The Size of the set of μ-Irregular Points of a measure μ

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Abstract

Let μ be a compactly suppported positive measure on the real line. A point $x \in$supp$[\mu]$ is said to be μ-regular, if, as $n \to \infty$,

$$\sup_{\deg(P) \geq n} \left(\frac{|P(x)|}{\|P\|_{L_2(d\mu)}} \right)^{1/n} \to 1.$$

Otherwise it is a μ-irregular point. We show that for any such measure, the set of μ-irregular points in $\{\mu' > 0\}$ (with a suitable definition of this set) has Hausdorff m_{h_β} measure 0, for $h_\beta(t) = \left(\log \frac{1}{t}\right)^{-\beta}$, any $\beta > 1$.

Orthogonal Polynomials on the real line, regular measures, irregular points

1 Introduction[1]

Let μ be a positive measure on the real line, with compact support supp$[\mu]$, and infinitely many points in its support. μ is said to be *regular in the sense of*

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Stahl, Totik, and Ullman, or just *regular,* [9, p. 68] if

$$\lim_{n\to\infty}\left(\sup_{\deg(P)\leq n}\frac{|P(x)|}{\|P\|_{L_2(d\mu)}}\right)^{1/n}\leq 1,$$

q.e. in supp[μ]. Here q.e. (quasi-everywhere) means except on a set of logarithmic capacity 0, while

$$\|P\|_{L_2(d\mu)}=\left(\int |P|^2\,d\mu\right)^{1/2}.$$

This should not be confused with the notion of a regular Borel measure. Regular measures play an important role in asymptotics of orthogonal polynomials, and in questions of weighted approximation. See the comprehensive monograph [9], and also [5], [6], [7], [10], [11]. Regular measures are those that permit localization of a whole host of properties.

In the monograph [9, p. 140], local and pointwise regularity are also investigated. We say that $x\in$supp[μ] is a *μ-regular point* (or, *regular point for μ*) if

$$\lim_{n\to\infty}\left(\sup_{\deg(P)\leq n}\frac{|P(x)|}{\|P\|_{L_2(d\mu)}}\right)^{1/n}=1. \tag{1}$$

Otherwise x is a *μ-irregular point.* It is known [9, p, 140] that μ is regular iff the set of μ-irregular points in the support has logarithmic capacity 0. Of course, this should not be confused with points that are irregular for the Dirichlet problem in classical potential theory.

Notice that since the polynomial $P=1$ is included in the sup, the left-hand side of (1) is ≥ 1. Thus x is a μ-irregular point iff

$$\limsup_{n\to\infty}\left(\sup_{\deg(P)\leq n}\frac{|P(x)|}{\|P\|_{L_2(d\mu)}}\right)^{1/n}>1.$$

This is easily formulated in terms of orthogonal polynomials. Let $\{p_n\}$ denote the orthonormal polynomials for μ, so that

$$\int p_n p_m d\mu=\delta_{mn}.$$

Define the reproducing kernels

$$K_n(x,t)=\sum_{j=0}^{n-1}p_j(x)\,p_j(t).$$

It is an easy consequence of Cauchy-Schwarz' inequality, that for any polynomial $P(x)$ of degree $\leq n-1$, we have

$$P^2(x)\leq K_n(x,x)\int P^2 d\mu.$$

In fact,

$$K_n(x,x) = \sup_{\deg(P)\leq n-1} \frac{P^2(x)}{\int P^2 d\mu};$$

the polynomial P attaining the supremum is $P(t) = K_n(x,t)$. We thus see that x is a μ-regular point iff

$$\lim_{n\to\infty} K_n(x,x)^{1/n} = 1.$$

In turn, since $K_n(x,x)$ increases with n, x is μ-regular iff

$$\lim_{j\to\infty} K_{2^j}(x,x)^{1/2^j} = 1. \tag{2}$$

We shall show that the set of μ-irregular points is thin, and "almost" has logarithmic capacity 0, in the sense of Hausdorff measures. Let $h : [0,\infty) \to [0,\infty]$ be an increasing right-continuous function that has limit 0 at 0. Given $E \subset \mathbb{R}$, its Hausdorff outer m_h measure is

$$m_h(E) = \inf \left\{ \sum_{j=1}^{\infty} h(meas(I_j)) : E \subset \bigcup_j I_j \right\},$$

where the inf is taken over all coverings of E by intervals $\{I_j\}$ with lengths $\{meas(I_j)\}$. For $h(t) = t^\alpha, \alpha > 0$, this leads to $\alpha-$dimensional Hausdorff measure. For $\beta > 0$,

$$h_\beta(t) = \begin{cases} \left(\log \frac{1}{t}\right)^{-\beta}, & t \in (0,1) \\ \infty, & t \geq 1, \end{cases}$$

we obtain $\beta-$logarithmic Hausdorff measure. Note that if for some $\beta > 0$,

$$m_{h_\beta}(E) = 0$$

then E has $\alpha-$dimensional Hausdorff measure 0 for all $\alpha > 0$. When $\beta = 1$, this in addition implies that E has logarithmic capacity 0 [1, p. 28], [2]. Even a set of $\sigma-$finite m_{h_1} measure has zero logarithmic capacity. Roughly speaking, if a set has m_{h_β} measure 0 for all $\beta > 1$, it is close to having logarithmic capacity 0, but not quite of logarithmic capacity 0.

Now

$$\{\mu' > 0\} = \{x : \mu'(x) > 0\}$$

is a Lebesgue measurable set that is unique only up to a set of linear Lebesgue measure 0. In defining this set, we use the absolutely continuous component μ_{ac} of μ. We say that $x \in \{\mu' > 0\}$ iff

$$\mu'_{ac}(x) = \lim_{meas(I)\to 0, I\ni x} \frac{\mu_{ac}(I)}{meas(I)} > 0,$$

where the limit is taken over intervals I. This ensures that for all $x \in \{\mu' > 0\}$, we have

$$\liminf_{meas(I)\to 0, I\ni x} \frac{\mu(I)}{meas(I)} > 0, \tag{3}$$

for the left-hand side of (3) is bounded below by $\mu'_{ac}(x)$.

Theorem 1.1
Assume that μ is a compactly supported measure on the real line. Then the set of μ-irregular points in $\{\mu' > 0\}$ has m_{h_β} measure 0 for all $\beta > 1$.

Remarks
(a) It seems unlikely that the set of irregular points can have zero capacity in $\{\mu' > 0\}$ for irregular measures. It certainly cannot be of zero capacity in the larger set supp$[\mu]$, for otherwise μ is regular [9, p. 140].
(b) If the set of irregular points in $\{\mu' > 0\}$ has finite (or even σ-finite) m_{h_1} measure, then it has zero capacity by a result of Kametani [1, p. 28], [2], and again, this is not possible in the larger set supp$[\mu]$, unless μ is regular.
(c) In a similar vein, any compact set E of finite positive (or $\sigma-$finite) m_{h_1} measure will have zero capacity. It is possible to construct regular measures that have E as their set of irregular points [12]. This again suggests that one cannot expect m_{h_1} measure 0 in Theorem 1.1.
(d) Let $h : [0, \infty) \to [0, \infty]$ be an increasing right continuous function such that for each $\varepsilon > 0$,

$$\sum_j 2^j h\left(e^{-\varepsilon 2^j}\right) < \infty.$$

Then the same proof shows that the set of μ-irregular points has Hausdorff m_h measure 0. As an example, we can let $\rho > 1$,

$$h(t) = \left(\log \frac{1}{t}\right)^{-1} (\log\log \frac{1}{t})^{-\rho}$$

for $t \in [0, e^{-e}]$, and define h to be constant in $[e^{-e}, \infty)$.

Theorem 1.1 is a special case of:

Theorem 1.2
Let $A \geq 1$. Assume that μ is a compactly supported measure on the real line. Let F be the set of points x satisfying

$$\liminf_{meas(I)\to 0, I\ni x} \frac{\mu(I)}{(meas(I))^A} > 0, \tag{4}$$

the limit being taken over intervals I. Then the set of μ-irregular points in F has $h_\beta-$measure 0 for all $\beta > 1$.

Note that the condition (4) holds at every point mass of the measure μ, with $A = 1$, and can hold in sets where μ is singularly continuous. Theorem 1.2 easily yields a result reminiscent of Criterion Λ for regularity of Stahl and Totik

[9, p. 108]:

Theorem 1.3
Assume that μ is a compactly supported measure on the real line. Let $\mathcal{F}$ be the set of points x satisfying

$$\limsup_{meas(I)\to 0, I\ni x} \left| \frac{\log \mu (I)}{\log meas (I)} \right| < \infty. \tag{5}$$

Then the set of μ-irregular points in $\mathcal{F}$ has $h_\beta-$measure 0 for all $\beta > 1$.

The condition (2) asserts subexponential growth of $K_n (x, x)$. For slower sorts of growth of K_n, the exceptional set is of course larger. Let $\{\delta_j\}$ be a decreasing sequence of positive numbers with $\sum_j \delta_j < \infty$. Then it is easily proven that for a.e. $x \in \{\mu' > 0\}$,

$$\lim_{n\to\infty} \frac{\delta_{[\log_2 n]}}{n} K_n (x, x) = 0.$$

Here $[x]$ denotes the greatest integer $\leq x$. A much more difficult question is whether

$$\limsup_{n\to\infty} \frac{1}{n} K_n (x, x) < \infty$$

for a.e. $x \in \{\mu' > 0\}$ or even in subintervals of this set, even when we impose additional conditions such as regularity of the measure μ, and a local condition - see the work of Totik [10], [11] and Simon [8].

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2 Proofs

Proof of Theorem 1.2
Step 1: Reduction to a special case
Fix $\beta > 1$, positive integers ℓ, m, and let $G = G (\ell, m)$ denote the set of points x in F such that

$$\text{x} \in I \text{ and } 0 < meas (I) \leq \frac{1}{m} \Rightarrow \frac{\mu (I)}{(meas (I))^A} > \frac{1}{\ell}. \tag{6}$$

Since

$$F = \bigcup_{\ell,m} G (\ell, m)$$

is a countable union, it suffices to show that $m_{h_\beta} (G (\ell, m)) = 0$ for a single ℓ, m. So in the sequel, we fix ℓ, m and let $G = G (\ell, m)$.

Step 2: The set E_n on which K_n is large
Next let $\eta > 0$ and

$$E_n = \{t : K_n(t,t) > e^{\eta n}\}. \tag{7}$$

As $K_n(t,t)$ is a polynomial of degree $2n-2$, the equation $K_n(t,t) - e^{\eta n} = 0$ has at most $2n-2$ roots. Then E_n consists of at most n disjoint open intervals $\{I_{nj}\}$. Write

$$E_n = \bigcup_j I_{nj}. \tag{8}$$

We have

$$\mu(E_n)\, e^{\eta n} \leq \int_{E_n} K_n(t,t)\, d\mu(t) \leq n,$$

so

$$\mu(E_n) \leq n e^{-\eta n}. \tag{9}$$

Step 3: Divide I_{nj} into disjoint intervals of equal length $\leq \frac{1}{m}$
Fix j, and divide I_{nj} into finitely many, but as few as possible, disjoint open or half open intervals $\{J_{njk}\}$ of equal length, subject to the restriction

$$meas(J_{njk}) \leq \frac{1}{m}. \tag{10}$$

Clearly if $meas(I_{nj}) \leq \frac{1}{m}$, there will be one interval J_{njk} for the given n and j. Otherwise, there will be at most $meas(I_{nj})\, m + 1$ such intervals. If there is more than one J_{njk}, the rightmost interval will have form (c,d), and the rest will all have form $(c,d]$.

Let $\left\{J^*_{njk}\right\}$ denote that subset of $\{J_{njk}\}$ which have non-empty intersection with G. Then the intervals $\left\{J^*_{njk}\right\}$ cover $I_{nj} \cap G$. Moreover, each $x \in I_{nj}$ can lie in at most three of the J^*_{njk}. Then

$$\sum_k \mu\left(J^*_{njk}\right) \leq 3\mu(I_{nj}),$$

while (6) gives

$$\sum_k \mu\left(J^*_{njk}\right) \geq \frac{1}{\ell} \sum_k \left(meas\left(J^*_{njk}\right)\right)^A.$$

Combining these inequalities gives

$$\sum_k \left(meas\left(J^*_{njk}\right)\right)^A \leq 3\ell\mu(I_{nj}). \tag{11}$$

Step 4 Counting the number of $\left\{J^*_{njk}\right\}_{j,k}$
Adding this last inequality over j, gives

$$\sum_{j,k} meas\left(J^*_{njk}\right)^A \leq 3\ell\mu(E_n) \leq 3\ell n e^{-\eta n}, \tag{12}$$

by (9). Let us now suppose that n is so large that

$$\frac{1}{(2m)^A} > 3\ell n e^{-\eta n}. \tag{13}$$

If there is an index j for which there is more than one J^*_{njk}, then by choice of the J_{njk}, at least one will have length at least $\frac{1}{2m}$, and (12) gives a contradiction to (13). It follows that when (12) is satisfied, there is at most one interval J^*_{njk} per j. Consequently, there are at most n such J^*_{njk} in all. Let us relabel the $\left\{J^*_{njk}\right\}$ as simply $\left\{J^{\#}_{nj}\right\}$ and summarize their properties:

(i) There are at most n intervals $\left\{J^{\#}_{nj}\right\}$;

(ii)

$$\sum_j meas\left(J^{\#}_{nj}\right)^A \leq 3\ell n e^{-\eta n}; \tag{14}$$

(iii)

$$E_n \cap G \subset \bigcup_j J^{\#}_{nj}. \tag{15}$$

Step 5 Estimate the Hausdorff measure

Let $\beta > 1$ and

$$E_\infty \cap G = \limsup_{k\to\infty} E_{2^k} \cap G = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} \left(E_{2^k} \cap G\right).$$

Now by (iii), $\left\{J^{\#}_{2^k,j}\right\}_{j,k}$ are intervals that cover $E_{2^k} \cap G$. Moreover, for large enough n, (i), (ii) give

$$\begin{aligned} & \sum_j h_\beta\left(measJ^{\#}_{nj}\right) \\ \leq\quad & n h_\beta\left(\left[3\ell n e^{-\eta n}\right]^{1/A}\right) \leq C n^{1-\beta}, \end{aligned}$$

where C is independent of n, but depends on ε, ℓ. (This very crude estimate suffices for our purposes). Then for large enough N,

$$\begin{aligned} & m_{h_\beta}\left(E_\infty \cap G\right) \\ \leq\quad & \sum_{k=N}^{\infty} \sum_j h_\beta\left(measJ^{\#}_{2^k,j}\right) \\ \leq\quad & C \sum_{k=N}^{\infty} 2^{k(1-\beta)} \to 0,\ N \to \infty, \end{aligned}$$

so $m_{h_\beta}\left(E_\infty \cap G\right) = 0$. Moreover, in $G \backslash E_\infty$, we have for large enough k,

$$K_{2^k}(t,t) \leq e^{\eta\left(2^k\right)}.$$

Thus, in $G \backslash E_\infty$,

$$\limsup_{k \to \infty} K_{2^k}(t,t)^{1/2^k} \le e^\eta.$$

Here $\eta > 0$ is arbitrary. The set E_∞ depends on η but increases as η decreases. By taking suitable countable unions, we deduce that in G, outside a set of h_β measure 0, we have

$$\limsup_{k \to \infty} K_{2^k}(t,t)^{1/2^k} = 1$$

Then Theorem 1.2 follows, recall the discussion after (2).■

Proof of Theorem 1.1
The condition (3) implies (4) with $A = 1$. ■

Proof of Theorem 1.3
It is easily seen that (5) holds iff (4) holds for large enough A. Then Theorem 1.3. follows from Theorem 1.2 by taking a countable union of exceptional sets, corresponding (for example) to integer values of A. ■

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