution $\mathbb{P}^{(n)}$ on the eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ of an $n \times n$ Hermitian matrix M. The probability density function for $\mathbb{P}^{(n)}$, which we denote by $\mathcal{P}^{(n)}$, takes the form

$$\mathcal{P}^{(n)}(\lambda_1, \lambda_2, ..., \lambda_n) = \frac{1}{Z_n} \left(\prod\nolimits_{1 \leq i < j \leq n} (\lambda_i - \lambda_j)^2\right) \left(\prod_{j=1}^{n} \mu_n'(\lambda_j)\right). \tag{1.1}$$

Here Z_n is a normalizing constant, often called the partition function, and μ_n is an absolutely continuous measure supported on the real line. In many cases, $\mu_n'(x) = e^{-nQ_n(x)}$, where Q_n is a given function. In the famous Gaussian Unitary Ensemble, $Q_n(x) = \frac{1}{2}x^2$. One particularly important quantity is the $m-$point correlation function [7, p. 112], where $1 \leq m \leq n$:

$$\mathcal{R}_m^{(n)}(\lambda_1, \lambda_2, ..., \lambda_m) = \frac{n!}{(n-m)!} \int ... \int \mathcal{P}^{(n)}(\lambda_1, \lambda_2, ..., \lambda_n)\, d\lambda_{m+1}\, d\lambda_{m+2}\, ...d\lambda_n.$$

Typically one analyzes this with m fixed and $n \to \infty$, using a remarkable connection to orthogonal polynomials, first discovered by Freeman Dyson.

If μ_n has all finite power moments, and the support supp$[\mu_n]$ of μ_n contains infinitely many points, we may define orthonormal polynomials

$$p_{n,m}(x) = \gamma_{n,m} x^m + ...,\ \gamma_{n,m} > 0,$$

$m = 0, 1, 2, ...$, satisfying the orthonormality conditions

$$\int p_{n,k}(x)\, p_{n,\ell}(x)\, d\mu_n(x) = \delta_{k\ell}. \tag{1.2}$$

Date: June 14, 2016.

Throughout we use μ_n' to denote the Radon-Nikodym derivative of μ_n. The nth reproducing kernel for μ_n is

$$K_n(x,y) = \sum_{k=0}^{n-1} p_{n,k}(x)\, p_{n,k}(y). \tag{1.3}$$

The nth Christoffel function is

$$\lambda_n(\mu_n, x) = K_n(x,x)^{-1}.$$

The n simple zeros of $p_{n,n}(x)$ are denoted by

$$x_{1n} < x_{2n} < ... < x_{nn}. \tag{1.4}$$

We note that in the orthogonal polynomial literature, it is more customary to index these in decreasing order, so that x_{1n} is the largest zero. However, we want to compare the jth eigenvalue λ_j (in increasing order) to x_{jn}, so adopt this unusual convention.

There is the basic formula for the probability density function $\mathcal{P}^{(n)}$:

$$\mathcal{P}^{(n)}(\lambda_1, \lambda_2, ..., \lambda_n) = \frac{1}{n!} \det\left(K_n(\lambda_i, \lambda_j)\, \mu_n'(\lambda_i)^{1/2}\, \mu_n'(\lambda_j)^{1/2}\right)_{1 \le i,j \le n}.$$

but somewhat deeper is Dyson's identity [7, p. 112]:

$$\mathcal{R}_m^{(n)}(\lambda_1, \lambda_2, ..., \lambda_m) = \det\left(K_n(\lambda_i, \lambda_j)\, \mu_n'(\lambda_i)^{1/2}\, \mu_n'(\lambda_j)^{1/2}\right)_{1 \le i,j \le m}. \tag{1.5}$$

In this paper, we focus on the fluctuations of the eigenvalues, extending earlier work of Jonas Gustavsson [11] and Deng Zhang [28], [29] to more general fixed and varying exponential weights, and also to fixed weights on $[-1,1]$. Their work in turn depended on earlier results of Costin and Leibovitz [5], and Soshnikov [22], [23]. The main innovation in this paper is that we use new technical ideas to analyze the expected number of eigenvalues in an interval, and consequently can allow more general measures. All our results follow from Theorems 2.1 to 2.3 stated in Section 2, which we believe are of independent interest.

Gustavsson considered the Hermite weight (or Gaussian Unitary Ensemble) $\mu_n'(t) = e^{-2nQ(t)}$, with $Q(t) = \frac{1}{2}t^2$, while Deng Zhang considered first $Q(t) = t^{2m}$, $m \ge 1$, and later polynomial Q. One of their main results is to show that for j/n bounded away from 0 and 1, the scaled difference between the jth eigenvalue λ_j and jth zero of $p_{n,n}(\cdot)$ satisfies [28, p. 1490, Thm. 1.4]

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2}} \frac{1}{n\sigma_n^*(x_{jn})}} \to N(0,1) \tag{1.6}$$

in distribution as $n \to \infty$. Here $N(0,1)$ is the normal distribution, that is, has probability density $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}, t \in (-\infty, \infty)$. A more precise statement of (1.6) is

$$\lim_{n\to\infty} \mathbb{P}^{(n)}\left(\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2}} \frac{1}{n\sigma_n^*(x_{jn})}} \le \xi\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-\frac{1}{2}t^2} dt, \tag{1.7}$$

for all $\xi \in \mathbb{R}$. In [11], [28], [29] the authors also establish limits involving multivariate normal distributions, and fluctuations at the edge.

The function σ_n^* is the contracted density of an equilibrium distribution for the external field Q, that we shall introduce later. In the work of Gustavsson and Zhang,

x_{kn} is replaced first by a different quantity involving the equilibrium density, but it does not seem possible to do this in our more general setting, as we do not have such precise asymptotics.

There is an extensive literature on the distribution of eigenvalues of random matrices, and we cannot survey this here. We simply mention that Gaussian fluctuations have been established in settings other than Hermitian matrices - see for example [19], [24], [26]. Recent work includes a law of large numbers for orthogonal polynomial ensembles [3] as well as mesoscopic, rather than microscopic fluctuations [4].

We next turn to equilibrium densities in potential theory. The equilibrium density for the interval $[-1,1]$ is

$$\sigma^*_{[-1,1]}(t) = \frac{1}{\pi\sqrt{1-t^2}}, t \in (-1,1) . \tag{1.8}$$

It has the property that

$$\int_{-1}^{1} \log|x-t|\, \sigma^*_{[-1,1]}(t)\, dt = -\log 2,\ x \in [-1,1],$$

and is the unique minimizer of the energy integral

$$\int_{-1}^{1}\int_{-1}^{1} \log \frac{1}{|x-y|} d\nu(x)\, d\nu(y)$$

amongst all probability measures ν supported on $[-1,1]$. It plays the role of σ^*_n in (1.6) when we consider fixed measures supported on $[-1,1]$. We can now formulate our result for such measures, which involves an implicit assumption about asymptotics of orthonormal polynomials:

Theorem 1.1
Let μ be an absolutely continuous measure supported on $[-1,1]$ that satisfies the Szegő condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty. \tag{1.9}$$

Let $\{p_n\}$ denote the system of orthonormal polynomials for the measure μ. Let $\mathcal{I}$ be a closed subinterval of $(-1,1)$ in which μ' is positive and continuous. Assume, moreover, that as $n \to \infty$, uniformly for $x = \cos\theta \in \mathcal{I}$,

$$p_n(x)\, \mu'(x)^{1/2} \left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \cos(n\theta + \psi(x)) + o(\log n)^{-1/2}, \tag{1.10}$$

where ψ is a function with modulus of continuity in $\mathcal{I}$ satisfying for $t \to 0+$,

$$\omega(\psi; \mathcal{I}; t) = \sup\{|\psi(x) - \psi(y)| : x, y \in \mathcal{I},\ |x-y| \le t\} = o\left(|\log t|^{-1/2}\right). \tag{1.11}$$

For $n \ge 1$, let $\mathbb{P}^{(n)}$ denote the probability distribution with density defined by (1.1) with $\mu_n = \mu$ for all $n \ge 1$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ denote the associated eigenvalues in increasing order. Also, let $\{x_{jn}\}$ denote the zeros of the nth orthogonal polynomial for μ, ordered as in (1.4). Let $\mathcal{J}$ be a subinterval of the interior of $\mathcal{I}$. Then for j, n with $x_{jn} \in \mathcal{J}$,

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2} \frac{1}{n\sigma^*_{[-1,1]}(x_{jn})}}} \to N(0,1) \tag{1.12}$$

in distribution as $n \to \infty$*.*

Remarks

It is indeed unfortunate to have an implicit assumption such as (1.10). In an earlier version of this paper, we assumed the weaker asymptotic

$$p_n(x)\,\mu'(x)^{1/2}\left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}}\cos\left(n\theta+\psi(x)\right)+o(1), \tag{1.13}$$

where ψ was only required to be continuous in $\mathcal{I}$. Maurice Duits as referee discovered that this is not enough to prove Theorem 2.3. The most general hypothesis for (1.13) is due to Badkov [1] (something the author learned from Leonid Golinskii) and involves a Dini condition on the modulus of continuity of μ' in $\mathcal{I}$, namely

$$\int_0^1 \frac{\omega\left(\mu';\mathcal{I};t\right)}{t}dt < \infty.$$

Unfortunately, as far as I am aware, there is no simple analogue of Badkov's theorem that guarantees (1.10). Most research on pointwise asymptotics of orthonormal polynomials associated with a measure on $(-1,1)$, does not include rates. This is true of the results in the book of Freud [6], and largely true of the results in the book of Geronimus [8], though Table V on page 200 ff. there doss contain some applicable results. Here are some examples of known hypotheses that do guarantee (1.10):

Example A

Let

$$f(\theta) = \mu'(\cos\theta)\left|\sin\theta\right|, \theta\in[-\pi,\pi],$$

and assume that f is positive and continuous in $[-\pi,\pi]$, with modulus of continuity satisfying, for some $\varepsilon>0$,

$$\omega\left(f;[-\pi,\pi];t\right) \le C\left|\log t\right|^{-3/2-\varepsilon}, t\to 0+.$$

Then [25, Thm. 12.1.3, p. 297] we have (1.10) uniformly in compact subsets of $(-1,1)$. Note that there results are proved for orthonormal polynomials on the unit circle, but classical methods permit their translation to $[-1,1]$. Moreover, the function ψ is essentially the same as γ in [25, p. 299, eq. (12.2.1)] and the required smoothness of the function ψ in (1.11) follows from Privalov type theorems for singular integrals.

Example B

Assume that f above satisfies

$$\int_{-\pi}^{\pi}\frac{1}{f(\theta)^2}d\theta < \infty;\ \sup_{[-\pi,\pi]} f < \infty$$

and in the closed subinterval $\mathcal{I}$ of $(-1,1)$, f is bounded below by a positive number, while the local modulus of continuity satisfies, for some $\varepsilon>0$,

$$\omega\left(f;\mathcal{I};t\right) \le C\left|\log t\right|^{-3/2-\varepsilon}, t\to 0+.$$

Then [9, Theorem 2.2, p. 396] we have (1.10) uniformly in compact subsets of $(-1,1)$. Again there results are proved for orthonormal polynomials on the unit circle, and again, the required smoothness of the function ψ in (1.11) follows from Privalov type theorems for singular integrals.

Example C
Let μ' be a generalized Jacobi weight,

$$\mu'(x) = (1-x)^{\alpha}(1+x)^{\beta} h(x) \prod_{j=1}^{p} |x-x_j|^{c_j}, x \in (-1,1),$$

where $\alpha, \beta > -1$, all $c_j > -1$, $-1 < x_1 < ... < x_p < 1$, and h is positive and analytic on $[-1,1]$. Vanlessen [27] obtained a complete asymptotic expansion for the coefficients in the three term recurrence relation for the orthonormal polynomials for μ. These imply the required asymptotics for the orthonormal polynomials, although the latter are stated only close to $\{x_j\}_{j=1}^{p}$ in [27]. It is also very likely that the many papers of Badkov on generalized Jacobi weights include rates that imply (1.10), but I have been unable to find these.

We next turn to fixed exponential weights e^{-2Q} on a bounded or unbounded interval I, of the type considered in [13, p. 7]. First we need a definition:

Definition 1.2
Let $I = (c,d)$ be a bounded or unbounded interval containing 0. Let $\mu'(x) = W^2(x) = e^{-2Q(x)}, x \in I$, where
(a) Q' is continuous in I and $Q(0) = 0$;
(b) Q'' exists and is positive in $I \backslash \{0\}$;
(c)

$$\lim_{t \to c+} Q(t) = \infty = \lim_{t \to d-} Q(t);$$

(d) The function $T(t) = \frac{tQ'(t)}{Q(t)}, t \in I \backslash \{0\}$ is quasi-increasing in $(0,d)$, in the sense that for some constant C and $0 \le x < y < d \Rightarrow$

$$T(x) \le CT(y);$$

T is also assumed quasi-decreasing in $(c,0)$. In addition we assume that T is bounded below in $I \backslash \{0\}$ by a constant larger than 1.
(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \le C_1 \frac{Q'(x)}{Q(x)}, \text{ a.e. } x \in I \backslash \{0\}.$$

Then we write $\mu' = e^{-2Q} \in \mathcal{F}\left(C^2\right)$.

Examples of Q satisfying the conditions above on $(-\infty,\infty)$ include [13, pp. 8-9]

$$Q(x) = \begin{cases} x^{\alpha}, & x \in [0,\infty) \\ |x|^{\beta}, & x \in (-\infty,0), \end{cases}$$

where $\alpha, \beta > 1$. A more general example is

$$Q(x) = \begin{cases} \exp_{\ell}(x^{\alpha}) - \exp_{\ell}(0), & x \in [0,\infty) \\ \exp_{k}\left(|x|^{\beta}\right) - \exp_{k}(0) & x \in (-\infty,0), \end{cases}$$

where $\alpha, \beta > t$, $k, \ell \geq 0$, and $\exp_k(x) = \exp(\exp(...(\exp(x)))$ is the kth iterated exponential. An example on $(-1,1)$ is

$$Q(x) = \begin{cases} \exp_\ell\left(\left(1-x^2\right)^{-\alpha}\right) - \exp_\ell(1), & x \in [0,1) \\ \exp_k\left(\left(1-x^2\right)^{-\beta}\right) - \exp_k(1), & x \in (-1,0), \end{cases}$$

where $\alpha, \beta > 0$ and $k, \ell \geq 0$. We could actually allow a more general (but more technical) class of weights, namely the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ from [13].

In considering orthogonal polynomials associated with the measure $d\mu(t) = e^{-2Q(t)}dt$, a crucial role is played by the Mhaskar-Rakhmanov-Saff interval $\Delta_n = [a_{-n}, a_n], n > 0$. The numbers a_{-n}, a_n are defined by the equations [13, p. 13]

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx; \tag{1.14}$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx. \tag{1.15}$$

In fact $a_{\pm n}$ are also defined for non-integer n. We note that a_n increases with n and $a_{-n} \to c$ and $a_n \to d$ as $n \to \infty$. The interval $\left(a_{-n-\frac{1}{2}}, a_{n+\frac{1}{2}}\right)$ contains the zeros of the nth orthonormal polynomial for $d\mu$ [13, p. 313] and the largest zero of this polynomial is close to a_n [13, p. 314]. As an example of Mhaskar-Rakhmanov-Saff numbers, let $Q(x) = |x|^\alpha$, $x \in \mathbb{R}$, $\alpha > 0$. It is known that then [17, p. 206, eqn. (1.14)], [18]

$$a_n = \left\{\frac{2^{\alpha-2}\Gamma\left(\frac{\alpha}{2}\right)^2}{\Gamma(\alpha)}\right\}^{1/\alpha} n^{1/\alpha}, n \geq 1.$$

Another important quantity associated with Q is the nth equilibrium density [13, p. 16]

$$\sigma_n(x) = \frac{1}{\pi^2}\sqrt{(x-a_{-n})(a_n-x)}\int_{a_{-n}}^{a_n} \frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}, x \in \Delta_n. \tag{1.16}$$

It has total mass n

$$\int_{a_{-n}}^{a_n} \sigma_n = n, \tag{1.17}$$

and satisfies the equilibrium equation

$$\int_{a_{-n}}^{a_n} \log\left|\frac{1}{x-s}\right| \sigma_n(s)ds + Q(x) = c_n, x \in \Delta_n. \tag{1.18}$$

Here c_n is a constant.

In many contexts, it is convenient to map σ_n onto a density function that is supported on $[-1,1]$. Let

$$\beta_n := \frac{1}{2}(a_n + a_{-n}); \delta_n = \frac{1}{2}(a_n + |a_{-n}|). \tag{1.19}$$

We can then define the linear map of Δ_n onto $[-1,1]$ by [13, p. 24]

$$L_n(x) = (x-\beta_n)/\delta_n, x \in \Delta_n$$

$$\Leftrightarrow x = L_n^{[-1]}(t) = \beta_n + \delta_n t, t \in [-1,1]. \tag{1.20}$$

The transformed (and renormalized) density is

$$\sigma_n^*(t) = \frac{\delta_n}{n}\sigma_n \circ L_n^{[-1]}(t), t \in [-1,1]. \tag{1.21}$$

It satisfies

$$\int_{-1}^{1} \sigma_n^* = 1.$$

Note that when Q is even, we have $\delta_n = a_n$, and $\sigma_n^*(x) = \frac{a_n}{n}\sigma_n(a_n x), x \in [-1,1]$.

We shall also scale the weight $W^2 = e^{-2Q}$ to $[-1,1]$. Thus for $n \geq 1$, we define [13, p. 25]

$$W_n^*(t) = W\left(L_n^{[-1]}(t)\right),\ t \in L_n(I), \tag{1.22}$$

and

$$d\mu_n(t) = W_n^*(t)^2\, dt, t \in L_n(I). \tag{1.23}$$

With this in hand, we can state our result for fixed exponential weights:

Theorem 1.3
Let $\mu' = e^{-2Q} \in \mathcal{F}(C^2)$. For $n \geq 1$, define a probability distribution $\hat{\mathbb{P}}^{(n)}$ with probability density function

$$\hat{\mathcal{P}}^{(n)}\left(\mu, \hat{\lambda}_1, \hat{\lambda}_2, ..., \hat{\lambda}_n\right) = \frac{1}{\hat{Z}_n}\left(\prod_{1\leq i<j\leq n}\left(\hat{\lambda}_i - \hat{\lambda}_j\right)^2\right)\left(\prod_{j=1}^{n} e^{-2Q(\hat{\lambda}_j)}\right). \tag{1.24}$$

where $\hat{Z}_n$ is a normalizing constant. Also, let $\{\hat{x}_{jn}\}$ denote the zeros of the nth orthogonal polynomial for μ, ordered as in (1.4). Let $\varepsilon \in \left(0, \frac{1}{2}\right)$ and $\{a_{\pm n}\}, \{\delta_n\}$, and $\{\sigma_n\}$ be as above. Then for j, n with $\hat{x}_{jn} \in L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]$, we have

$$\frac{\hat{\lambda}_j - \hat{x}_{jn}}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{\sigma_n(\hat{x}_{jn})}}} \to N(0,1) \tag{1.25}$$

in distribution as $n \to \infty$.

Remark
We shall derive (1.25) from its analogue for the contracted measures $\{d\mu_n\}$ of (1.22) and (1.23). If $\mathcal{P}^{(n)}$ is defined by (1.1) and $\{x_{jn}\}$ are the zeros of $p_{n,n}$, the nth orthonormal polynomial for μ_n, then for j, n with $j/n \in [-1+\varepsilon, 1-\varepsilon]$, we shall show

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{n\sigma_n^*(x_{jn})}}} \to N(0,1) \tag{1.26}$$

and then derive (1.25) via the linear transformation $\hat{\lambda}_j = L_n^{[-1]}(\lambda_j)$; $\hat{x}_{jn} = L_n^{[-1]}(x_{jn})$.

Our final class of weights is varying exponential weights. For these, we use the work of McLaughlin and Miller [16]. We assume that we are given a function $Q : \mathbb{R} \to \mathbb{R}$ that grows faster than $(\log|x|)^{1+\varepsilon}$ as $|x| \to \infty$, for some $\varepsilon > 0$. We let

$$\mu_n'(x) = e^{-2nQ(x)}, x \in \mathbb{R}, n \geq 1, \tag{1.27}$$

so that

$$\int_{-\infty}^{\infty} p_{n,j}(x)\, p_{n,k}(x)\, e^{-2nQ(x)} dx = \delta_{jk}.$$

We shall also assume that Q'' is continuous and satisfies a Lipschitz condition of order 1 and Q is strictly convex. In addition, we assume that the equilibrium densities have support $[-1,1]$. Note that because of the special form of the weight e^{-2nQ}, σ_n has a very special form:

$$\sigma_n(x) = n\sigma(x) = \frac{n}{\pi^2}\sqrt{1-x^2}\int_{-1}^{1}\frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{1-s^2}}. \tag{1.28}$$

Moreover, the contracted density σ_n^* is independent of n: for $n \geq 1$,

$$\sigma_n^*(x) = \sigma(x) = \frac{1}{\pi^2}\sqrt{1-x^2}\int_{-1}^{1}\frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{1-s^2}}, x \in [-1,1]. \tag{1.29}$$

We can now state:

Theorem 1.4
Let $Q : \mathbb{R} \to \mathbb{R}$ be stricly convex. Assume that Q'' is continuous in $\mathbb{R}$, and satisfies a Lipschitz condition of order 1 in each compact set. Assume moreover, that the equilibrium density σ for the field Q has support $[-1,1]$. Let μ_n' be given by (1.27), and let $\{p_{n,j}\}$ denote the corresponding orthonormal polynomials, as in (1.2). For $n \geq 1$, let $\mathbb{P}^{(n)}$ denote the probability distribution defined by (1.1) for all $n \geq 1$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues in increasing order. Also, let $\{x_{jn}\}$ denote the zeros of the nth orthogonal polynomial $p_{n,n}$ for μ_n, ordered as in (1.4). Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Then for j, n with $\frac{j}{n} \in [-1+\varepsilon, 1-\varepsilon]$, we have in distribution, as $n \to \infty$,

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{n\sigma(x_{jn})}}} \to N(0,1). \tag{1.30}$$

The paper is organized as follows. In Section 2, we state three general results, Theorems 2.1-2.3 from which Theorems 1.1, 1.3, and 1.4 will follow. Theorems 2.1 and 2.3 will be proved in Section 2, while Theorem 2.2 will be proved in Section 3. We believe these results have independent interest, and will have application beyond Theorem 1.1, 1.3 and 1.4. We prove Theorem 1.1 in Section 4, Theorem 1.3 in Section 5, and Theorem 1.4 in Section 6.

In the sequel $C, C_1, C_2, ...$ denote positive constants independent of n, x, polynomials of degree $\leq n$, and possibly other parameters. We use $\sim$ in the following sense: given sequences of non-zero real numbers $\{x_n\}$ and $\{y_n\}$, we write $x_n \sim y_n$ if there exists a constant $C > 1$ such that

$$C^{-1} \leq x_n/y_n \leq C \text{ for } n \geq 1.$$

Similar notation is used for functions and sequences of functions.

Acknowledgement
The author is very grateful to Maurice Duits, who as a very thorough referee, found an error in the proof of Theorem 2.2, in an earlier version of this paper. This led to an extensive rewrite of the paper, and required stricter hypotheses, including asymptotics for orthonormal polynomials with a rate, such as in (1.10) or in (2.1). The author also thanks Peter Miller for clarifying the form of the asymptotic (6.4), and his comments on asymptotics from [16], and Leonid Golinskii for helpful references to the work of Badkov and B.L. Golinskii.

2. The Auxiliary General Case

In this section, we prove a general result assuming appropriate asymptotics relating to orthonormal polynomials. We assume that we are given a sequence of measures $\{\mu_n\}$ with support on the real line, and corresponding orthonormal polynomials $p_{n,j}(x) = \gamma_{n,j}x^j + ...$, satisfying

$$\int_{-\infty}^{\infty} p_{n,j}p_{n,k}d\mu_n = \delta_{jk},\ \ j,k = 0,1,2,... \ .$$

Assume, as in Section 1, that the zeros of $p_{n,n}$ are $x_{1n} < x_{2n} < ... < x_{nn}$, and

$$K_n(x,y) = \sum_{j=0}^{n-1} p_{n,j}(x)\, p_{n,j}(y)\,.$$

We also denote the zeros of $p_{n,n-1}$ by $y_{1n} < y_{2n} < ... < y_{n,n-1}$, so that for $j \leq n-1$, $x_{jn} < y_{jn} < x_{j+1,n}$. We now list our four technical assumptions:

(I) Pointwise asymptotics of orthonormal polynomials with a rate

$\mathcal{I}$ is a closed subinterval of $(-1,1)$ in which each μ_n is absolutely continuous, and that as $n \to \infty$, uniformly for $x = \cos\theta \in \mathcal{I}$, and $m = n-1, n$,

$$\begin{aligned} &p_{n,m}(x)\,\mu_n'(x)^{1/2}\left(1-x^2\right)^{1/4} \\ = \quad & \sqrt{\frac{2}{\pi}}\cos\left(\left(m-n+\frac{1}{2}\right)\theta + n\pi\int_x^1 \sigma_n^* + g(x) + \zeta_n\right) + o\left((\log n)^{-1/2}\right), \end{aligned} \tag{2.1}$$

where $g : [-1,1] \to \mathbb{R}$ is continuous in $\mathcal{I}$, with modulus of continuity in $\mathcal{I}$ satisfying for $t \to 0+$,

$$\omega(g;\mathcal{I};t) = \sup\{|g(x) - g(y)| : x,y \in \mathcal{I},\ |x-y| \leq t\} = o\left(|\log t|^{-1/2}\right);$$

ζ_n is a number independent of x, and for $n \geq 1$, $\sigma_n^* : (-1,1) \to (0,\infty)$ is a function with $\int_{-1}^1 \sigma_n^* = 1$, satisfying for some $C > 1$, $n \geq 1$, $x \in \mathcal{I}$,

$$C^{-1} \leq \sigma_n^*(x) \leq C. \tag{2.2}$$

We also assume that the $\{\sigma_n^*\}$ are equicontinuous in $\mathcal{I}$, with

$$\omega(t) = \sup\{|\sigma_n^*(x) - \sigma_n^*(y)| : n \geq 1, x,y \in \mathcal{I}, |x-y| \leq t\} = o\left(|\log t|^{-1/2}\right) \tag{2.3}$$

for $t \to 0+$. We shall often use the notation

$$f_n(x) = n\pi\int_x^1 \sigma_n^* + g(x) + \zeta_n. \tag{2.4}$$

(II) Asymptotics of Leading Coefficients

We assume that

$$\frac{\gamma_{n,n-1}}{\gamma_{n,n}} = \frac{1}{2} + o(1)\,, n \to \infty. \tag{2.5}$$

(III) Asymptotic Spacing of Zeros

Uniformly for j,n with $x_{jn} \in \mathcal{I}$,

$$\lim_{n\to\infty} n\sigma_n^*(x_{j,n})(x_{jn} - x_{j+1,n}) = 1. \tag{2.6}$$

(IV) Asymptotics and Bounds for Reproducing Kernels
Uniformly for $x \in \mathcal{I}$, we have

$$\lim_{n\to\infty} K_n(x,x)\,\mu_n'(x)/\left(n\sigma_n^*(x)\right) = 1. \tag{2.7}$$

Consequently for some $C > 1$, and $x \in \mathcal{I}$,

$$C^{-1} \le \frac{1}{n} K_n(x,x)\,\mu_n'(x) \le C. \tag{2.8}$$

Remarks on the assumptions
(a) These assumptions hold for a wide variety of fixed and varying exponential weights. The function σ_n^* is typically a contracted form of the equilibrium density of an external field, formed when the Mhaskar-Rakhmanov-Saff interval is contracted to $[-1,1]$. The function g allows us also to handle the case of a fixed measure on a compact interval.
(b) The assumptions are not independent of one another. For example, the pointwise asymptotic (2.1) for the orthonormal polynomials implies the asymptotic spacing of the zeros via the intermediate value theorem, though one also needs to assume a crude lower bound on spacing of successive zeros to ensure distinct limits for distinct scaled zeros. The bound on the reproducing kernel (2.8) is also essentially implied by the pointwise asymptotic, while the asymptotic for the leading coefficients would follow if we assume (2.1) in every compact subset of (-1,1).

We shall prove:

Theorem 2.1
Assume (I) - (IV). Let $\mathcal{J}$ be a closed interval contained in the interior of $\mathcal{I}$. Uniformly for $\xi \in \mathcal{J}$, as $n \to \infty$,

$$\int_{-\infty}^{\xi} \int_{\xi}^{\infty} K_n^2(x,y)\,d\mu_n(x)\,d\mu_n(y) = \frac{\log n}{2\pi^2}\left(1+o(1)\right). \tag{2.9}$$

Remark
For this theorem we could replace the error term in (2.1) with $o(1)$. Moroever, we need only g to be continuous, and $\{\sigma_n^*\}$ to be equicontinuous.

Theorem 2.2
Assume (I) - (IV). Let $\mathcal{J}$ be a closed interval contained in the interior of $\mathcal{I}$. Uniformly for j, n with $y_{j,n} \in \mathcal{J}$, as $n \to \infty$,

$$\Upsilon_j = \int_{y_{jn}}^{\infty} K_n(t,t)\,d\mu_n(t) = n - j + o\left(\sqrt{\log n}\right). \tag{2.10}$$

Recall here that y_{jn} is the jth zero of $p_{n,n-1}$. We deduce that the eigenvalues of random Hermitian matrices have Gaussian fluctuations:

Theorem 2.3
Assume (I) - (IV) and that $\{\mu_n\}$ are absolutely continuous. Let $\mathcal{J}$ be a closed interval contained in the interior of $\mathcal{I}$. For $n \ge 1$, let $\mathbb{P}^{(n)}$ denote the probability distribution with density defined by (1.1) for all $n \ge 1$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ denote the eigenvalues in increasing order. Also, let $\{x_{jn}\}$ denote the zeros of the nth

orthogonal polynomial for μ_n, ordered as in (1.4). Then for j, n with $x_{jn} \in \mathcal{J}$, we have in distribution as $n \to \infty$,

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2} \frac{1}{n\sigma_n^*(x_{jn})}}} \to N(0,1). \tag{2.11}$$

We emphasize that our methods of proof, especially for Theorem 2.2, are different from those in Zhang and Gustavsson, as we are assuming much weaker asymptotics for the orthogonal polynomials and related quantities. This section is organized as follows: we first prove three preliminary lemmas, and then prove Theorem 2.1. Then, we deduce Theorem 2.3, assuming Theorem 2.2. We prove Theorem 2.2 in Section 3.

We begin this section with estimates for some tail integrals:

Lemma 2.4
Uniformly for $\xi \in \mathcal{J}$,
(a)

$$\left(\int_{-\infty}^{\xi - \frac{1}{(\log n)^{1/4}}} \int_{\xi}^{\infty} + \int_{-\infty}^{\xi} \int_{\xi + \frac{1}{(\log n)^{1/4}}}^{\infty} \right) K_n^2(x,y)\, d\mu_n(x)\, d\mu_n(y) \le C (\log n)^{1/2}.$$

(2.12)

(b)

$$\int_{\xi - \frac{\log n}{n}}^{\xi} \int_{\xi}^{\xi + \frac{1}{(\log n)^{1/4}}} K_n^2(x,y)\, d\mu_n(x)\, d\mu_n(y) \le C \log(\log n + 1). \tag{2.13}$$

Proof
(a) By the Christoffel-Darboux formula, and Cauchy-Schwarz,

$$K_n^2(x,y) \le \left(\frac{\gamma_{n-1,n}}{\gamma_{n,n}} \right)^2 \frac{\left(p_{n,n}^2(x) + p_{n,n-1}^2(x)\right)\left(p_{n,n}^2(y) + p_{n,n-1}^2(y)\right)}{(x-y)^2}$$

so that using (2.5),

$$\begin{aligned} &\left(\int_{-\infty}^{\xi - \frac{1}{(\log n)^{1/4}}} \int_{\xi}^{\infty} + \int_{-\infty}^{\xi} \int_{\xi + \frac{1}{(\log n)^{1/4}}}^{\infty} \right) K_n^2(x,y)\, d\mu_n(x)\, d\mu_n(y) \\ \le\ & (\log n)^{1/2} \left(\frac{1}{4} + o(1) \right) \int \int \left(p_{n,n}^2(x) + p_{n,n-1}^2(x)\right)\left(p_{n,n}^2(y) + p_{n,n-1}^2(y)\right) d\mu_n(x)\, d\mu_n(y) \\ =\ & (\log n)^{1/2} (1 + o(1)). \end{aligned}$$

(b) Our asymptotics for the orthogonal polynomials shows that for $n \ge 1$.

$$\sup_{x \in \mathcal{I}} \left| p_{n,n}(x)\, \mu_n'(x)^{1/2} \right| \le C. \tag{2.14}$$

(Recall that $\mathcal{I}$ is a positive distance from ± 1). The Christoffel-Darboux formula and (2.5) then give for $x, y \in \mathcal{I}$,

$$K_n^2(x,y)\, \mu_n'(x)\, \mu_n'(y) \le \frac{C}{(x-y)^2}. \tag{2.15}$$

In addition, (2.8) and Cauchy-Schwarz give for $x, y \in \mathcal{I}$,

$$K_n^2(x,y)\,\mu_n'(x)\,\mu_n'(y) \le Cn^2. \tag{2.16}$$

Now assume $\xi \in \mathcal{J} \subset \mathcal{I}^o$. Then for n large enough, making the substitutions $x = \xi + \frac{u}{n}$ and $y = \xi - \frac{v}{n}$,

$$\begin{aligned}
&\int_{\xi-\frac{\log n}{n}}^{\xi} \int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} K_n^2(x,y)\,d\mu_n(x)\,d\mu_n(y) \\
\le\ & C\int_{\xi-\frac{\log n}{n}}^{\xi} \int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} \min\left\{n^2, \frac{1}{(x-y)^2}\right\} dxdy \\
=\ & C\int_0^{\log n} \int_0^{n(\log n)^{-1/4}} \min\left\{1, \frac{1}{(u+v)^2}\right\} du\ dv \\
\le\ & C\int_0^{\log n} \int_0^{\infty} \frac{1}{(1+u+v)^2} du\ dv \\
=\ & C\int_0^{\log n} \frac{1}{1+v}\ dv = C\log(\log n + 1).
\end{aligned}$$

■

Taking into account the three ranges of integration in Lemma 2.4(a), (b), we need to analyze

$$\Gamma = \int_{\xi-\frac{1}{(\log n)^{1/4}}}^{\xi-\frac{\log n}{n}} \int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} K_n^2(x,y)\,d\mu_n(x)\ \ d\mu_n(y). \tag{2.17}$$

We first prove a scaling limit over a non-compact range. Recall that f_n is defined by (2.4).

Lemma 2.5
For $n \ge 1$, let $\mathcal{I}_n \subset \mathcal{I}$ with

$$\text{diameter}(\mathcal{I}_n) \to 0 \text{ as } n \to \infty.$$

Then for $n \ge 1$ and $x, y \in \mathcal{I}_n$,

$$K_n^2(x,y)\,\mu_n'(x)\,\mu_n'(y) = \frac{1-\cos[2f_n(x) - 2f_n(y)] + o(1)}{2\pi^2(x-y)^2}. \tag{2.18}$$

The $o(1)$ term is uniform in x, y.

Proof

Let $x = \cos\theta; y = \cos\phi$, where $\theta, \phi \in (0, \pi)$, and consider the numerator in the Christoffel-Darboux formula,

$$\psi_n(x,y) = p_{n,n}(x)\,p_{n,n-1}(y) - p_{n,n-1}(x)\,p_{n,n}(y).$$

Note that $x, y \in \mathcal{I}_n \Rightarrow x - y = o(1) \Rightarrow \theta - \phi = o(1)$ (recall that $\mathcal{I}$ is a compact subset of $(-1,1)$). Then we see from (2.1) that

$$\begin{aligned}
&\frac{\pi}{2}\psi_n(x,y)\,\mu_n'(x)^{1/2}\,\mu_n'(y)^{1/2}\left(1-x^2\right)^{1/4}\left(1-y^2\right)^{1/4} \\
= \;& \cos\left(\frac{\theta}{2}+f_n(x)\right)\cos\left(-\frac{\phi}{2}+f_n(y)\right) - \cos\left(-\frac{\theta}{2}+f_n(x)\right)\cos\left(\frac{\phi}{2}+f_n(y)\right) + o(1) \\
= \;& \left[\cos\frac{\theta}{2}\cos f_n(x) - \sin\frac{\theta}{2}\sin f_n(x)\right]\left[\cos\frac{\phi}{2}\cos f_n(y) + \sin\frac{\phi}{2}\sin f_n(y)\right] \\
& - \left[\cos\frac{\theta}{2}\cos f_n(x) + \sin\frac{\theta}{2}\sin f_n(x)\right]\left[\cos\frac{\phi}{2}\cos f_n(y) - \sin\frac{\phi}{2}\sin f_n(y)\right] + o(1) \\
= \;& 2\left\{\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos f_n(x)\sin f_n(y) - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\sin f_n(x)\cos f_n(y)\right\} + o(1) \\
= \;& 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left\{\cos f_n(x)\sin f_n(y) - \sin f_n(x)\cos f_n(y)\right\} + o(1) \\
= \;& \sqrt{1-x^2}\left\{\cos f_n(x)\sin f_n(y) - \sin f_n(x)\cos f_n(y)\right\} + o(1).
\end{aligned}$$

Then, again as $\sqrt{1-y^2} = \sqrt{1-x^2} + o(1)$ and both are bounded below by a constant depending only on $\mathcal{I}$,

$$\begin{aligned}
&\pi^2\psi_n^2(x,y)\,\mu_n'(x)\,\mu_n'(y) \\
= \;& 4\cos^2 f_n(x)\sin^2 f_n(y) + 4\sin^2 f_n(x)\cos^2 f_n(y) \\
& -8\cos f_n(x)\sin f_n(y)\sin f_n(x)\cos f_n(y) + o(1) \\
= \;& (1+\cos 2f_n(x))(1-\cos 2f_n(y)) + (1-\cos 2f_n(x))(1+\cos 2f_n(y)) \\
& -2\sin 2f_n(x)\sin 2f_n(y) + o(1) \\
= \;& 2 - 2\cos 2f_n(x)\cos 2f_n(y) - 2\sin 2f_n(x)\sin 2f_n(y) + o(1).
\end{aligned}$$

(2.19)

Finally,

$$K_n^2(x,y)\,\mu_n'(x)\,\mu_n'(y) = \left(\frac{\frac{\gamma_{n,n-1}}{\gamma_{n,n}}}{x-y}\right)^2 \psi_n^2(x,y)\,\mu_n'(x)\,\mu_n'(y),$$

so on applying (2.5), we obtain the result. ■

We next establish a Riemann-Lebesgue type estimate, that will also be used in Section 3:

Lemma 2.6
For $n \geq 1$, let $[\alpha_n, \beta_n] \subset \mathcal{I}$, and let

$$c_n \leq \alpha_n - \frac{A}{n}, \tag{2.20}$$

for some fixed $A>0$. Let $\phi:\mathcal{I}\rightarrow\mathbb{R}$ be continuous, with modulus of continuity $\omega(\phi;\mathcal{I};\cdot)$. Let f_n be defined by (2.4). Then

$$\left|\int_{\alpha_n}^{\beta_n}\phi(x)\frac{\cos 2f_n(x)}{(x-c_n)^2}dx\right|$$

$$\leq \frac{C}{\alpha_n-c_n}\left\{\omega(\phi;\mathcal{I};\frac{1}{n})+\omega(g;\mathcal{I};\frac{1}{n})+\omega(\sigma_n^*;\mathcal{I};\frac{1}{n})\right\}+\frac{C}{n(\alpha_n-c_n)^2}. \tag{2.21}$$

The same estimate holds if we replace cos *by* sin*. The constant C is independent of $n,\alpha_n,\beta_n,c_n,g,\sigma_n^*$ but depends on A and ϕ.*

Proof

We do the proof for cos. For $n\geq 1$, let

$$G_n(x)=\int_{\alpha_n}^{x}\sigma_n^*; \tag{2.22}$$

$$h(x)=\phi(x)\sin(2g(x)) \text{ or } h(x)=\phi(x)\cos(2g(x)); \tag{2.23}$$

and

$$\rho_n=-2\zeta_n-2n\pi\int_{\alpha_n}^{1}\sigma_n^*. \tag{2.24}$$

Then

$$\begin{aligned}\phi(x)\cos(2f_n(x)) &= \phi(x)\cos(-2f_n(x))\\ &= \phi(x)\cos(2n\pi G_n(x)-2g(x)+\rho_n)\end{aligned}$$

is a sum of 8 terms of the form

$$\pm\begin{pmatrix}\sin\rho_n\\ \cos\rho_n\end{pmatrix}h(x)\begin{pmatrix}\sin 2n\pi G_n(x)\\ \cos 2n\pi G_n(x)\end{pmatrix},$$

where h is one of the functions in (2.23). Since $|\sin\rho_n|\leq 1$ and $|\cos\rho_n|\leq 1$, it suffices to estimate

$$J:=\int_{\alpha_n}^{\beta_n}\frac{h(x)}{(x-c_n)^2}\begin{pmatrix}\sin 2n\pi G_n(x)\\ \cos 2n\pi G_n(x)\end{pmatrix}dx.$$

We shall do this with cos, the one for sin is similar. Straightforward estimation shows that

$$\omega(h;\mathcal{I};t)\leq C\left(\omega(\phi;\mathcal{I};t)+\omega(g;\mathcal{I};t)\right), t\geq 0. \tag{2.25}$$

Here C depends on $\|\phi\|_{L_\infty(\mathcal{I})}$. Now G_n increases in $[\alpha_n,\beta_n]$ from $G_n(\alpha_n)=0$ to $G_n(\beta_n)$. Choose a nonnegative integer L such that

$$\alpha_n=t_0<t_1<t_2<...<t_L\leq\beta_n<t_{L+1},$$

with

$$nG_n(t_j)=j,\ 0\leq j\leq L. \tag{2.26}$$

Then in view of (2.2),

$$t_{j+1}-t_j\sim\frac{1}{n} \tag{2.27}$$

uniformly in j and n. The constants in $\sim$ depend only on the bounds on σ_n^* in (2.2). Then

$$\begin{aligned} J &= \sum_{j=0}^{L-1} \int_{t_j}^{t_{j+1}} \frac{h(x)}{(x-c_n)^2} \begin{pmatrix} \sin 2n\pi G_n(x) \\ \cos 2n\pi G_n(x) \end{pmatrix} dx + \int_{t_L}^{\beta_n} \frac{h(x)}{(x-c_n)^2} \begin{pmatrix} \sin 2n\pi G_n(x) \\ \cos 2n\pi G_n(x) \end{pmatrix} dx \\ &= \sum_{j=0}^{L-1} J_j + J_L, \end{aligned}$$

say. For $0 \le j \le L-1$,

$$\begin{aligned} & J_j \\ = & \frac{1}{2n\pi} \int_{t_j}^{t_{j+1}} \frac{h(x)}{\sigma_n^*(x)(x-c_n)^2} \frac{d}{dx} \begin{pmatrix} -\cos 2n\pi G_n(x) \\ \sin 2n\pi G_n(x) \end{pmatrix} dx \\ = & \frac{1}{2n\pi} \int_{t_j}^{t_{j+1}} \left[\frac{d}{dx} \begin{pmatrix} -\cos 2n\pi G_n(x) \\ \sin 2n\pi G_n(x) \end{pmatrix} \right] \left\{ \frac{h(x)}{\sigma_n^*(x)(x-c_n)^2} - \frac{h(t_j)}{\sigma_n^*(t_j)(t_j-c_n)^2} \right\} dx, \end{aligned}$$

since, from (2.26),

$$\int_{t_j}^{t_{j+1}} \left[\frac{d}{dx} \begin{pmatrix} -\cos 2n\pi G_n(x) \\ \sin 2n\pi G_n(x) \end{pmatrix} \right] dx = 0.$$

Since $G_n'(x) = \sigma_n(x) \sim 1$, (recall (2.2)) we see that

$$\begin{aligned} |J_j| &\le \frac{C}{n} \sup_{x\in[t_j,t_{j+1}]} \left| \frac{h(x)}{\sigma_n^*(x)(x-c_n)^2} - \frac{h(t_j)}{\sigma_n^*(t_j)(t_j-c_n)^2} \right| \\ &\le \frac{C}{n} \left\{ \frac{w\left(h;\mathcal{I};\frac{1}{n}\right) + w\left(\sigma_n^*;\mathcal{I};\frac{1}{n}\right)}{(t_j-c_n)^2} + \frac{1}{n(t_j-c_n)^3} \right\} \\ &\le \frac{C}{n} \left\{ \frac{w\left(\phi;\mathcal{I};\frac{1}{n}\right) + w\left(g;\mathcal{I};\frac{1}{n}\right) + w\left(\sigma_n^*;\mathcal{I};\frac{1}{n}\right)}{(t_j-c_n)^2} + \frac{1}{n(t_j-c_n)^3} \right\}. \end{aligned}$$

Here we have also used the doubling property of moduli, namely $w(\phi;\mathcal{I};2t) \le 2w(\phi;\mathcal{I};t), t \ge 0$. Next, as $\beta_n \ge \alpha_n \ge c_n + \frac{A}{n}$,

$$|J_L| \le \frac{C}{n(\beta_n - c_n)^2}.$$

Again, C depends on $\|\phi\|_{L_\infty(\mathcal{I})}$. Then

$$\begin{aligned} |J| &\le \sum_{j=0}^{L-1} |J_j| + |J_L| \\ &\le C\left\{ w\left(\phi;\mathcal{I};\frac{1}{n}\right) + w\left(g;\mathcal{I};\frac{1}{n}\right) + w\left(\sigma_n^*;\mathcal{I};\frac{1}{n}\right) \right\} \frac{1}{n} \sum_{j=0}^{L-1} \frac{1}{(t_j-c_n)^2} \\ &\quad + C\frac{1}{n^2} \sum_{j=0}^{L-1} \frac{1}{(t_j-c_n)^3} + \frac{C}{n(\beta_n-c_n)^2}. \end{aligned}$$

Here, using our spacing (2.27), and our hypothesis $t_0 - c_n = \alpha_n - c_n \geq \frac{A}{n}$,

$$\frac{1}{n}\sum_{j=0}^{L-1}\frac{1}{(t_j - c_n)^2} \leq \frac{C}{\alpha_n - c_n}$$

and

$$\frac{1}{n}\sum_{j=0}^{L-1}\frac{1}{(t_j - c_n)^3} \leq \frac{C}{(\alpha_n - c_n)^2}.$$

Then the result follows. ■

Proof of Theorem 2.1
We have to show that Γ defined by (2.17) satisfies

$$\Gamma = \int_{\xi - \frac{1}{(\log n)^{1/4}}}^{\xi - \frac{\log n}{n}} \int_{\xi}^{\xi + \frac{1}{(\log n)^{1/4}}} K_n^2(x,y)\, d\mu_n(x)\, d\mu_n(y) = \frac{1+o(1)}{2\pi^2}\log n.$$

Note that in this integral,

$$|y - x| \leq \frac{2}{(\log n)^{1/4}} = o(1),$$

so Lemma 2.5 is applicable. By that lemma,
(2.28)

$$\Gamma = \frac{1}{2\pi^2}\int_{\xi - \frac{1}{(\log n)^{1/4}}}^{\xi - \frac{\log n}{n}} \int_{\xi}^{\xi + \frac{1}{(\log n)^{1/4}}} \frac{\{1 - \cos[2f_n(x) - 2f_n(y)] + o(1)\}}{(x-y)^2} dx\ dy.$$

Here

$$\begin{aligned}
&\frac{1}{2\pi^2}\int_{\xi - \frac{1}{(\log n)^{1/4}}}^{\xi - \frac{\log n}{n}} \int_{\xi}^{\xi + \frac{1}{(\log n)^{1/4}}} \frac{1}{(x-y)^2} dx\ dy \\
=\ & \frac{1}{2\pi^2}\int_{\xi - \frac{1}{(\log n)^{1/4}}}^{\xi - \frac{\log n}{n}} \left[\frac{1}{\xi - y} - \frac{1}{\xi + (\log n)^{-1/4} - y}\right] dy \\
=\ & \frac{1}{2\pi^2}\left[\log\left[\frac{n}{(\log n)^{5/4}}\right] - \log\left[\frac{2(\log n)^{-1/4}}{\frac{\log n}{n} + (\log n)^{-1/4}}\right]\right] \\
=\ & \frac{1}{2\pi^2}\log n + O(\log\log n). \qquad (2.29)
\end{aligned}$$

Next, we use Lemma 2.6 with $\phi(x) = 1$, $\alpha_n = \xi$, $\beta_n = \xi + \frac{1}{(\log n)^{1/4}}$, and $c_n = y\left(\leq \alpha_n - \frac{\log n}{n} \leq \alpha_n - \frac{1}{n}\right)$. That lemma gives

$$\begin{aligned}
&\left|\int_{\xi-\frac{1}{(\log n)^{1/4}}}^{\xi-\frac{\log n}{n}} \int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} \frac{\cos\left[2f_n(x) - 2f_n(y)\right]}{(x-y)^2} dx\, dy\right| \\
\leq\ & \int_{\xi-\frac{1}{(\log n)^{1/4}}}^{\xi-\frac{\log n}{n}} \left(\left|\int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} \frac{\cos 2f_n(x)}{(x-y)^2} dx\right| + \left|\int_{\xi}^{\xi+\frac{1}{(\log n)^{1/4}}} \frac{\sin 2f_n(x)}{(x-y)^2} dx\right|\right) dy \\
\leq\ & C\left[\omega(g;\mathcal{I};\frac{1}{n}) + \omega(\frac{1}{n})\right] \int_{\xi-\frac{1}{(\log n)^{1/4}}}^{\xi-\frac{\log n}{n}} \frac{dy}{\xi - y} + C\int_{\xi-\frac{1}{(\log n)^{1/4}}}^{\xi-\frac{\log n}{n}} \frac{dy}{n(\xi-y)^2} \\
\leq\ & C\left[\omega(g;\mathcal{I};\frac{1}{n}) + \omega(\frac{1}{n})\right] \log\left[\frac{n}{(\log n)^{5/4}}\right] + C\frac{1}{\log n} = o(\log n)^{-1/2},
\end{aligned}$$

recall (2.3), and our hypothesis on $\omega(g;\mathcal{I};\cdot)$. Substituting this and (2.29) into (2.28), gives the result. ■

Next, we prove Theorem 2.3 assuming Theorem 2.2:

Proof of Theorem 2.3 assuming Theorem 2.2

We shall apply a central limit theorem that combines results of Costin, Leibowitz [5] and Soshnikov [22]. Let $\xi \in \mathbb{R}$, and choose j, n such that $x_{jn} \in \mathcal{J}$, and let

$$a_n = \sqrt{\frac{\log n}{2\pi^2}} \frac{1}{n\sigma_n^*(x_{jn})},$$

and

$$\mathcal{I}_n = [x_{jn} + a_n\xi, \infty).$$

From the definition of $\mathcal{R}_1^{(n)}$ and (1.5), if $\#\mathcal{I}_n$ denotes the number of elements of $\mathcal{I}_n$, its expected value is

$$\mathbb{E}(\#\mathcal{I}_n) = \int_{\mathcal{I}_n} \mathcal{R}_1^{(n)}(x)\, dx = \int_{x_{jn}+a_n\xi}^{\infty} K_n(x,x)\, d\mu_n(x).$$

Choose $k = k(j,n)$ such that

$$x_{kn} \leq x_{jn} + a_n\xi \leq x_{k+1,n}.$$

Note that for large enough n, $k < n$ and $x_{kn} \in \mathcal{J}$. By Theorem 2.2, (2.6), and (2.8), and the interlacing of $\{x_{in}\}, \{y_{in}\}$,

$$\begin{aligned}
\mathbb{E}(\#\mathcal{I}_n) &= \int_{y_{kn}}^{\infty} K_n(x,x)\, d\mu_n(x) + O\left(\int_{x_{kn}}^{x_{k+1,n}} K_n(x,x)\, d\mu_n(x)\right) \\
&= n - k + o\left(\sqrt{\log n}\right) + O(1). \qquad (2.30)
\end{aligned}$$

Assume now $\xi \neq 0$. In view of our uniform spacing (2.6), and the equicontinuity and boundedness above and below of $\{\sigma_n^*\}$,

$$\begin{aligned}
k - j &= n\sigma_n^*(x_{jn})(x_{kn} - x_{jn})(1 + o(1)) = n\sigma_n^*(x_{jn})\, a_n\xi(1 + o(1)) \\
&= \sqrt{\frac{\log n}{2\pi^2}}\xi(1 + o(1)) = \sqrt{\frac{\log n}{2\pi^2}}\xi + o\left(\sqrt{\log n}\right).
\end{aligned}$$

If $\xi = 0$, we still trivially have this last relation since then $j = k$. Thus, using (2.30),

$$\mathbb{E}\left(\#\mathcal{I}_n\right) = n - j - \sqrt{\frac{\log n}{2\pi^2}}\xi + o\left(\sqrt{\log n}\right)$$

$$\Rightarrow n - j - \mathbb{E}\left(\#\mathcal{I}_n\right) = \sqrt{\frac{\log n}{2\pi^2}}\xi + o\left(\sqrt{\log n}\right). \tag{2.31}$$

Also, if χ_n denotes the characteristic function of $\mathcal{I}_n$, the variance of $\#\mathcal{I}_n$ is

$$\begin{aligned}
Var\left(\#\mathcal{I}_n\right) &= \mathbb{E}\left(\left(\#\mathcal{I}_n\right)^2\right) - \left(\mathbb{E}\left(\#\mathcal{I}_n\right)\right)^2 \\
&= \mathbb{E}\left(\left[\sum_{k=1}^{n}\chi_n\left(x_k\right)\right]^2\right) - \left(\sum_{k=1}^{n}\mathbb{E}\left(\chi_n\left(x_k\right)\right)\right)^2 \\
&= \sum_{k=1}^{n}\mathbb{E}\left(\chi_n\left(x_k\right)\right) + \sum_{j\neq k}\mathbb{E}\left(\chi_n\left(x_j\right)\chi_n\left(x_k\right)\right) - \left(\sum_{k=1}^{n}\mathbb{E}\left(\chi_n\left(x_k\right)\right)\right)^2 \\
&= \int_{-\infty}^{\infty}\chi_n\left(x\right)\mathcal{R}_1^{(n)}\left(x\right)dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_n\left(x\right)\chi_n\left(y\right)\mathcal{R}_2^{(n)}\left(x,y\right)dxdy \\
&\quad -\left(\int_{-\infty}^{\infty}\chi_n\left(x\right)\mathcal{R}_1^{(n)}\left(x\right)dx\right)^2 \\
&\quad \text{(by definition of } \mathcal{R}_2^{(n)}\text{)} \\
&= \int_{-\infty}^{\infty}\chi_n\left(x\right)K_n\left(x,x\right)d\mu_n\left(x\right) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_n\left(x\right)\chi_n\left(y\right)K_n\left(x,y\right)^2 d\mu_n\left(x\right)d\mu_n\left(y\right) \\
&\quad \text{(by (1.51))} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_n\left(x\right)\left(1-\chi_n\left(y\right)\right)K_n\left(x,y\right)^2 d\mu_n\left(x\right)d\mu_n\left(y\right) \\
&= \int_{x_{jn}+a_n\xi}^{\infty}\int_{-\infty}^{x_{jn}+a_n\xi}K_n^2\left(x,y\right)d\mu_n\left(x\right)d\mu_n\left(y\right) \\
&= \frac{\log n}{2\pi^2}\left(1+o\left(1\right)\right),
\end{aligned}$$

(2.32)

by Theorem 2.1. Since

$$\begin{aligned}
\frac{\lambda_j - x_{jn}}{a_n} &< \xi \\
&\Leftrightarrow \lambda_j \notin \mathcal{I}_n \Leftrightarrow \#\mathcal{I}_n \leq n - j \\
&\Leftrightarrow \#\mathcal{I}_n - \mathbb{E}\left(\#\mathcal{I}_n\right) \leq n - j - \mathbb{E}\left(\#\mathcal{I}_n\right) = \sqrt{\frac{\log n}{2\pi^2}}\xi + o\left(\sqrt{\log n}\right)
\end{aligned}$$

(by (2.31)) we see using (2.32) that

$$\mathbb{P}_n\left(\frac{\lambda_j - x_{jn}}{a_n} < \xi\right) = \mathbb{P}_n\left(\frac{\#\mathcal{I}_n - \mathbb{E}\left(\#\mathcal{I}_n\right)}{\sqrt{Var\left(\#\mathcal{I}_n\right)}} \leq \xi + o\left(1\right)\right).$$

Since $Var(\#\mathcal{I}_n) \to \infty$ as $n \to \infty$, the result now follows from the aforementioned result of Costin, Leibowitz in a form stated by Soshnikov [23, p. 4]. Alternatively, see [11, p. 155, Theorem 2.1]. ■

3. Proof of Theorem 2.2

The basic idea in the proof of Theorem 2.2 is contained in the following identity. We use the notation $\lambda_{kn} = \frac{1}{K_n(x_{kn}, x_{kn})}$ for the weights in the Gauss quadrature formula. Recall too that x_{jn} is a zero of $p_{n,n}$ while y_{jn} is a zero of $p_{n,n-1}$.

Lemma 3.1

$$\begin{aligned} \Upsilon_j &= \int_{y_{jn}}^{\infty} K_n(t,t)\, d\mu_n(t) \\ &= n - j + \left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^2 \{\sum_{k=1}^{j} \lambda_{kn} p_{n,n-1}^2(x_{kn}) \int_{y_{jn}}^{\infty} \left(\frac{p_{n,n}(t)}{t - x_{kn}}\right)^2 d\mu_n(t) \\ &\quad - \sum_{k=j+1}^{n} \lambda_{kn} p_{n,n-1}^2(x_{kn}) \int_{-\infty}^{y_{jn}} \left(\frac{p_{n,n}(t)}{t - x_{kn}}\right)^2 d\mu_n(t)\}. \end{aligned}$$

(3.1)

Proof

We use the reproducing kernel property, and then Gauss quadrature to write

$$\begin{aligned} \Upsilon_j &= \int_{y_{jn}}^{\infty} \left(\int_{-\infty}^{\infty} K_n(s,t)^2\, d\mu_n(s)\right) d\mu_n(t) \\ &= \int_{y_{jn}}^{\infty} \sum_{k=1}^{n} \lambda_{kn} K_n(x_{kn}, t)^2\, d\mu_n(t) \\ &= \sum_{k=1}^{n} \lambda_{kn} \int_{y_{jn}}^{\infty} K_n(x_{kn}, t)^2\, d\mu_n(t) \\ &= \sum_{k=1}^{j} \lambda_{kn} \int_{y_{jn}}^{\infty} K_n(x_{kn}, t)^2\, d\mu_n(t) \\ &\quad + \sum_{k=j+1}^{n} \lambda_{kn} \left[K_n(x_{kn}, x_{kn}) - \int_{-\infty}^{y_{jn}} K_n(x_{kn}, t)^2\, d\mu_n(t)\right] \\ &= n - j + \sum_{k=1}^{j} \lambda_{kn} \int_{y_{jn}}^{\infty} K_n(x_{kn}, t)^2\, d\mu_n(t) - \sum_{k=j+1}^{n} \lambda_{kn} \int_{-\infty}^{y_{jn}} K_n(x_{kn}, t)^2\, d\mu_n(t). \end{aligned}$$

Finally, by the Christoffel-Darboux formula,

$$K_n(x_{kn}, t) = \frac{\gamma_{n,n-1}}{\gamma_{n,n}} p_{n,n-1}(x_{kn}) \frac{p_{n,n}(t)}{t - x_{kn}}.$$

■

Remark

The two sums on the right-hand side in (3.1) cancel each other out, and are relatively small, since x_{kn} lies outside the range of integration. Unfortunately each sum is really $O(\log n)$, so we really do need to use cancellation to obtain $o\left(\sqrt{\log n}\right)$, and this requires precise estimation. For $k \leq j$, let

$$I_{k,n} = \int_{y_{jn}}^{\infty} \left(\frac{p_{n,n}(t)}{t - x_{kn}}\right)^2 d\mu_n(t) \tag{3.2}$$

and for $k > j$ let

$$I_{k,n} = \int_{-\infty}^{y_{jn}} \left(\frac{p_{n,n}(t)}{t - x_{kn}}\right)^2 d\mu_n(t).$$

We shall show that for $k \leq j$,

$$I_{k,n} = \frac{1}{\pi} \frac{1}{\sqrt{1 - y_{jn}^2}} \frac{1}{y_{jn} - x_{kn}} + \tau_{kn}, \tag{3.3}$$

and for $k > j$,

$$I_{k,n} = \frac{1}{\pi} \frac{1}{\sqrt{1 - y_{jn}^2}} \frac{1}{x_{kn} - y_{jn}} - \tau_{kn},$$

where τ_{kn} are "tail terms". The estimation of τ_{kn} is non-trivial. However, the main idea is that when we substitute into (3.1), we obtain

$$\begin{aligned}
\Upsilon_j &= n - j + \left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^2 \{\frac{1}{\pi} \frac{1}{\sqrt{1 - y_{jn}^2}} \sum_{k=1}^{n} \lambda_{kn} \frac{p_{n,n-1}^2(x_{kn})}{y_{jn} - x_{kn}} + \sum_{k=1}^{n} \lambda_{kn} p_{n,n-1}^2(x_{kn}) \tau_{kn}\} \\
&= n - j + \left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^2 \{\frac{1}{\pi} \frac{1}{\sqrt{1 - y_{jn}^2}} \int_{-\infty}^{\infty} \frac{p_{n,n-1}^2(t)}{y_{jn} - t} d\mu_n(t) + \sum_{k=1}^{n} \lambda_{kn} p_{n,n-1}^2(x_{kn}) \tau_{kn}\} \\
&= n - j + \left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^2 \sum_{k=1}^{n} \lambda_{kn} p_{n,n-1}^2(x_{kn}) \tau_{kn},
\end{aligned}$$

(3.4)

by first the Gauss quadrature formula and then orthogonality of $p_{n,n-1}(t)$ to polynomials of degree less than $n-1$.

Lemma 3.2

$$y_{jn} - x_{jn} \sim \frac{1}{n} \sim x_{j+1,n} - y_{jn}. \tag{3.5}$$

Proof

We use a weaker form of our asymptotics (2.1), namely that for $m = n-1, n$,

$$\begin{aligned}
&p_{n,m}(x) \mu_n'(x)^{1/2} \left(1 - x^2\right)^{1/4} \\
&= \sqrt{\frac{2}{\pi}} \cos\left(\left(m - n + \frac{1}{2}\right)\theta + f_n(x)\right) + o(1).
\end{aligned}$$

Then if $x_{jn} = \cos\theta_{jn}$ and $y_{jn} = \cos\phi_{jn}$, we have

$$0 = \sqrt{\frac{2}{\pi}} \cos\left(\frac{1}{2}\theta_{jn} + f_n(x_{jn})\right) + o(1)$$

and

$$0=\sqrt{\frac{2}{\pi}}\cos\left(-\frac{1}{2}\phi_{jn}+f_n\left(y_{jn}\right)\right)+o\left(1\right).$$

If for some subsequence $\mathcal{N}$ of integers, $y_{jn}-x_{jn}=o\left(\frac{1}{n}\right)$, or equivalently $\left|\phi_{jn}-\theta_{jn}\right|=o\left(\frac{1}{n}\right)$, then in that subsequence,

$$f_n\left(y_{jn}\right)=f_n\left(x_{jn}\right)+o\left(1\right),$$

using (2.4), (2.2) and just continuity of g. Then

$$\cos\left(\frac{1}{2}\theta_{jn}+f_n\left(x_{jn}\right)\right)=o\left(1\right)=\cos\left(-\frac{1}{2}\phi_{jn}+f_n\left(x_{jn}\right)\right)$$

so

$$\begin{aligned}
&\sin\left(\frac{1}{2}\theta_{jn}+\frac{1}{2}\phi_{jn}\right)\\
=\;&\sin\left(\left(\frac{1}{2}\theta_{jn}+f_n\left(x_{jn}\right)\right)-\left(-\frac{1}{2}\phi_{jn}+f_n\left(x_{jn}\right)\right)\right)\\
=\;&\sin\left(\frac{1}{2}\theta_{jn}+f_n\left(x_{jn}\right)\right)\cos\left(-\frac{1}{2}\phi_{jn}+f_n\left(x_{jn}\right)\right)\\
&-\cos\left(\frac{1}{2}\theta_{jn}+f_n\left(x_{jn}\right)\right)\sin\left(-\frac{1}{2}\phi_{jn}+f_n\left(x_{jn}\right)\right)=o\left(1\right),
\end{aligned}$$

so as $\theta_{jn},\phi_{jn}\in[0,\pi]$ and are $o\left(\frac{1}{n}\right)$ apart, this forces

$$\sin\left(\theta_{jn}\right)=o\left(1\right),$$

so that either $\theta_{jn}=o\left(1\right)$ or $\theta_{jn}=\pi-o\left(1\right)$. But this is impossible, as x_{jn} lies in the compact subset $\mathcal{I}$ of $(-1,1)$. Consequently, we must have $y_{jn}-x_{jn}\geq\frac{C}{n}$. Since $x_{j+1,n}-x_{jn}\leq C/n$, we also have the other direction, so $y_{jn}-x_{jn}\sim\frac{1}{n}$. The other half of (3.5) is proved similarly. ■

We proceed to estimate $I_{k,n}$ of (3.2) for $k\leq j$. The case $k>j$ is similar. First we deal with terms arising from x_{kn} that are not too close to y_{jn}.

Lemma 3.3
Suppose $y_{jn}\in I$ and $k\leq j$ is such that $y_{jn}-x_{kn}\geq(\log n)^{-1/5}$. Then

$$I_{k,n}=\frac{1}{\pi}\frac{1}{\sqrt{1-y_{jn}^2}}\frac{1}{y_{jn}-x_{kn}}+O\left(\left(\log n\right)^{2/5}\right).$$

Proof

$$I_{k,n}=\int_{y_{jn}}^{\infty}\left(\frac{p_{n,n}\left(t\right)}{t-x_{kn}}\right)^2d\mu_n\left(t\right)\leq\frac{1}{\left(y_{jn}-x_{kn}\right)^2}\leq\left(\log n\right)^{2/5}.$$

Moreover, as the distance from y_{jn} to ±1 is bounded below by the distance from $\mathcal{I}$ to ±1,

$$\frac{1}{\pi}\frac{1}{\sqrt{1-y_{jn}^2}}\frac{1}{y_{jn}-x_{kn}}\leq C\left(\log n\right)^{1/5}.$$

■

Next, we deal with "central" terms.

Lemma 3.4
Suppose $y_{jn} \in I$ and $k \leq j$ is such that $y_{jn} - x_{kn} < (\log n)^{-1/5}$. Then

$$I_{k,n} = \frac{1}{\pi\sqrt{1-y_{jn}^2}} \frac{1}{y_{jn} - x_{kn}} + \tau_{kn},$$

where

$$\begin{aligned} \tau_{kn} &= O\left((\log n)^{2/5}\right) + \frac{o(\log n)^{-1/2}}{y_{jn} - x_{kn}} \\ &+ O\left(\frac{1}{n(y_{jn} - x_{kn})^2}\right) + O\left(\log\left(1 + \frac{1}{y_{jn} - x_{kn}}\right)\right). \end{aligned} \tag{3.6}$$

Proof
We split

$$\begin{aligned} I_{k,n} &= \int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \left(\frac{p_{n,n}(x)}{x - x_{kn}}\right)^2 d\mu_n(x) + O\left((\log n)^{2/5}\right). \\ &= I_{k,n}^* + O\left((\log n)^{2/5}\right), \end{aligned} \tag{3.7}$$

say. Next, we use our asymptotic (2.1) in the form

$$\begin{aligned} p_{n,n}(x)^2 \mu_n'(x) \left(1 - x^2\right)^{1/2} &= \frac{2}{\pi} \cos^2\left(\frac{\arccos x}{2} + f_n(x)\right) + o(\log n)^{-1/2} \\ = \ \frac{1}{\pi} + \frac{1}{\pi} \cos(\arccos x + 2f_n(x)) &+ o(\log n)^{-1/2}. \end{aligned}$$

This holds uniformly in $\mathcal{I}$ and hence uniformly in the range of integration in $I_{k,n}^*$ in (3.7) (and uniformly in j such that $y_{jn} \in \mathcal{I}$). Then

$$I_{k,n}^* = \frac{1}{\pi} \int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \frac{1 + x\cos 2f_n(x) - \sqrt{1-x^2}\sin 2f_n(x) + o(\log n)^{-1/2}}{\sqrt{1-x^2}\,(x - x_{kn})^2} dx. \tag{3.8}$$

We split this into three integrals, that we estimate separately: first, integrating by parts,

$$\begin{aligned} &\frac{1}{\pi} \int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \frac{1}{\sqrt{1-x^2}\,(x - x_{kn})^2} dx \\ = \ &\frac{1}{\pi} \left[\frac{1}{\sqrt{1-x^2}} \frac{-1}{x - x_{kn}}\right]_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} + \frac{1}{\pi} \int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \frac{x}{(1-x^2)^{3/2}} \frac{dx}{x - x_{kn}} \\ = \ &\frac{1}{\pi\sqrt{1-y_{jn}^2}} \left[\frac{1}{y_{jn} - x_{kn}} + O\left((\log n)^{1/5}\right)\right] + O\left(\int_{y_{jn}}^{y_{jn}+1} \frac{dx}{x - x_{kn}}\right) \\ = \ &\frac{1}{\pi\sqrt{1-y_{jn}^2}} \frac{1}{y_{jn} - x_{kn}} + O\left((\log n)^{1/5}\right) + O\left(\log\left(1 + \frac{1}{y_{jn} - x_{kn}}\right)\right). \end{aligned} \tag{3.9}$$

Next,

$$\frac{1}{\pi}\int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \frac{o(\log n)^{-1/2}}{\sqrt{1-x^2}\,(x-x_{kn})^2}dx = o(\log n)^{-1/2}\frac{1}{y_{jn}-x_{kn}}. \tag{3.10}$$

Next, using Lemma 2.6, with $\phi(x) = x/\sqrt{1-x^2}$ or 1, $\alpha_n = y_{jn}$, $\beta_n = y_{jn} + (\log n)^{-1/5}$, $c_n = x_{kn} \le x_{jn} \le y_{jn} - \frac{A}{n}$ (recall Lemma 3.2), and using our hypotheses (I) in Section 2,

$$\begin{aligned}&\left|\frac{1}{\pi}\int_{y_{jn}}^{y_{jn}+(\log n)^{-1/5}} \frac{x\sin 2f_n(x) - \sqrt{1-x^2}\cos 2f_n(x)}{\sqrt{1-x^2}\,(x-x_{kn})^2}dx\right| \\ \le\ & \frac{o\left((\log n)^{-1/2}\right)}{y_{jn}-x_{kn}} + \frac{C}{n\,(y_{jn}-x_{kn})^2}.\end{aligned}$$

Combining this and (3.7-3.10) yields the result. ■

Lemma 3.5
Suppose $y_{jn} \in \mathcal{I}$. Then

$$\begin{aligned}&\sum_{k=1}^{j}\lambda_{kn}p_{n,n-1}^2(x_{kn})\int_{y_{jn}}^{\infty}\left(\frac{p_{n,n}(t)}{t-x_{kn}}\right)^2 d\mu_n(t) \\ =\ & \frac{1}{\pi\sqrt{1-y_{jn}^2}}\sum_{k=1}^{j}\frac{\lambda_{kn}p_{n,n-1}^2(x_{kn})}{y_{jn}-x_{kn}} + o(\log n)^{1/2}.\end{aligned} \tag{3.11}$$

Proof
Using (3.2) and (3.3), the sum equals

$$\sum_{k=1}^{j}\frac{1}{\pi\sqrt{1-y_{jn}^2}}\frac{\lambda_{kn}p_{n,n-1}^2(x_{kn})}{y_{jn}-x_{kn}} + \sum_{k=1}^{j}\lambda_{kn}p_{n,n-1}^2(x_{kn})\,\tau_{kn}.$$

By Lemma 3.3,

$$\sum_{k\le j,\ y_{jn}-x_{kn}\ge(\log n)^{-1/5}}\lambda_{kn}p_{n,n-1}^2(x_{kn})\,\tau_{kn} = O\left((\log n)^{2/5}\right) = o(\log n)^{1/2}.$$

Also, using (2.8) and the asymptotics for orthonormal polynomials that imply bounds on $p_{n,n-1}$, followed by the uniform spacing of the zeros and Lemma 3.4,

$$\begin{aligned}
& \sum_{k\leq j,\ y_{jn}-x_{kn}\leq(\log n)^{-1/5}} \lambda_{kn}p_{n,n-1}^2\left(x_{kn}\right)\left|\tau_{kn}\right| \\
\leq\ & \frac{C}{n}\sum_{k\leq j,\ y_{jn}-x_{kn}\leq(\log n)^{-1/5}}\left[\begin{array}{c}(\log n)^{2/5}+\frac{o(\log n)^{-1/2}}{y_{jn}-x_{kn}} \\ +O\left(\frac{1}{n(y_{jn}-x_{kn})^2}\right)+O\left(\log\left(1+\frac{1}{y_{jn}-x_{kn}}\right)\right)\end{array}\right] \\
\leq\ & C\left(\log n\right)^{2/5}+o\left(\log n\right)^{-1/2}\int_{-1}^{y_{jn}-\frac{C}{n}}\frac{dt}{y_{jn}-t}dt \\
& +O\left(\frac{1}{n}\int_{-1}^{y_{jn}-\frac{C}{n}}\frac{dt}{\left(y_{jn}-t\right)^2}dt\right)+O\left(\int_{-1}^{y_{jn}-\frac{C}{n}}\log\left(1+\frac{1}{y_{jn}-t}\right)dt\right) \\
\leq\ & C\left(\log n\right)^{2/5}+o\left(\log n\right)^{1/2}+O\left(1\right)+O\left(1\right)=o\left(\log n\right)^{1/2}.
\end{aligned}$$

■

Proof of Theorem 2.2
By proceeding similarly as in Lemmas 3.3-3.5,

$$\begin{aligned}
& \sum_{k=j+1}^{n}\lambda_{kn}p_{n,n-1}^2\left(x_{kn}\right)\int_{-\infty}^{y_{jn}}\left(\frac{p_{n,n}\left(t\right)}{t-x_{kn}}\right)^2 d\mu_n\left(t\right) \\
=\ & \frac{1}{\pi\sqrt{1-y_{jn}^2}}\sum_{k=j+1}^{n}\frac{\lambda_{kn}p_{n,n-1}^2\left(x_{kn}\right)}{x_{kn}-y_{jn}}+o\left(\log n\right)^{1/2}.
\end{aligned}$$

Combining this and Lemma 3.5 in (3.1) yields (as in (3.4)),

$$\begin{aligned}
\Upsilon_j &= n-j+\left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^2\sum_{k=1}^{n}\frac{\lambda_{kn}p_{n,n-1}^2\left(x_{kn}\right)}{y_{jn}-x_n}+o\left(\log n\right)^{1/2} \\
&= n-j+o\left(\log n\right)^{1/2}.
\end{aligned}$$

■

4. Proof of Theorem 1.1

As we are dealing with a fixed measure, we let $p_n\left(x\right)=\gamma_n x^n+...$ denote the nth orthonormal polynomial for μ, so that

$$\int_{-1}^{1}p_n p_m d\mu=\delta_{mn}.$$

We must verify the four asymptotic assumptions (I)-(IV) in Section 2.
(I) Pointwise Asymptotics of Orthonormal Polynomials
This is our hypothesis (1.10) - (1.11).
(II) Asymptotics of Leading Coefficients
The limit

$$\lim_{n\to\infty}\frac{\gamma_{n-1}}{\gamma_n}=\frac{1}{2}$$

is an immediate consequence of the fact that μ satisfies Szegő's condition [25, p. 309].

(III) Asymptotic Spacing of Zeros
It is known that if $x_{jn} \to x$

$$\lim_{n\to\infty} n\sigma_n^*(x)(x_{jn}-x_{j+1,n}) = \lim_{n\to\infty} n\pi\sqrt{1-x^2}(x_{jn}-x_{j+1,n})\pi\sqrt{1-x^2}=1,$$

uniformly for $x \in \mathcal{I}$, see [21, Thm, 3.11.11, p. 221].
(IV) Asymptotics for Reproducing Kernels
It is known [15], [21, Thm. 3.11.9, p. 220] that uniformly for $x \in \mathcal{I}$,

$$\lim_{n\to\infty} \frac{1}{n\sigma_n^*(x)} K_n(x,x)\mu'(x) = \lim_{n\to\infty} \pi\sqrt{1-x^2}K_n(x,x)\mu'(x)=1.$$

5. Proof of Theorem 1.3

We shall first verify the assumptions (I) - (IV) in Section 2 for the measures $\{\mu_n\}$, defined by

$$W_n^*(t) = W\left(L_n^{[-1]}(t)\right), t \in L_n(I), \tag{5.1}$$

and

$$d\mu_n(t) = W_n^*(t)^2\,dt, t \in L_n(I). \tag{5.2}$$

Here the linear transformations L_n of $[a_{-n}, a_n]$ onto $[-1,1]$, are defined by (1.19-1.20). If $p_m(x) = \gamma_m x^m + ...$ denotes the mth orthonormal polynomial for $d\mu(x) = W^2(x)\,dx$, then a substitution shows that the mth orthonormal polynomial $p_{n,m}$ for $d\mu_n$ satisfies

$$p_{n,m}(x) = \delta_n^{1/2} p_m\left(L_n^{[-1]}(x)\right), x \in L_n(I). \tag{5.3}$$

Consequently, if x_{jn} denotes the jth zero of $p_{n,n}(x)$ and $\hat{x}_{jn}$ denotes the jth zero of $p_n(x)$, then

$$x_{jn} = L_n(\hat{x}_{jn}). \tag{5.4}$$

Moreover, if $K_n(W^2,x,y)$ denotes the nth reproducing kernel for the measure $d\mu$, while $K_n(x,y)$ denotes the nth reproducing kernel for $d\mu_n$, it is easily seen that

$$K_n(x,y) = \delta_n K_n\left(W^2, L_n^{[-1]}(x), L_n^{[-1]}(y)\right). \tag{5.5}$$

(I) Pointwise Asymptotics of Orthonormal Polynomials
The asymptotic (2.1) with $g(x) = 0$, and $\zeta_n = -\frac{\pi}{4}$ follows immediately from Theorem 15.3 in [13, (15.11), p. 403] and the identity (5.3). Note that the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ in the hypothesis of the theorem in [13] contains the class $\mathcal{F}\left(C^2\right)$. The error term there is $O(n^{-\eta})$ for some $\eta > 0$, which is stronger than the $o(\log n)^{-1/2}$ required in (2.1) of this paper. The bound (2.2) on $\{\sigma_n^*\}$ follows from Theorem 6.1 in [13, p. 146]. Finally, the required uniform smoothness (2.3) follows from Theorem 6.3(b) in [13, p. 148], where it is shown that $\{\sigma_n^*\}$ satisfy a uniform Lipschitz condition of order $\frac{1}{4}$ in compact subsets of $(-1,1)$.
(II) Asymptotics of Leading Coefficients
From (5.3), we see that

$$\frac{\gamma_{n,n-1}}{\gamma_{n,n}} = \delta_n^{-1}\frac{\gamma_{n-1}}{\gamma_n},$$

so (1.124) of Theorem 1.23 in [13, p. 26], where $A_n = \frac{\gamma_{n-1}}{\gamma_n}$, gives

$$\lim_{n\to\infty} \frac{\gamma_{n,n-1}}{\gamma_{n,n}} = \frac{1}{2}.$$

(III) Asymptotic Spacing of Zeros

In [14, Thm. 1.4, p. 75], it is shown that uniformly for j, n with $\hat{x}_{jn} \in [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]$,

$$\lim_{n\to\infty} \sigma_n(\hat{x}_{jn})(\hat{x}_{jn} - \hat{x}_{j+1,n}) = 1.$$

Now using (5.4) and (1.21), we see that

$$\begin{aligned} & n\sigma_n^*(x_{jn})(x_{jn} - x_{j+1,n}) \\ = \; & n\frac{\delta_n}{n}\sigma_n(\hat{x}_{jn})\left(\frac{\hat{x}_{jn} - \hat{x}_{j+1,n}}{\delta_n}\right) = 1 + o(1), \end{aligned}$$

uniformly for $x_{jn} \in L_n([a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]) = [-1+\varepsilon, 1-\varepsilon]$.

(IV) Asymptotics and Bounds for Reproducing Kernels

Let $0 < \alpha < 1$. In [13, Theorem 1.25, p. 26], it is shown that

$$\frac{1}{K_n(W^2, t, t) W^2(t)} = \frac{1}{\sigma_n(t)}(1 + o(1)),$$

uniformly for $t \in [a_{-\alpha n}, a_{\alpha n}]$. Then from (5.5), we see that uniformly for $x \in L_n([a_{-\alpha n}, a_{\alpha n}])$,

$$\begin{aligned} & K_n(x,x)\mu_n'(x)/(n\sigma_n^*(x)) \\ = \; & \delta_n K_n\left(W^2, L_n^{[-1]}(x), L_n^{[-1]}(y)\right) W^2\left(L_n^{[-1]}(x)\right) / \left[\delta_n\sigma_n\left(L_n^{[-1]}(x)\right)\right] = 1 + o(1), \end{aligned}$$

that is we have (2.7). Note that for some C and C_1 independent of α and n, [13, (3.50), p. 81]

$$1 - L_n(a_{\alpha n}) = \frac{a_n - a_{\alpha n}}{\delta_n} \le C\frac{a_n}{\delta_n T(a_n)}(1-\alpha) \le C_1(1-\alpha),$$

with C_1 independent of n, α. A similar inequality holds for $L_n(a_{-\alpha n}) + 1$. Thus we can choose α close enough to 1 to ensure that for a given $\varepsilon > 0$, $L_n([a_{-\alpha n}, a_{\alpha n}])$ contains $[-1+\varepsilon, 1-\varepsilon]$. So we have (2.7) in the desired range.

Theorem 2.3 now gives (1.26), the alternate form of Theorem 1.3. Since

$$\hat{\lambda}_j = L_n^{[-1]}(\lambda_j); \hat{x}_{jn} = L_n^{[-1]}(x_{jn}); n\sigma_n^*(x_{jn}) = \delta_n\sigma_n(\hat{x}_{jn}),$$

we see that

$$\frac{\lambda_j - x_{jn}}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{n\sigma_n^*(x_{jn})}}} = \frac{\left(\hat{\lambda}_j - \hat{x}_{jn}\right)/\delta_n}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{\delta_n\sigma_n(\hat{x}_{jn})}}} = \frac{\hat{\lambda}_j - \hat{x}_{jn}}{\sqrt{\frac{\log n}{2\pi^2}\frac{1}{\sigma_n(\hat{x}_{jn})}}},$$

so (1.25) also follows. ■

6. Proof of Theorem 1.4

We note that the notation of [16] is somewhat different. There Q is denoted by V. The formulation there involves a parameter c, which we take to be 2. There, in the case where the support of the equilibrium measures is a single interval, it is denoted by $[\alpha, \beta]$. We take it to be $[-1, 1]$. The leading coefficient of $p_{n,n}$ is denoted there by $\kappa_{n,n}$, while we use $\gamma_{n,n}$. They abbreviate $p_{n,n}$ as p_n. Moreover, the coefficient of $p_{n,n-1}$ (p_{n-1} there) is denoted by $\kappa_{n-1,n-1}$, while we use $\gamma_{n,n-1}$.

(I) Pointwise Asymptotics of Orthonormal Polynomials
In [16], the asymptotics are stated in terms of entries of a 2 by matrix. Thus in their notation [16, p. 46, eqn. (209)]

$$A_{11}(x) = \frac{1}{\kappa_{n,n}} p_n(x) \text{ and } A_{21}(x) = -\frac{2\pi i}{\kappa_{n-1,n-1}} p_{n-1}(x). \tag{6.1}$$

They also use the error term [16, p. 45, eqn. (202)]

$$\Delta_n = n^{-1/3} \log n,\ n \geq 2.$$

Their asymptotic for the leading coefficients [16, p. 45, eqn. (207)] is

$$\kappa_{n-1,n-1}^2 = \frac{1}{4\pi} e^{-n\ell} (1 + O(\Delta_n)) \text{ and } \kappa_{n.n}^2 = \frac{1}{\pi} e^{-n\ell} (1 + O(\Delta_n)), \tag{6.2}$$

where ℓ is a number that appears in an equilibrium relation. We do not need an explicit expression for it. In [16, p. 48, eqns. (223-4)], they show that uniformly for x in compact subsets of $(-1,1)$,

$$A_{11}(x) = e^{n(2V(x)+\ell-\phi(x))/2} a(x) \left[\cos\left(\frac{1}{2}(n\theta(x) - \varphi(x))\right) + O(\Delta_n)\right]. \tag{6.3}$$

and

$$A_{21}(x) = -ie^{n(2V(x)-\ell-\phi(x))/2} a(x) \left[\sin\left(\frac{1}{2}(n\theta(x) + \varphi(x))\right) + O(\Delta_n)\right]. \tag{6.4}$$

We note that there is a typo in the statement of (224) there, $n\theta(x) + \varphi(x)$ is mistakenly listed as $n\theta(x) - \varphi(x)$. This can be seen from equation (222) there, and has been confirmed to the author by Peter Miller. The functions $\phi(x), a(x), \theta(x), \varphi(x)$ are as follows: $\phi(x) = 0$ [16, pp. 15-16, eqns. (47), (51)] Moreover,

$$\begin{aligned} a(x) &= \sqrt{2}\left(1-x^2\right)^{-1/4}; \\ \theta(x) &= 2\pi \int_x^1 \sigma^*(x)\, dx; \\ \varphi(x) &= \arcsin(x). \end{aligned}$$

See [16, p. 46, eqn. (214)], and [16, p. 63, eqn. (A29)]. Combining with (6.1) and (6.2), and using

$$\arcsin x = \frac{\pi}{2} - \arccos x,$$

we obtain the desired forms

$$p_{n,n}(x)\, \mu_n'(x)^{1/2} \left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \cos\left(\frac{1}{2}\theta + n\pi \int_x^1 \sigma^* - \frac{\pi}{4}\right) + O(\Delta_n);$$

$$p_{n,n-1}(x)\, \mu_n'(x)^{1/2} \left(1-x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}} \cos\left(-\frac{1}{2}\theta + n\pi \int_x^1 \sigma^* - \frac{\pi}{4}\right) + O(\Delta_n).$$

These have the form (2.1). Moreover, [16, p. 63, Lemma 3] σ^* is positive and continuously differentiable in $(-1,1)$.

(II) Asymptotics of Leading Coefficients
Directly from (6.2), we have

$$\lim_{n\to\infty} \frac{\gamma_{n,n-1}}{\gamma_{n,n}} = \lim_{n\to\infty} \frac{\kappa_{n-1,n-1}}{\kappa_{n,n}} = \frac{1}{2}.$$

(IV) Asymptotics for Reproducing Kernels
In [16, p. 6, eqn. (20)], McLaughlin and Miller show that when $\sigma^*(x) > 0$, then for $u, v \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n\sigma^*(x)} e^{-nQ(x)} K_n\left(x + \frac{u}{n\sigma^*(x)}, x + \frac{v}{n\sigma^*(x)}\right) = \mathbb{S}(u-v). \tag{6.5}$$

In the special case we consider, this holds for all $x \in (-1, 1)$. In particular,

$$\lim_{n\to\infty} \frac{1}{n\sigma^*(x)} e^{-nQ(x)} K_n(x, x) = 1. \tag{6.6}$$

The proofs in [16] show that (6.5) and (6.6) hold uniformly for x in compact subsets of $(-1, 1)$, and u, v in compact intervals. Thus recalling that $\sigma_n^*(x) = n\sigma^*(x)$, we have (2.7). Finally,

(III) Asymptotic Spacing of Zeros
The universality limit (6.5) and limits of the Christoffel functions imply the required asymptotic spacing for the zeros. The proof of this is exactly the same as that of Theorem 1.4 in [14, p. 86], so is omitted.

References

[1] V.M. Badkov, *The Asymptotic Behavior of Orthogonal Polynomials*, Math. USSR Sbornik, 37(1980), 39-51.

[2] V.M. Badkov, *Uniform Asymptotic Representations of Orthogonal Polynomials*, Proc. Steknlov Inst. Math., 164(1985), 5-41.

[3] J. Breuer, M. Duits, *The Nevai condition and a local law of large numbers for orthogonal polynomial ensembles*, Adv. Math. 265 (2014), 441–484.

[4] J. Breuer, M. Duits, *Universality of mesocopic fluctuation in orthogonal polynomial ensembles*, to appear in Comm. Math. Phys.

[5] O. Costin, J. Leibowitz, *Gaussian Fluctuations in Random Matrices*, Physics Review Letters, 75(1995), 69-72.

[6] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[7] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[8] L. Ya. Geronimus, *Orthogonal Polynomials,* Consultants Bureau, New York, 1961.

[9] B. Golinskii, *On the Rapidity of Convergence of Sequences of Orthogonal Polynomials to a Limit Function,* Ukrainian Mathematical Journal, 19(1967), 385-402.

[10] B. L. Golinskii, *Asymptotic Representation of Orthogonal Polynomials,* Russian Math. Surveys, 35(1980), 169-229.

[11] J. Gustavsson, *Gaussian Fluctuations of Eigenvalues in the GUE,* Annales Institut H. Poincare, 41(2005), 151-178.

[12] A. Kuijlaars, K. T-R McLaughlin, W. Van Assche, and M. Vanlessen, *The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on* [-1,1], Advances in Mathematics, 188 (2004), 337-398.

[13] Eli Levin and D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.

[14] Eli Levin and D.S. Lubinsky, *Applications of Universality Limits to Zeros and Reproducing Kernels of Orthogonal Polynomials*, J. Approx. Theory, 150(2008), 69-95.

[15] A. Mate, P. Nevai, V. Totik, *Szegő's Extremum Problem on the Unit Circle*, Annals of Math., 134(1991), 433-453.

[16] K. T.-R McLaughlin and P.D. Miller, *The $\bar{\partial}$ Steepest Descent Method for Orthogonal Polynomials on the Real Line with Varying Weights*, Int. Math. Res. Notices, (2008), Article ID rnn075, 66 pages.

[17] H.N. Mhaskar and E.B. Saff, *Extremal Problems for Polynomials with Exponential Weights,* Trans. Amer. Math. Soc., 285(1984), 203-234.

[18] H.N. Mhaskar and E.B. Saff, *Where does the L_p Norm of a Weighted Polynomial Live?,* Trans. Amer. Math. Soc., 303(1987), 109-124.

[19] S. O'Rourke, *Gaussian Fluctuations of Eigenvalues in Wigner Random Matrices*, Jour. Stat. Phys., 138(2009), 1045-1066.
[20] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer, New York, 1997.
[21] B. Simon, *Szegő's Theorem and its descendants*, Princeton University Press, Princeton, 2011.
[22] A.B. Soshnikov, *Gaussian Fluctuation for the Number of Particles in Airy, Bessel, Sine, and Other Determinantal Random Point Fields,* Journal of Statistical Physics, 100(2000), 491-522.
[23] A.B. Soshnikov, *Gaussian Limit for Determinantal Random Point Fields,* The Annals of Probability, 30(2001), 1-17.
[24] Z. Su, *Gaussian Fluctuations in Complex Sample Covariance Matrices,* Electr. J. Probab., 11(2006), 1284-1320.
[25] G. Szegő, *Orthogonal Polynomials*, American Mathematical Society, Providence, 1939.
[26] T. Tao, V. Vu, *Random matrices: universality of local eigenvalues statistics*, Acta Math., 206(2011), 127-204.
[27] M. Vanlessen, *Strong asymptotics of the recurrence coefficients of orthogonal polynomials associated to the generalized Jacobi weight*, J. Approx. Theory, 125(2003), 198-237.
[28] Deng Zhang, *Gaussian Fluctuations of Eigenvalues in Log-gas Ensemble: Bulk Case I*, Acta Math. Sinica, 31(2015), 1487-1500.
[29] Deng Zhang, *Gaussian Fluctuations of Eigenvalues in the Unitary Ensemble*, manuscript.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu