Jackson and Bernstein Theorems for the Weight $\exp(-|x|)$ on $\mathbb{R}$

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Abstract

In 1978, Freud, Giroux and Rahman established a weighted L_1 Jackson theorem for the weight $\exp(-|x|)$ on the real line, using methods that work only in L_1. This weight is somewhat exceptional, for it sits on the boundary between weights like $\exp(-|x|^\alpha)$, $\alpha \geq 1$, where weighted polynomials are dense, and the case $\alpha < 1$, where weighted polynomials are not dense. We obtain the first L_p Jackson theorem for $\exp(-|x|)$, valid in all L_p, $0 < p \leq \infty$, as well as for higher order moduli of continuity. We also establish a converse Bernstein type theorem, characterizing rates of approximation in terms of smoothness of the approximated function.

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1 Statement of Results

Let $W : \mathbb{R} \to (0, \infty)$. Bernstein's Approximation Problem asks when is it true that for every continuous $f : \mathbb{R} \to \mathbb{R}$ with

$$\lim_{|x| \to \infty} (fW)(x) = 0,$$

there exist a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ with

$$\lim_{n \to \infty} \|(f - P_n) W\|_{L_\infty(\mathbb{R})} = 0.$$

This problem was resolved independently by Pollard, Mergelyan and Achieser in the 1950's, all with the aid of some sort of regularization of W. If $W \leq 1$ and is even, and is regular in the sense that $\ln 1/W(e^x)$ is even and convex, a simple necessary and sufficient condition for density of the polynomials is [11, p. 170]

$$\int_0^\infty \frac{\ln 1/W(x)}{1 + x^2} dx = \infty.$$

In particular, for

$$W_\alpha(x) = \exp(-|x|^\alpha), \tag{1.1}$$

the polynomials are dense iff $\alpha \geq 1$.

Once the density question had been resolved, the natural next step was to look for analogues of the Jackson-Bernstein theorems on the degree (or rate) of approximation. Dzrbasjan took the first steps in the 1950's, with major strides being made by Freud and Nevai in the 1970's. The simplest theorems are the Jackson-Favard inequalities, which involve the derivative of the function being approximated. For the weights W_α, where $\alpha > 1$, and $1 \leq p \leq \infty$, these take the form

$$\begin{aligned} E_n[f; W_\alpha]_p &:= \inf_{P \in \mathcal{P}_n} \|(f - P) W_\alpha\|_{L_p(\mathbb{R})} \\ &\leq C n^{-1+\frac{1}{\alpha}} \|f' W_\alpha\|_{L_p(\mathbb{R})}, \end{aligned} \tag{1.2}$$

where $\mathcal{P}_n$ is the set of polynomials of degree $\leq n$. Here C is independent of f and n [7, p. 185, (11.3.5)] [20, p. 81, (4.1.5a)]. The rate is best possible for the class of absolutely continuous functions f with $\|f'W\|_{L_p(\mathbb{R})}$ finite. Freud proved these for $\alpha \geq 2$, and later E. Levin and the author provided the necessary technical estimates to extend this to all $\alpha > 1$. More general Jackson type theorems involving weighted moduli of continuity for various classes of weights were proved in [2], [5], [7], [8], [10], [16], [17], [20], [23].

One particularly interesting case is $\alpha = 1$, namely $W_1(x) = \exp(-|x|)$. For this weight Bernstein's approximation problem has a positive solution, that is, the polynomials are dense. However, (1.2) suggests that there may not be an analogue of a Jackson theorem, because $n^{-1+\frac{1}{\alpha}}$ has limit 1, as $\alpha \to 1+$. On the other hand, a result of Freud, Giroux and Rahman [9, p. 360] for L_1 suggests that possibly (1.2) is true with $n^{-1+\frac{1}{\alpha}}$ replaced by $\frac{1}{\log n}$.

In a recent paper [18], we characterized those weights on $\mathbb{R}$ that admit a Jackson or Jackson-Favard inequality. In particular, we showed that there is no Jackson-Favard inequality like (1.2) for the weight $\exp(-|x|)$. The reason for the failure of such a theorem, is that there is no suitable estimate for "tails". There does not exist a function $\eta : [0,\infty) \to (0,\infty)$ with limit 0 at ∞, such that there is an inequality of the form

$$\|fW_1\|_{L_p(\mathbb{R}\setminus[-\lambda,\lambda])} \leq \eta(\lambda) \|f'W_1\|_{L_p(\mathbb{R})}$$

valid for all $\lambda > 0$, and for all absolutely continuous functions f with $f(0) = 0$.

Despite this inherent problem for the weight W_1, Freud, Giroux and Rahman [9] did establish an L_1 Jackson theorem back in 1978, by avoiding estimation of the tail. Their technique was a classical one, that involves Christoffel functions, and gives estimates for the rate of one-sided weighted L_1 approximation. They used the modulus of continuity

$$\omega(f,\varepsilon) = \sup_{|h|\leq\varepsilon} \int_{-\infty}^{\infty} |(fW_1)(x+h) - (fW_1)(x)|\, dx + \varepsilon \int_{-\infty}^{\infty} |fW_1|$$

and proved that

$$E_n[f;W_1]_p \leq C\left[\omega\left(f,\frac{1}{\log n}\right)+\int_{|x|\geq\sqrt{n}}|fW|(x)\,dx\right].$$

Here C is independent of f and n. More generally, they allowed the tail integral to be over the range $|x| \geq n^{1-\delta}$, any $\delta \in (0,1)$. Ditzian, the author, Nevai and Totik later extended this result [6] to a characterization of smoothness in L_1, also involving higher order moduli of continuity.

In this paper, we establish the first L_p analogue of this result, by substantially modifying techniques from [5]. We also treat higher order moduli of continuity. The approach involves approximating f by a spline (or piecewise polynomial), representing the piecewise polynomial in terms of certain characteristic functions, and then approximating the characteristic functions (in a suitable sense) by polynomials. For unweighted approximation on a bounded interval, this method has been used extensively [3], [25]. In the weighted setting, this method has also been used in [2], [17]. All previous attempts by this author to adapt the technique of [5] in a simple way failed; our novelty here is to use the reproducing kernel for the weight W_1 to get a peaking kernel, rather than modified Chebyshev polynomials.

Our modulus of continuity is similar to that in [5], [7], [20], and involves two parts, a "main part" and a "tail". The "main part" involves rth symmetric differences over a suitable interval, and the tail involves an error of weighted polynomial approximation over the remainder of the real line. Fix

$$\varepsilon \in (0,1).$$

For $h > 0$, an interval J, and $r \geq 1$, we define the rth symmetric difference

$$\Delta_h^r(f,x,J) := \sum_{i=0}^{r}\binom{r}{i}(-1)^i f\left(x+\frac{rh}{2}-ih\right),$$

provided all arguments of f lie in J, and 0 otherwise. Our rth order modulus of continuity is

$$\omega_{r,p}(f,W_1,t) := \sup_{0<h\leq t} \|W_1\Delta_h^r(f,x,\mathbb{R})\|_{L_p(-\exp(\frac{1-\varepsilon}{t}),\exp(\frac{1-\varepsilon}{t}))} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W_1\|_{L_p(\mathbb{R}\setminus(-\exp(\frac{1-\varepsilon}{t})+1,\exp(\frac{1-\varepsilon}{t})-1))}. \quad (1.3)$$

The inf in the tail is at first disconcerting, but note that it is over polynomials of degree at most $r-1$, not n. Its presence ensures that if $\omega_{r,p}(f,W_1,t)\equiv 0$, then $f\in\mathcal{P}_{r-1}$ — as one expects for an rth order modulus. The modulus of continuity is not easy to assimilate, but still simpler than that for more general Freud weights [5], [7] let alone those for Erdös weights [2], or for exponential weights on $[-1,1]$ [17].

Following is our main Jackson theorem:

Theorem 1 *Let $r\geq 1$ and $0<p\leq\infty$. Let $f:\mathbb{R}\to\mathbb{R}$ and $fW_1\in L_p(\mathbb{R})$. For $p=\infty$, we also require f to be continuous. For $n\geq C_3$,*

$$E_n[f;W_1]_p \leq C_1\omega_{r,p}\left(f,W_1,\frac{1}{\log(C_2n)}\right), \quad (1.4)$$

where C_j, $j=1,2,3$, do not depend on f or n.

When $p\geq 1$, C_2 can be dropped. It is not clear if it can be dropped for $p<1$. We emphasize that there can be no corollary of the form (1.2) with $n^{-1+\frac{1}{\alpha}}$ replaced by $\frac{1}{\log n}$. It is instructive to compare this to the results in [5, Thm 1.2, p. 102] or [7] for $W_\alpha, \alpha>1$. The results there become

$$E_n[f;W_\alpha]_p \leq C_1\omega_{r,p,}(f,W_\alpha,n^{\frac{1}{\alpha}-1}),$$

where now

$$\omega_{r,p}(f,W_\alpha,t) := \sup_{0<h\leq t} \|W_\alpha\Delta_h^r(f,x,\mathbb{R})\|_{L_p(-h^{-\frac{1}{\alpha-1}},h^{-\frac{1}{\alpha-1}})} + \inf_{P\in\mathcal{P}_{r-1}} \|(f-P)W_\alpha\|_{L_p(\mathbb{R}\setminus(-t^{-\frac{1}{\alpha-1}},t^{-\frac{1}{\alpha-1}}))}.$$

We also prove a Bernstein type theorem:

Theorem 2 *Let $r \geq 1$ and $1 \leq p \leq \infty$. Let $0 < \alpha < r$ and $fW \in L_p(\mathbb{R})$. The following are equivalent:*

(a) *As $n \to \infty$,*

$$E_n[f; W_1]_p = O\left((\log n)^{-\alpha}\right). \tag{1.5}$$

(b) *As $h \to 0+$, both*

$$\|W_1 \triangle_h^r(f, x, \mathbb{R})\|_{L_p(\mathbb{R})} = O(h^\alpha) \tag{1.6}$$

and

$$\|W_1 f\|_{L_p\left(|x| \geq \exp\left(\frac{1}{h}\right)\right)} = O(h^\alpha). \tag{1.7}$$

For $p = 1$, this theorem appeared in [6]. The paper is organised as follows: In Section 2, we present some technical estimates. In Section 3, we present our crucial approximations to characteristic functions. We prove Theorem 1 in Section 4 and Theorem 2 in Section 5.

2 Technical Lemmas

We begin with some notation that will be used in the sequel. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x and $P \in \mathcal{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. In Sections 3 and 4, these constants will be independent of a parameter τ as well. We write $C \neq C(L)$ to indicate that C is independent of L. The notation $c_n \sim d_n$ means that $C_1 \leq c_n/d_n \leq C_2$ for the relevant range of n. Similar notation is used for functions and sequences of functions.

We define the Mhaskar-Rakhmanov-Saff numbers a_u, $u > 0$, for the weight W_1 by

$$a_u = \frac{\pi}{2}u, \qquad u > 0. \tag{2.1}$$

One of their features is the Mhaskar-Saff identity

$$\|PW_1\|_{L_\infty(\mathbb{R})} = \|PW_1\|_{L_\infty[-a_n, a_n]},$$

valid for all polynomials P of degree $\leq n$ [15], [21], [22], [26]. These numbers play an important descriptive role in all aspects of weighted approximation, and asymptotics of orthogonal polynomials.

The orthonormal polynomials for the weight W_1^2 are denoted by $\{p_n\}$. Thus

$$\int_{-\infty}^{\infty} p_n(x)\, p_m(x)\, W_1^2(x)\, dx = \delta_{mn}.$$

The positive leading coefficient of p_n is γ_n, and the nth reproducing kernel is

$$\begin{aligned} K_n(x,t) &= \sum_{j=0}^{n-1} p_j(x)\, p_j(t) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(t) - p_{n-1}(x)\, p_n(t)}{x-t}. \end{aligned} \tag{2.2}$$

The nth Christoffel function is

$$\lambda_n\left(W^2, x\right) = 1/K_n(x,x). \tag{2.3}$$

We begin by listing some known general results associated with the weight W_1^2 :

Lemma 2.1 *Let $L > 0$ and $0 < p \leq \infty$.*
(a) *For $n \geq 1$, and polynomials P of degree $\leq Ln$,*

$$\|P' W_1^L\|_{L_p(\mathbb{R})} \leq C \log n \|P W_1^L\|_{L_p(\mathbb{R})}, \tag{2.4}$$

where C depends only on p, L, W.
(b) *For $n \geq 1$ and polynomials of degree $\leq Ln$,*

$$\|P W_1^L\|_{L_p(\mathbb{R})} \leq C \|P W_1^L\|_{L_p[-a_n, a_n]}. \tag{2.5}$$

Moreover, if $r > 1$,

$$\|P W_1^L\|_{L_p(\mathbb{R} \setminus [-a_{rn}, a_{rn})} \leq C_1 e^{-C_2 n} \|P W_1^L\|_{L_p[-a_n, a_n]}. \tag{2.6}$$

Here C, C_1, C_2 depend only on p, L, W.

Proof

(a) For $L = 1$, this was first proved in Nevai and Totik [24], see also [26, Theorem VI.5.5, p. 338]. The case of general L follows by a substitution $x \to Lx$ in the integrals defining the L_p norm.

(b) Firstly (2.5) is a special case of Theorem 1.8 in [13, p. 469], while (2.6) follows, for example, from Theorem VI.5.1 in [26, p. 334]. □

Next, we list bounds on orthogonal polynomials, and reproducing kernels:

Lemma 2.2

(a) *For* $n \geq 1$ *and* $t \in \mathbb{R}$,

$$|p_n W_1|(t) \leq C n^{-1/2} \left(\left| 1 - \frac{|t|}{a_n} \right| + n^{-2/3} \right)^{-1/4}. \tag{2.7}$$

(b) *Uniformly for* $n \geq 1$ *and* $|t| \leq a_n \left(1 + L n^{-2/3}\right)$,

$$K_n(t,t) W_1^2(t) \sim \log \frac{\pi n}{1 + |t|}. \tag{2.8}$$

(c) *For* $n \geq 1$,

$$\sup_{x,t \in \mathbb{R}} |K_n(x,t)| W_1(x) W_1(t) \sim \log n \tag{2.9}$$

and

$$\sup_{t \in \mathbb{R}} K_n(t,t) W_1^2(t) \sim \log n. \tag{2.10}$$

(d) *For* $n \geq 1$ *and* $x, t \in [-2a_n, 2a_n]$,

$$\begin{aligned} &|K_n(x,t)| W_1(x) W_1(t) \\ &\leq C \frac{\log n}{1 + \log n \left| x - t \right|} \left(\left| 1 - \frac{|x|}{a_n} \right| + n^{-2/3} \right)^{-1/4} \\ &\quad \times \left(\left| 1 - \frac{|t|}{a_n} \right| + n^{-2/3} \right)^{-1/4}. \end{aligned} \tag{2.11}$$

(e) *Let* $\varepsilon > 0$. *There exists* $C > 0$ *such that for* $|t| \le n^{1-\varepsilon}$ *and* $|x-t| \le \frac{C}{\log n}$,

$$\frac{K_n(x,t)}{K_n(t,t)} \ge \frac{1}{2}. \tag{2.12}$$

Proof

(a) Let $\varepsilon \in (0,1)$. It was shown in [14, Corollary 1.4, p. 222], that $n \ge 1$ and $|x| \in [\varepsilon a_n, a_n]$,

$$|p_n(x) W_1(x)| \le C n^{-1/2} \left(\left| 1 - \frac{|x|}{a_n} \right| + n^{-2/3} \right)^{-1/4}.$$

A sharper asymptotic for p_n has been given in [12, Theorem 1.16, p. 303]. For $|x| \le \varepsilon a_n$, where ε is small, the bound can be deduced from the asymptotic in [12]. In Theorem 1.16(v) (with $\beta = 1$ and the range D_δ there), they establish a uniform asymptotic of the form

$$p_n(a_n x) W_1(a_n x) = \sqrt{\frac{2}{\pi n}} \left(1 - x^2\right)^{-1/4} \left(\cos \phi_n(x) + O\left(\frac{1}{\log n}\right)\right),$$

valid for $x \in [-\varepsilon, \varepsilon]$. Here ϕ_n is an explicitly given real valued function. This implies

$$|p_n(x) W_1(x)| \le C n^{-1/2}, |x| \le \varepsilon a_n.$$

Together these two estimates give for $x \in [-a_n, a_n]$,

$$\left| p_n^8(x) \right| \left| \left(1 - \left(\frac{x}{a_n} \right)^2 \right)^2 + n^{-4/3} \right| W_1^8(x) \le C n^{-4}. \tag{2.13}$$

The polynomial

$$P(x) = p_n^8(x) \left[\left(1 - \left(\frac{x}{a_n} \right)^2 \right)^2 + n^{-4/3} \right]$$

is of degree $m = 8n + 4$. As a_n for W_1 is a_{8n} for the weight W_1^8, we can use Theorem 1.8 in [13, p. 469], to deduce that

$$\|P W_1^8\|_{L_\infty(\mathbb{R})} \le C \|P W_1^8\|_{L_\infty\left(|x| \le a_{m/8}\left(1 - C m^{-2/3}\right)\right)}.$$

Since

$$a_{m/8}\left(1-Cm^{-2/3}\right)=a_{n+1/2}\left(1-Cm^{-2/3}\right)\leq a_n,$$

for large n, we see that (2.13) holds for all $x\in\mathbb{R}$. So we obtain (2.7).
(b) The estimate (2.8) for the Christoffel function was established in [14, Theorem 1.1, p. 221]. (For a smaller range of x, this was established in the 1978 paper of Freud, Giroux and Rahman [9].)
(c) Cauchy-Schwarz gives for $|x|,|t|\leq a_n$,

$$\begin{aligned}\left|K_n(x,t)W_1(x)W_1(t)\right| &\leq \left[K_n(x,x)W_1^2(x)\right]^{1/2}\left[K_n(t,t)W_1^2(t)\right]^{1/2}\\ &\leq C\log n.\end{aligned}$$

The restricted range inequality Lemma 2.1(b), applied separately in x,t, then gives this result for all $x,t\in\mathbb{R}$. In particular, for all $t\in\mathbb{R}$,

$$K_n(t,t)W_1^2(t)\leq C\log n.$$

Then we have the upper bounds implicit in (2.9) and (2.10). Finally (2.8) gives the matching lower bounds required for the $\sim$ relation in (2.9), (2.10).
(d) It is known [19] that

$$\frac{\gamma_{n-1}}{\gamma_n}=\frac{1}{2}a_n\left(1+o(1)\right),$$

and more precise asymptotics are given in [12]. Then

$$\frac{\gamma_{n-1}}{\gamma_n}\sim n,$$

so (a), (c) and the Christoffel-Darboux formula (2.2) give for all $x,t\in\mathbb{R}$,

$$\begin{aligned}&\left|K_n(x,t)W_1(x)W_1(t)\right|\\ &\quad\leq C\min\left\{\log n,\frac{1}{|x-t|}\left(\left|1-\frac{|x|}{a_n}\right|+n^{-2/3}\right)^{-1/4}\right.\\ &\qquad\left.\times\left(\left|1-\frac{|t|}{a_n}\right|+n^{-2/3}\right)^{-1/4}\right\}.\end{aligned}$$

For $|x|, |t| \leq 2a_n$, this implies (2.11).

(e) We use (2.9) and the Bernstein-inequality Lemma 2.1(a). For $|x-t| \leq 1$, and some ξ between x and t,

$$
\begin{aligned}
&|K_n(x,t) W_1(x) W_1(t) - K_n(t,t) W_1(x) W_1(t)| \\
&= \left| (x-t) W_1(\xi) \frac{\partial}{\partial x} K_n(x,t)_{|x=\xi} \right| W_1^{-1}(\xi) W_1(x) W_1(t) \\
&\leq |x-t| C \log n \sup_{y \in \mathbb{R}} |W_1(y) K_n(y,t)| W_1(t) \\
&\leq C |x-t| (\log n)^2 .
\end{aligned}
$$

Since

$$K_n(t,t) W_1(x) W_1(t) \geq C K_n(t,t) W_1^2(t) \geq C_1 \log n,$$

for $|t| \leq n^{1-\varepsilon}$, we obtain for such t,

$$\frac{K_n(x,t)}{K_n(t,t)} \geq 1 - C|x-t| \log n.$$

For

$$|x-t| \leq \frac{1}{2C \log n},$$

we then obtain (2.12). □

3 Polynomials approximating characteristic functions

Our Jackson theorem is based on polynomial approximations to the characteristic function $\chi_{[a,b]}$ of an interval $[a,b]$. We believe the following result is of independent interest, because the technique used in [5, Theorem 4.1, p. 117], based on the peaking polynomials $V_{n,\xi}$ totally fails.

Theorem 3 *Let L, l be even positive integers with $L \geq \ell + 2$ and $L > \frac{6}{5}\ell$. Let $\varepsilon \in (0,1)$. There exists n_0 such that for $n \geq n_0$ and*

$|\tau| \leq n^{1-\varepsilon}$, *there exist polynomials* $R_{n,\tau}$ *of degree at most* Ln *such that for* $x \in \mathbb{R}$,

$$|\chi_{[\tau,2n^{1-\varepsilon}]} - R_{n,\tau}|(x)W_1^L(x)/W_1^L(\tau) \leq C_1(1+\log n\,|x-\tau|)^{-\ell}. \quad (3.1)$$

We emphasise that the constants n_0 and C_1 are independent of n, τ, x. The sequence $\{\log n\}_{n=n_0}^{\infty}$ cannot be replaced by a faster growing sequence, as this would lead to a faster Jackson rate, which would contradict Theorem 2.

Lemma 3.1 *Let* $\varepsilon > 0$. *There exists* n_0 *such that for* $n \geq n_0$ *and* $0 \leq \tau \leq n^{1-\varepsilon}$,

$$W_1^{2L}(\tau)\int_{-1}^{\tau+1} K_n(s,\tau)^L\,ds \geq C(\log n)^{L-1}. \quad (3.2)$$

Proof This follows as the integrand is nonnegative, and since for $|s-\tau| \leq \frac{C}{\log n}$,

$$K_n(s,\tau) \geq \frac{1}{2}K_n(\tau,\tau) \geq CW_1^{-2}(\tau)\log n,$$

by (2.8) and (2.12). Here C is independent of s, τ, n. □

The Proof of Theorem 3 for $\tau \in [0, n^{1-\varepsilon}]$.

We fix $\tau \in [0, n^{1-\varepsilon}]$ and set

$$R_{n,\tau}(x) := \frac{\int_{-1}^{x} K_n(s,\tau)^L\,ds}{\int_{-1}^{\tau+1} K_n(s,\tau)^L\,ds},$$

a polynomial of degree $\leq L(n-1)+1 \leq Ln$. We also set

$$\Delta(x) = |\chi_{[\tau,2n^{1-\varepsilon}]} - R_{n,\tau}|(x)W_1^L(x)/W_1^L(\tau),$$

and consider several different ranges of x, taking account of whether $x < \tau$ or $x > \tau$, and whether or not $x \in [\tau, 2n^{1-\varepsilon}]$. We frequently use the fact that for $|s|, |\tau| \leq \frac{1}{2}a_n = \frac{\pi}{4}n$, Lemma 2.2(d) simplifies to

$$|K_n(s,\tau)|\,W_1(s)\,W_1(\tau) \leq C\frac{\log n}{1+\log n\,|s-\tau|}. \quad (3.3)$$

Range I: $x \in [0, \tau)$

Here

$$\Delta(x) = \frac{W_1(x)^L W_1(\tau)^L \int_{-1}^{x} K_n(s,\tau)^L ds}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds}$$
$$\leq C(\log n) \int_{-1}^{x} \left(\frac{1}{1+\log n\,|s-\tau|}\right)^L e^{L(s-x)} ds,$$

by Lemma 3.1, and (3.3). We have also used the fact that $W_1^{\pm 1}$ are bounded in $[-1, 0]$. We continue this as

$$\leq C(\log n) \int_{-\infty}^{x} \left(\frac{1}{1+\log n\,(\tau-s)}\right)^L ds$$
$$\leq C(1+\log n\,(\tau-x))^{-L+1},$$

recall that $\tau > x$. Since $L \geq \ell + 2$, we obtain (3.1).

Range II: $x \in [\tau, \tau+1)$

Here for n large enough, as $\tau + 1 \leq \frac{1}{2}a_n$, and $\tau + 1 \leq 2n^{1-\varepsilon}$, Lemma 3.1 and (3.3) again give

$$\Delta(x) = \frac{W_1(x)^L W_1(\tau)^L \int_{x}^{\tau+1} K_n(s,\tau)^L ds}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds}$$
$$\leq C(\log n) \int_{x}^{\tau+1} \left(\frac{1}{1+\log n\,|s-\tau|}\right)^L e^{L(s-x)} ds$$
$$\leq \frac{C(\log n)}{(1+\log n\,(x-\tau))^{L-2}} \int_{-\infty}^{\infty} \left(\frac{1}{1+\log n\,|s-\tau|}\right)^2 dx$$
$$\leq \frac{C}{(1+\log n\,(x-\tau))^{\ell}},$$

as $L \geq \ell + 2$.

Range III: $x \in (\tau+1, 2n^{1-\varepsilon}]$

We assume that n is so large that $2n^{1-\varepsilon} \leq \frac{1}{2}a_n$. Here Lemma 3.1 and (3.3) again give

$$\begin{aligned}
\Delta(x) &= \frac{W_1(x)^L W_1(\tau)^L \int_{\tau+1}^{x} K_n(s,\tau)^L ds}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds} \\
&\leq C(\log n) \int_{\tau+1}^{x} \left(\frac{1}{1+\log n\,|s-\tau|}\right)^L e^{L(s-x)} ds \\
&\leq C(\log n) \left[\begin{array}{c} \left(\frac{1}{1+\log n}\right)^L \int_{-\infty}^{\frac{x+\tau}{2}} e^{L(s-x)} ds \\ + \left(\frac{1}{1+\log n\left|\frac{x-\tau}{2}\right|}\right)^{L-2} \int_{\frac{x+\tau}{2}}^{x} \left(\frac{1}{1+\log n|s-\tau|}\right)^2 ds \end{array}\right] \\
&\leq C(\log n)^{-L+1} e^{L\frac{\tau-x}{2}} + C\left(\frac{1}{1+\log n\left|\frac{x-\tau}{2}\right|}\right)^{L-2}.
\end{aligned}$$

Since the function $e^{-Lu}u^{L-2}$ is bounded for $u \in [\frac{1}{2}, \infty)$, while $\frac{x-\tau}{2} \geq \frac{1}{2}$, we can continue this as

$$\begin{aligned}
&\leq C(\log n)^{-L+2} |x-\tau|^{-L+2} \\
&\leq C(1+\log n\,|x-\tau|)^{-\ell},
\end{aligned}$$

recall that $L \geq \ell + 2$.

Range IV: $x \in (2n^{1-\varepsilon}, 2a_n]$

Here we split

$$\begin{aligned}
\Delta(x) &= \frac{W_1(x)^L W_1(\tau)^L \left(\int_{-1}^{\frac{x+\tau}{2}} + \int_{\frac{x+\tau}{2}}^{\frac{1}{2}a_n} + \int_{\frac{1}{2}a_n}^{x}\right) K_n(s,\tau)^L ds}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds} \\
&=: T_1 + T_2 + T_3.
\end{aligned}$$

If $x \leq \frac{1}{2}a_n$, we drop the third integral and replace $\frac{1}{2}a_n$ by x in the integral in T_2. The terms T_1 and T_2 can be handled much as before, using Lemma 3.1 and (3.3). We also use that $x-\tau \geq x/2 \geq n^{1-\varepsilon}$.

$$
\begin{aligned}
& T_1 + T_2 \\
& \leq C\,(\log n)\left(\int_{-1}^{\frac{x+\tau}{2}} + \int_{\frac{x+\tau}{2}}^{\frac{1}{2}a_n}\right)\left(\frac{1}{1+\log n\,|s-\tau|}\right)^L e^{L(s-x)}ds \\
& \leq C\,(\log n)\left[\int_{-\infty}^{\frac{x+\tau}{2}} e^{L(s-x)}ds + \left(\frac{1}{1+\log n\left|\frac{x-\tau}{2}\right|}\right)^{L-2}\right. \\
& \qquad \left. \times \int_{\frac{x+\tau}{2}}^{\infty}\left(\frac{1}{1+\log n\,|s-\tau|}\right)^2 ds\right] \\
& \leq C\,(\log n)\, e^{L\frac{\tau-x}{2}} + C\left(\frac{1}{1+\log n\left|\frac{x-\tau}{2}\right|}\right)^{L-2} \\
& \leq C\,(\log n)\, e^{-Ln^{1-\varepsilon}/2} + C\left(\frac{1}{1+\log n\left|\frac{x-\tau}{2}\right|}\right)^{L-2} \\
& \leq C\,(1+\log n\,|x-\tau|)^{-\ell},
\end{aligned}
$$

since $L \geq \ell + 2$ and

$$1 + \log n\,|x-\tau| \leq Cn\log n.$$

Next, in T_3, we cannot drop the factor involving $\left|1-\frac{|s|}{a_n}\right| + n^{-2/3}$ from the estimate in (2.11), but $s-\tau \geq \frac{1}{2}a_n - n^{1-\varepsilon} \sim n$, so

$$
\begin{aligned}
T_3 & \leq C\,(\log n)\int_{\frac{1}{2}a_n}^{x}\left(\frac{\left(\left|1-\frac{|s|}{a_n}\right| + n^{-2/3}\right)^{-1/4}}{1+\log n\,|s-\tau|}\right)^L e^{L(s-x)}ds \\
& \leq C\,(\log n)\left(\frac{n^{1/6}}{n\log n}\right)^L \int_{-\infty}^{x} e^{L(s-x)}ds \\
& \leq C\,(\log n)^{1-L}\, n^{-\frac{5}{6}L} \\
& \leq C\,(n\log n)^{-\ell} \leq C(1+(\log n)\,|x-\tau|)^{-\ell},
\end{aligned}
$$

as $\frac{5}{6}L > \ell$ and $L > \ell + 1$, while $|x-\tau| \leq Cn$.

Range V: $x \in [-\frac{1}{2}a_n, 0)$

Here

$$\Delta(x) = \frac{W_1(x)^L W_1(\tau)^L \left|\int_{-1}^{x} K_n(s,\tau)^L ds\right|}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds}$$
$$\leq C(\log n)\left|\int_{-1}^{x}\left(\frac{1}{1+\log n(|s|+\tau)}\right)^L e^{L(x-s)} ds\right| \qquad (3.4)$$

If $x \in [-1,0]$, we have $|s| \geq x$ in the integral, so we continue this as

$$\leq C(\log n)\left(\frac{1}{1+\log n(|x|+\tau)}\right)^{L-2}\int_{-1}^{x}\left(\frac{1}{1+\log n|s|}\right)^2 ds$$
$$\leq C(1+\log n|x-\tau|)^{-\ell}.$$

If $x \in \left[-\frac{1}{2}a_n, -1\right]$, we instead continue (3.4) as

$$\Delta(x) \leq C(\log n)\begin{pmatrix} \left(\frac{1}{1+\log n(|x|+\tau)}\right)^{L-2}\int_{x}^{x/2}\left(\frac{1}{1+\log n|s|}\right)^2 ds \\ +\left(\frac{1}{1+\log n(1+\tau)}\right)^{L-2} e^{Lx/2}\int_{x/2}^{-1}\left(\frac{1}{1+\log n|s|}\right)^2 ds \end{pmatrix}$$
$$\leq C\Bigg(\left(\frac{1}{1+\log n(|x|+\tau)}\right)^{L-2}$$
$$+\left(\frac{1}{1+\log n(1+\tau)}\right)^{L-2} e^{Lx/2}\Bigg). \qquad (3.5)$$

Here we have as $x \leq -1$,

$$e^{Lx/2} \leq C(1+|x|)^{-(L-2)}$$

while

$$(1+|x|)(1+\tau) > |x|+\tau,$$

and hence we can continue (3.5) as

$$\Delta(x) \leq C\left(\frac{1}{1+\log n(|x|+\tau)}\right)^{L-2} \leq C(1+(\log n)|x-\tau|)^{-\ell}.$$

Range VI: $x \in [-2a_n, -\frac{1}{2}a_n)$

Here

$$\Delta(x) = \frac{W_1(x)^L W_1(\tau)^L \left(\int_{-\frac{1}{4}a_n}^{-1} + \int_x^{-\frac{1}{4}a_n}\right) K_n(s,\tau)^L ds}{W_1(\tau)^{2L} \int_{-1}^{\tau+1} K_n(s,\tau)^L ds}$$
$$=: T_1 + T_2. \tag{3.6}$$

As before,

$$\begin{aligned} T_1 &\le C(\log n)\left(\frac{1}{1+\log n(1+\tau)}\right)^{L-2} e^{Lx/2} \\ &\quad \times \int_{-\frac{1}{4}a_n}^{-1} \left(\frac{1}{1+\log n|s|}\right)^2 ds \\ &\le C\left(\frac{1}{1+\log n(1+\tau)}\right)^{L-2} (1+|x|)^{-(L-2)} \\ &\le C\left(\frac{1}{1+\log n(|x|+\tau)}\right)^{L-2} \le C(1+(\log n)|x-\tau|)^{-\ell}. \end{aligned}$$

Next,

$$\begin{aligned} T_2 &\le C(\log n)\int_x^{-\frac{1}{4}a_n} \left(\frac{\left(\left|1-\frac{|s|}{a_n}\right| + n^{-2/3}\right)^{-1/4}}{1+\log n(|s|+\tau)}\right)^L e^{L(x-s)} ds \\ &\le C(\log n)\left(\frac{n^{1/6}}{n\log n}\right)^L \int_x^{\infty} e^{L(x-s)} ds \\ &\le C(\log n)^{1-L} n^{-\frac{5}{6}L} \\ &\le C(n\log n)^{-\ell} \le C(1+(\log n)|x-\tau|)^{-\ell}. \end{aligned}$$

In summary, we have proved that for $x \in [-2a_n, 2a_n]$,

$$|\chi_{[\tau,2n^{1-\varepsilon}]} - R_{n,\tau}|(x) W_1^L(x)/W_1^L(\tau) \le C(1+(\log n)|x-\tau|)^{-\ell}.$$

Since $\chi_{[\tau,2n^{1-\varepsilon}]}$ vanishes outside $[\tau, 2n^{1-\varepsilon}]$ and $W_1^L(x)/W_1^L(\tau) \le 1$ for $x \in [\tau, 2n^{1-\varepsilon}]$, this also gives

$$\left|R_{n,\tau}(x) W_1^{-L}(\tau)\right| W_1^L(x) \le C, \qquad x \in [-2a_n, 2a_n],$$

for some C independent of x, τ, n. Then, recalling that $a_{2n} = 2a_n$,

$$\left|\left(\frac{x}{a_{2n}}\right)^n R_{n,\tau}(x) W_1^{-L}(\tau)\right| W_1^L(x) \leq C, \qquad x \in [-a_{2n}, a_{2n}].$$

As $\left(\frac{x}{a_{2n}}\right)^n R_{n,\tau}(x)$ has degree $\leq (L+1)n = \delta L2n$, with $\delta = \frac{L+1}{2L} < 1$, (2.6) of Lemma 2.1(b) then gives

$$\left|\left(\frac{x}{a_{2n}}\right)^n R_{n,\tau}(x) W_1^{-L}(\tau)\right| W_1^L(x) \leq e^{-C_1 n}, |x| \geq a_{2n}.$$

Then for $|x| \geq 2a_n$, we obtain

$$\begin{aligned}\left|R_{n,\tau}(x) W_1^{-L}(\tau)\right| W_1^L(x) &\leq e^{-C_1 n - n\log\frac{|x|}{a_{2n}}} \\ &\leq C(1 + (\log n)|x-\tau|)^{-\ell}.\end{aligned}$$

Thus (3.1) is true for all $x \in \mathbb{R}$. □

Proof of Theorem 3 for negative τ

Let $\tau \in (0, n^{1-\varepsilon})$. We set

$$R_{n,-\tau}(x) := 1 - R_{n,\tau}(-x), \qquad x \in \mathbb{R}.$$

Note the identity

$$\chi_{[-\tau,2n^{1-\varepsilon}]}(x) = 1 - \chi_{(\tau,2n^{1-\varepsilon}]}(-x), \qquad x \in \left[-2n^{1-\varepsilon}, 2n^{1-\varepsilon}\right].$$

Then for $|x| \leq 2n^{1-\varepsilon}$, $x \neq -\tau$,

$$\begin{aligned}&\left|\chi_{[-\tau,2n^{1-\varepsilon}]}(x) - R_{n,-\tau}(x)\right| W_1^L(x)/W_1^L(-\tau) \\ &= \left|\chi_{(\tau,2n^{1-\varepsilon}]}(-x) - R_{n,\tau}(-x)\right| W_1^L(-x)/W_1^L(\tau) \\ &\leq C(1 + \log n\,|(-x) - \tau|)^\ell \\ &= C(1 + \log n\,|x - (-\tau)|)^\ell.\end{aligned}$$

This estimate also holds for $x = -\tau$, since $R_{n,\tau}$ is continuous. For $|x| > 2n^{1-\varepsilon}$,

$$\begin{aligned}&\left|\chi_{[-\tau,2n^{1-\varepsilon}]}(x) - R_{n,-\tau}(x)\right| W_1^L(x)/W_1^L(-\tau) \\ &= |1 - R_{n,\tau}(-x)| W_1^L(x)/W_1^L(\tau) \qquad (3.7) \\ &\leq W_1^L(x)/W_1^L(\tau) + |R_{n,\tau}(-x)| W_1^L(-x)/W_1^L(\tau) \\ &\leq C(1 + \log n\,|x - (-\tau)|)^\ell,\end{aligned}$$

by what we proved for $\tau > 0$, and since $|x| - \tau > \frac{|x|}{2} > n^{1-\varepsilon}$. $\square$

Remark

In [5, p. 118, last line], there was a small mistake, the 1 inside the modulus was omitted in the analogue of (3.7). But this is easily fixed. $\square$

In the next section, we shall use the following consequence of Theorem 3:

Theorem 4 *Let ℓ be an even positive integer, $A > 1$ and $\varepsilon \in (0, 1)$. There exist $L > 0$ and n_0 such that for $n \geq n_0$ and $|\tau| \leq n^{1-\varepsilon}$, there exist polynomials $S_{n,\tau}$ of degree at most Lnsuch that for $x \in \mathbb{R}$,*

$$|\chi_{[\tau, An^{1-\varepsilon}]} - S_{n,\tau}|(x) W_1(x)/W_1(\tau) \leq C_1(1 + \log n\, |x - \tau|)^{-\ell}. \quad (3.8)$$

Here C_1 is independent of n, τ, x.

Proof

By Theorem 3, there exist polynomials $R_{n,\tau}$ of degree$\leq Ln$ with

$$|\chi_{[\tau, 2n^{1-\varepsilon}]}(x) - R_{n,\tau}(x)| e^{-L|x| + L|\tau|} \leq C_1(1 + \log n\, |x - \tau|)^{-\ell},$$

for all $x \in \mathbb{R}$. We now make the substitutions

$$y = Lx; \ \sigma = L\tau; \ S_{n,\sigma}(y) = R_{n,\tau}(x).$$

As

$$\chi_{[\tau, 2n^{1-\varepsilon}]}(x) = \chi_{[\sigma, 2Ln^{1-\varepsilon}]}(y),$$

we obtain for $y \in \mathbb{R}, |\sigma| \leq Ln^{1-\varepsilon}$,

$$|\chi_{[\sigma, 2Ln^{1-\varepsilon}]}(y) - S_{n,\sigma}(y)|(x) e^{-|y| + |\sigma|} \leq C_1(1 + \log n\, |y - \sigma|)^{-\ell}. \quad (3.9)$$

In Theorem 3, L could be as large as we please. We assume $L > A/2$ and restrict $|\sigma| \leq n^{1-\varepsilon}$. We obtain for $y \in \mathbb{R} \backslash (An^{1-\varepsilon}, 2Ln^{1-\varepsilon}]$, and such σ,

$$|\chi_{[\sigma, An^{1-\varepsilon}]}(y) - S_{n,\sigma}(y)|(x) W_1(y) / W_1(\sigma) \leq C_1(1 + \log n\, |y - \sigma|)^{-\ell}. \quad (3.10)$$

For $y \in (An^{1-\varepsilon}, 2Ln^{1-\varepsilon}]$, we obtain instead from (3.9),

$$\begin{aligned}
&|\chi_{[\sigma,An^{1-\varepsilon}]}(y) - S_{n,\sigma}(y)|(x)W_1(y)/W_1(\sigma) \\
&\quad = |S_{n,\sigma}(y)|(x)W_1(y)/W_1(\sigma) \\
&\quad \leq e^{-|y|+|\sigma|} + C_1(1+\log n\,|y-\sigma|)^{-\ell}.
\end{aligned}$$

Since $y - \sigma \geq \left(1 - \frac{1}{A}\right) y \geq \frac{A-1}{A} n^{1-\varepsilon}$, we again get (3.10). Thus (3.10) holds for all $y \in \mathbb{R}$. Finally, replace σ by τ and y by x to get the result. □

4 The Proof of Theorem 1

In this section, we prove Theorem 1. We need some unweighted, semi-classical moduli of continuity. If I is an interval, and $f : I \to \mathbb{R}$, we define for $t > 0$, $0 < p < \infty$,

$$\Lambda_{r,p}(f,t,I) := \sup_{0<h\leq t} \left(\int_I |\Delta_h^r(f,x,I)|^p dx\right)^{1/p}$$

and its averaged cousin

$$\Omega_{r,p}(f,t,I) := \left(\frac{1}{t}\int_0^t \int_I |\Delta_s^r(f,x,I)|^p dx\, ds\right)^{1/p}.$$

(There are obvious modifications if $p = \infty$.) Note that for some C_1, C_2 depending only on r and p, (not on f, I, t) [4], [25, Lemma 7.2, p. 191],

$$C_1 \leq \Lambda_{r,p}(f,t,I)/\Omega_{r,p}(f,t,I) \leq C_2. \tag{4.1}$$

For large enough n, we choose a partition by equally spaced points,

$$-n^{1-\varepsilon} = \tau_{0,n} < \tau_{1,n} < \cdots < \tau_{m,n} = n^{1-\varepsilon},$$

where for each $0 \leq j \leq m-1$,

$$\tau_{j+1,n} - \tau_{j,n} = h = \frac{2n^{1-\varepsilon}}{[10n^{1-\varepsilon}\log n]}$$

and where $[x]$ denotes the greatest integer $\leq x$. Since $mh = 2n^{1-\varepsilon}$, we see that

$$m = [10n^{1-\varepsilon} \log n]$$

and

$$\frac{1}{5\log n} \leq h \leq \frac{1}{5\log n - 1/(2n^{1-\varepsilon})}.$$

We set

$$I_{kn} := [\tau_{kn}, \tau_{k+1,n}], \qquad 0 \leq k \leq m-1,$$

so that for $k \leq m-1$, and large n,

$$\frac{1}{5\log n} \leq |I_{kn}| \leq \frac{1}{4\log n}. \tag{4.2}$$

($|I|$ denotes the length of the interval I.) We also set $I_{mn} := I_{m-1,n}$. Let us set

$$I_n := [-n^{1-\varepsilon}, n^{1-\varepsilon}] = \bigcup_{k=0}^{m-1} I_{kn}$$

and

$$\theta_{kn}(x) := \chi_{[\tau_{kn}, 2n^{1-\varepsilon}]}(x) = \chi_{(n^{1-\varepsilon}, 2n^{1-\varepsilon}]} + \chi_{\bigcup_{i=k}^{m-1} I_{in}}(x).$$

We set

$$I^*_{kn} := I_{kn} \cup I_{k+1,n}, \qquad 0 \leq k \leq m-1,$$

and $I^*_{mn} = I^*_{m-1,n}$. By Whitney's theorem [25, p. 195], we can find a polynomial p_k of degree at most r, such that

$$\|f - p_k\|_{L_p(I^*_{kn})} \leq C_2 \Lambda_{r,p}(f, |I^*_{kn}|, I^*_{kn}) \leq C_3 \Omega_{r,p}(f, |I^*_{kn}|, I^*_{kn}) \tag{4.3}$$

with C_2, C_3 independent of f, n, k, I^*_{kn}.

Now define an approximating piecewise polynomial/spline by

$$L_n[f](x) := p_0(x)\theta_{0n}(x) + \sum_{k=1}^{m-1} (p_k - p_{k-1})(x)\theta_{kn}(x) - p_{m-1}\theta_{mn}(x). \tag{4.4}$$

We first show that $L_n[f]$ is a good approximation to f:

Lemma 4.1 *For $0 < p < \infty$,*

$$\begin{aligned}
&\|(f - L_n[f])W_1\|^p_{L_p(\mathbb{R})} \\
&\leq C_1[\log n \int_0^{1/(2\log n)} \|W_1\Delta_s^r(f,x,\mathbb{R})\|^p_{L_p[-n^{1-\varepsilon},n^{1-\varepsilon}]}ds \\
&\quad + \|fW_1\|^p_{L_p(|x|\geq n^{1-\varepsilon})}]. \qquad (4.5)
\end{aligned}$$

Here $C_j \neq C_j(f,n)$, $j = 1,2,3$. For $p = \infty$, (4.5) holds if we remove the exponents p.

Proof

We first deal with $p < \infty$. Now

$$\|(f - L_n[f])W_1\|^p_{L_p(\mathbb{R})} = \sum_{j=0}^{m} \Delta_{jn} + \|fW_1\|^p_{L_p(\mathbb{R}\setminus[-n^{1-\varepsilon},2n^{1-\varepsilon}])}, \qquad (4.6)$$

where for $0 \leq j \leq m-1$,

$$\Delta_{jn} := \int_{I_{jn}} |f - L_n[f]|^p W_1^p,$$

while

$$\Delta_{mn} = \int_{n^{1-\varepsilon}}^{2n^{1-\varepsilon}} |f - L_n[f]|^p W_1^p.$$

Note that for $j \leq m-1$, in $(\tau_{jn}, \tau_{j+1,n})$, $L_n[f] = p_j$, so that Whitney's Theorem (4.3) gives

$$\begin{aligned}
\Delta_{jn} &= \int_{I_{jn}} |f - p_j|^p W_1^p \\
&\leq \|W_1\|^p_{L_\infty(I_{jn})} C_3^p \Omega^p_{r,p}(f, |I^*_{jn}|, I^*_{jn}) \\
&\leq \|W_1\|^p_{L_\infty(I^*_{jn})} \|W_1^{-1}\|^p_{L_\infty(I^*_{jn})} \frac{C_4}{|I^*_{jn}|} \int_0^{|I^*_{jn}|} \\
&\qquad \times \int_{I^*_{jn}} |W_1\Delta_s^r(f,x,I^*_{jn})|^p dx\ ds.
\end{aligned}$$

Since $C/(\log n) \leq \left|I_{jn}^*\right| \leq 1/(2\log n)$, we can continue this as

$$\leq C_5 \log n \int_0^{1/(2\log n)} \int_{I_{jn}^*} |W_1 \Delta_s^r(f, x, I_{jn}^*)|^p dx\, ds.$$

Adding over j gives

$$\sum_{j=0}^{m-1} \Delta_{jn} \leq C_5 \log n \int_0^{1/(2\log n)} \int_{I_n} |W_1 \Delta_s^r(f, x, \mathbb{R})|^p dx\, ds. \tag{4.7}$$

Next, in $(n^{1-\varepsilon}, 2n^{1-\varepsilon})$, $L_n[f] = 0$, so

$$\Delta_{mn} = \int_{n^{1-\varepsilon}}^{2n^{1-\varepsilon}} |fW_1|^p.$$

This, (4.6) and (4.7) give the result. Note that we have also effectively shown that

$$\sum_{j=0}^{m-1} \Omega_{r,p}^p(f, |I_{jn}^*|, I_{jn}^*) W_1^p(\tau_{jn})$$
$$\leq C_5 \log n \int_0^{1/(2\log n)} \int_{I_n} |W_1 \Delta_s^r(f, x, \mathbb{R})|^p dx\, ds. \tag{4.8}$$

For $p = \infty$, the proof is similar, but easier: We see that

$$\|(f - L_n[f])W_1\|_{L_\infty(\mathbb{R})}$$
$$\leq \max\left\{\max_{0\leq j\leq m-1} \|(f-p_j)W_1\|_{L_\infty(I_{jn})}, \|fW_1\|_{L_\infty(|x|\geq n^{1-\varepsilon})}\right\}.$$

The rest of the proof is as before. □

Now we can define our polynomial approximation to f:

$$P_n[f] := p_0(x) S_{n,\tau_{0n}}(x) + \sum_{k=1}^{m-1} (p_k - p_{k-1})(x) S_{n,\tau_{kn}}(x) - p_{m-1} S_{n,\tau_{mn}}(x).$$

Note that this has been formed from $L_n[f]$ of (4.4) by replacing the characteristic function $\theta_{kn}(x) = \chi_{[\tau_{kn}, 2n^{1-\varepsilon}]}(x)$ by its polynomial approximation $S_{n,\tau_{kn}}(x)$ from Theorem 4 with $A = 2$. We shall assume that ℓ in Theorem 4 is chosen large enough (depending only on r, p) and then L accordingly.

Lemma 4.2 *For* $0 < p \leq \infty$,

$$\begin{aligned}&\|(L_n[f] - P_n[f])W_1\|_{L_p(\mathbb{R})}\\&\leq C\Big\{\sup_{0<h\leq 1/(2\log n)} \|W_1\Delta_h^r(f,x,\mathbb{R})\|_{L_p[-n^{1-\varepsilon},n^{1-\varepsilon}]}\\&\quad + \|fW_1\|_{L_p(I_{0n}^*)} + \|fW_1\|_{L_p(I_{mn}^*)}\Big\}. \qquad (4.9)\end{aligned}$$

Proof

We see that if we define $p_{-1}(x) \equiv 0$, and $p_m(x) \equiv 0$,

$$\begin{aligned}&(L_n[f] - P_n[f])(x)\\&= \sum_{k=0}^{m}(p_k - p_{k-1})(x)(\theta_{kn}(x) - S_{n,\tau_{kn}}(x)). \qquad (4.10)\end{aligned}$$

We shall make substantial use of the following inequality: let S be a polynomial of degree at most r, and $[a,b]$ be a real interval. Then for all $x \in \mathbb{R}$,

$$|S(x)| \leq C(b-a)^{-1/p}\left(1 + \frac{\min\{|x-a|,|x-b|\}}{b-a}\right)^r \|S\|_{L_p[a,b]}.$$

Here $C \neq C(a,b,x,S)$ but $C = C(p,r)$. This follows from standard Nikolskii inequalities and the Bernstein-Walsh inequality. See for example [25, p. 193] for the relevant Nikolskii inequality. Hence for $x \in \mathbb{R}$, and $1 \leq k \leq m-1$,

$$|p_k - p_{k-1}|(x) \leq C(\log n)^{1/p}(1 + \log n|x - \tau_{kn}|)^r \|p_k - p_{k-1}\|_{L_p(I_{kn})}. \qquad (4.11)$$

This is still true for $k = 0$ and $k = m$ if we recall that $p_{-1} \equiv 0 \equiv p_m$. Now for $1 \leq k \leq m-1$, (4.3) gives

$$\|p_k - p_{k-1}\|_{L_p(I_{kn})} \leq C_1 \sum_{i=k-1}^{k} \Omega_{r,p}(f, |I_{in}^*|, I_{in}^*)$$

where $C_1 \neq C_1(f,k,n)$. For $k = 0$, we instead obtain

$$\|p_k - p_{k-1}\|_{L_p(I_{kn})} \leq C_1\Omega_{r,p}(f, |I_{0n}^*|, I_{0n}^*) + \|f\|_{L_p(I_{0n}^*)}$$

and for $k = m$, we instead obtain

$$\|p_k - p_{k-1}\|_{L_p(I_{kn})} \leq C_1 \Omega_{r,p}(f, |I^*_{m-1,n}|, I^*_{m-1,n}) + \|f\|_{L_p(I^*_{m-1,n})}.$$

For notational convenience, we set

$$\Omega_{r,p}(f, |I^*_{-1,n}|, I^*_{-1,n}) := \|f\|_{L_p(I^*_{0n})};$$
$$\Omega_{r,p}(f, |I^*_{m,n}|, I^*_{m,n}) := \|f\|_{L_p(I^*_{mn})};$$
$$\tau_{-1,n} = \tau_{0,n}. \tag{4.12}$$

Since uniformly in k, n, and $x \in \mathbb{R}$,

$$1 + \log n\, |x - \tau_{k,n}| \sim 1 + \log n |x - \tau_{k-1,n}|$$

we obtain from (4.11) and Theorem 4, uniformly for $0 \leq k \leq m$ and $x \in \mathbb{R}$,

$$|(p_k - p_{k-1})(x)(\theta_{kn}(x) - S_{n,\tau_{kn}}(x))| \frac{W_1(x)}{W_1(\tau_{kn})}$$
$$\leq C_2 (\log n)^{1/p} \sum_{i=k-1}^{k} (1 + \log n |x - \tau_{in}|)^{r-\ell} \Omega_{r,p}(f, |I^*_{in}|, I^*_{in}). \tag{4.13}$$

We consider three different ranges of p:

(I) $0 < p < 1$

Here from (4.10) and then (4.13),

$$\int_{\mathbb{R}} (|L_n[f] - P_n[f]| W_1)^p \leq \sum_{k=0}^{m} \int_{\mathbb{R}} (|p_k - p_{k-1}||\theta_{kn} - S_{n,\tau_{kn}}| W_1)^p$$
$$\leq C \log n \sum_{k=-1}^{m} \Omega^p_{r,p}(f, |I^*_{kn}|, I^*_{kn}) W_1^p(\tau_{kn})$$
$$\times \int_{\mathbb{R}} (1 + \log n\, |x - \tau_{kn}|)^{(r-\ell)p} dx.$$

Here if ℓ is so large that $(r - \ell)p < -1$,

$$\log n \int_{\mathbb{R}} (1 + \log n\, |x - \tau_{kn}|)^{(r-\ell)p} dx = \int_{\mathbb{R}} (1 + |u|)^{(r-\ell)p} du =: C_3 < \infty.$$

So assuming this,

$$\int_{\mathbb{R}} (|L_n[f] - P_n[f]|W_1)^p \le C_4 \sum_{k=-1}^{m} \Omega^p_{r,p}(f, |I^*_{kn}|, I^*_{kn}) W^p_1(\tau_{kn}).$$

This is the same as the sum in (4.8), except for the terms for $k = -1$ and $k = m$. So the estimate (4.8) gives the estimate (4.9), keeping in mind our conventions (4.12).

(II) $1 \le p < \infty$

From (4.10) and (4.1)) and then Hölder's inequality,

$$\begin{aligned}
&\{|L_n[f] - P_n[f]|(x) W_1(x)\}^p \\
&\le C \log n \left\{ \sum_{k=-1}^{m-1} (1 + \log n |x - \tau_{k,n}|)^{r-\ell} \Omega_{r,p}(f, |I^*_{kn}|, I^*_{kn}) W_1(\tau_{kn}) \right\}^p \\
&\le C \log n \sum_{k=-1}^{m-1} (1 + \log n |x - \tau_{k,n}|)^{(r-\ell)p/2} \\
&\quad \times \Omega^p_{r,p}(f, |I^*_{kn}|, I^*_{kn}) W^p_1(\tau_{kn}) \cdot \sigma_n(x)^{p/q} \qquad (4.14)
\end{aligned}$$

where $q := p/(p-1)$ and

$$\sigma_n(x) := \sum_{k=0}^{m-1} (1 + \log n \, |x - \tau_{k,n}|)^{(r-\ell)q/2}.$$

We shall show that if $(r - \ell)q/2 < -1$, then

$$\sup_{n \ge 1} \sup_{x \in \mathbb{R}} \sigma_n(x) \le C_1 < \infty. \qquad (4.15)$$

Note that $\sigma_n(x)$ is a decreasing function of x for $x \ge \tau_{mn}$, and increasing for $x < \tau_{0n}$, so it suffices to consider $x \in [\tau_{0n}, \tau_{m,n}]$. Recall that for $k \le m - 1$,

$$|I_{kn}| \sim \frac{1}{\log n}.$$

It is then not difficult to see that

$$\begin{aligned}
\sigma_n(x) &\le C \log n \int_{-n^{1-\varepsilon}}^{n^{1-\varepsilon}} (1 + \log n \, |x - \tau|)^{(r-\ell)q/2} \, d\tau \\
&\le C \log n \int_{-\infty}^{\infty} (1 + \log n \, |x - \tau|)^{(r-\ell)q/2} \, d\tau = C,
\end{aligned}$$

as $(r-\ell)\,q/2 < -1$. Now assume also $(r-\ell)\,p/2 < -1$. Then integrating (4.14) and using (4.8), (4.15) gives our result.

(III) $p=\infty$

Now

$$|L_n[f]-P_n[f]|(x)W_1(x) \le C\sum_{k=0}^{m}|p_k-p_{k-1}|(x)|\theta_{kn}-S_{n,\tau_{kn}}|(x)W_1(x)$$

$$\le C\max_{-1\le k\le m}\Omega_{r,p}(f,|I_{kn}^*|,I_{kn}^*)W_1(\tau_{kn})\cdot\sum_{k=0}^{m-1}(1+\log n\,|x-\tau_{k,n}|)^{r-\ell}.$$

As before, the sum is bounded if ℓ is large enough. Then we can continue this as

$$\le C_1\Big\{\sup_{0\le k\le m-1}\sup_{0<h\le|I_{kn}^*|}\|\Delta_h^r(f,x,I_{kn}^*)W_1\|_{L_\infty(I_{kn}^*)}+\|fW_1\|_{L_\infty(I_{0n}^*)}$$
$$+\|fW_1\|_{L_\infty(I_{mn}^*)}\Big\}$$
$$\le C_2\Big\{\sup_{0\le k\le m-1}\sup_{0<h\le 1/(2\log n)}\|\Delta_h^r(f,x,I_{kn}^*)W_1\|_{L_\infty(I_{kn}^*)}+\|fW_1\|_{L_\infty(I_{0n}^*)}$$
$$+\|fW_1\|_{L_\infty(I_{mn}^*)}\Big\}$$
$$\le C_3\Big\{\sup_{0<h\le 1/(2\log n)}\|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_\infty(-n^{1-\varepsilon},n^{1-\varepsilon})}$$
$$+\|fW_1\|_{L_\infty(I_{0n}^*)}+\|fW_1\|_{L_\infty(I_{mn}^*)}\Big\}.$$

□

We can now turn to the

Proof of Theorem 1

Recall that $S_{n,\tau}$ has degree at most Ln, so $P_n[f]$ has degree at most $Ln+r\le 2Ln$, for $n\ge r$. So, for such n,

$$E_{2Ln}[f;W_1]_p \le \|(f-P_n[f])W_1\|_{L_p(\mathbb{R})}$$
$$\le C\Big\{\|(f-L_n[f])W_1\|_{L_p(\mathbb{R})}$$
$$+\|(L_n[f]-P_n[f])W_1\|_{L_p(\mathbb{R})}\Big\}$$

$$\leq C_1 \left\{ \sup_{0<h\leq 1/2\log n} \|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_p(-n^{1-\varepsilon},n^{1-\varepsilon})} + \|fW_1\|_{L_p(|x|\geq n^{1-\varepsilon}-\frac{1}{\log n})} \right\},$$

by Lemmas 4.1 and 4.2. For large enough k, choose n such that

$$2Ln \leq k < 2L\,(n+1)\,.$$

Here

$$\left(\frac{k}{2L}\right)^{1-\varepsilon} + O\left(n^{-\varepsilon}\right) \leq n^{1-\varepsilon} \leq \left(\frac{k}{2L}\right)^{1-\varepsilon},$$

while for large enough k,

$$\frac{1}{2\log n} \leq \frac{1}{\log \frac{k}{2L}}\,.$$

(The lower bound on k is independent of f.) Thus with $c_0 = \frac{1}{2L}$, we have

$$\begin{aligned} E_k\,[f;W_1]_p &\leq E_{2Ln}\,[f;W_1]_p \\ &\leq C_1 \left\{ \sup_{0<h\leq 1/\log(c_0 k)} \|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_p(-(c_0k)^{1-\varepsilon},(c_0k)^{1-\varepsilon})} + \|fW_1\|_{L_p(|x|\geq (c_0k)^{1-\varepsilon}-1)} \right\} \\ &= C_1\omega_{r,p}\left(f,W_1,\frac{1}{\log c_0 k}\right). \end{aligned}$$

□

5 Proof of Theorem 2

We begin with some simple estimates:

Lemma 5.1 *Let $1 \leq p \leq \infty$ and $r \geq 1$.*

(a) *For $|h| \leq 1$,*

$$\|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_p(\mathbb{R})} \leq C\|fW_1\|_{L_p(\mathbb{R})}, \tag{5.1}$$

where C is independent of f and h.

(b) *If $f^{(r)}W_1 \in L_p(\mathbb{R})$ and $f^{(r-1)}$ is absolutely continuous, then for $h \in \mathbb{R}$,*

$$\|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_p(\mathbb{R})} \leq C\,|h|^r\,\|f^{(r)}W_1\|_{L_p(\mathbb{R})}, \tag{5.2}$$

where C is independent of f and h.

Proof

(a) For $p < \infty$,

$$\begin{aligned}
&\|\Delta_h^r(f,x,\mathbb{R})W_1\|_{L_p(\mathbb{R})}^p \\
&\quad = \int_{-\infty}^{\infty}\left|\sum_{i=0}^{r}\binom{r}{i}(-1)^i f(x+\frac{rh}{2}-ih)\right|^p W_1^p(x)\,dx \\
&\quad \leq C\sum_{i=0}^{r}\int_{-\infty}^{\infty}\left|f(x+\frac{rh}{2}-ih)\right|^p W_1^p\left(x+\frac{rh}{2}-ih\right)dx \\
&\quad = C\,(r+1)\,\|fW_1\|_{L_p(\mathbb{R})}^p.
\end{aligned}$$

We have used here that $|h| \leq 1$, and the fact that for $|t-x| \leq 1, W_1(t) \leq eW_1(x)$. The case $p = \infty$ is easier.

(b) Assume $p < \infty$ and $r = 1$, and let $q = \frac{p}{p-1}$. Then

$$\begin{aligned}
&\|\Delta_h(f,x,\mathbb{R})W_1\|_{L_p(\mathbb{R})}^p \\
&\quad = \int_{-\infty}^{\infty}\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}} f'(s)\,ds\right|^p W_1^p(x)\,dx \\
&\quad \leq \int_{-\infty}^{\infty}|h|^{\frac{p}{q}}\left|\int_{x-\frac{h}{2}}^{x+\frac{h}{2}}|f'(s)|^p\,ds\right|W_1^p(x)\,dx \\
&\quad \leq C\,|h|^{\frac{p}{q}+1}\int_{-\infty}^{\infty}|f'(s)\,W_1(s)|^p\,ds,
\end{aligned}$$

by first Hölder's inequality and then Fubini's Theorem. Taking pth roots gives the result for $r = 1$. Induction then gives the case of general r. Again, the case $p = \infty$ is easier. □

Proof of Theorem 2

(b) ⇒ (a)

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. With $0 < h \leq \frac{1}{\log n}$, and C_2 as in (1.4), (1.6) gives

$$\|\Delta_h^r(f, x, \mathbb{R})W_1\|_{L_p(-(C_2 n)^{1-\varepsilon},(C_2 n)^{1-\varepsilon})} = O\left((\log n)^{-\alpha}\right)$$

while (1.7) with $h = \frac{2}{\log n}$ gives

$$\|fW_1\|_{L_p(|x| \geq n^{1/2})} = O\left((\log n)^{-\alpha}\right).$$

Applying Theorem 1 gives (1.5).

(a) ⇒ (b)

We proceed as in [6].

Proof of (1.6)

For $k \geq 0$, let P_k be a polynomial of degree $\leq k$ such that

$$\|\left(f - P_k\right) W_1\|_{L_p(R)} \leq C\left(\log\left(k+2\right)\right)^{-\alpha}, \tag{5.3}$$

where C is independent of k. Let $0 < h < 1$. Choose $m \geq 1$ such that

$$2^{-m} \leq h < 2^{1-m}$$

and let

$$n := 2^{2^m}.$$

We have

$$\begin{aligned} &\|\Delta_h^r(f, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} \\ &\quad \leq \|\Delta_h^r(f - P_n, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} + \|\Delta_h^r(P_n, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})}. \end{aligned} \tag{5.4}$$

By Lemma 5.1(a),

$$\begin{aligned} &\|\Delta_h^r(f - P_n, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} \\ &\leq C\|(f - P_n)W_1\|_{L_p(\mathbb{R})} \\ &\leq C\left(\log n\right)^{-\alpha} \leq C2^{-m\alpha}. \end{aligned} \tag{5.5}$$

Moreover, by Lemma 5.1(b), if we set $P_{2^{2^{-1}}} := P_0$,

$$\begin{aligned}
&\|\Delta_h^r(P_n, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} \\
&= \|\sum_{k=0}^{m} \Delta_h^r(P_{2^{2^k}} - P_{2^{2^{k-1}}}, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} \\
&\leq Ch^r \sum_{k=0}^{m} \|(P_{2^{2^k}} - P_{2^{2^{k-1}}})^{(r)} W_1\|_{L_p(\mathbb{R})}.
\end{aligned}$$

Using the Bernstein inequality Lemma 2.1(a), and then (5.3), we continue this as

$$\begin{aligned}
&\leq Ch^r \sum_{k=0}^{m} 2^{kr} \|(P_{2^{2^k}} - P_{2^{2^{k-1}}})W_1\|_{L_p(\mathbb{R})} \\
&\leq Ch^r \sum_{k=0}^{m} 2^{kr} 2^{-k\alpha} \leq Ch^r 2^{m(r-\alpha)} \leq C2^{-m\alpha}.
\end{aligned}$$

(Recall that $r > \alpha$.) Combined with (5.4) and (5.5), this gives

$$\|\Delta_h^r(f, x, \mathbb{R})W_1\|_{L_p(\mathbb{R})} \leq C2^{-m\alpha} \leq Ch^\alpha.$$

So we have (1.6).

Proof of (1.7)

Assume that $h > 0$ is so small that $\exp\left(\frac{1}{h}\right) \leq 16\pi$. Choose M such that for

$$n = 2^{2^M}$$

we have

$$a_{2n} \leq \exp\left(\frac{1}{h}\right) \leq \pi^{-1} a_{2n}^2. \tag{5.6}$$

Note that this is possible, since it is equivalent to

$$\pi 2^{2^M} \leq \exp\left(\frac{1}{h}\right) \leq \pi 2^{2^{M+1}}.$$

We have

$$\begin{aligned}&\|fW_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}\\&\quad\leq \|\left(f-P_n\right)W_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}+\|P_nW_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}.\end{aligned}\tag{5.7}$$

Here by (5.3),

$$\|\left(f-P_n\right)W_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}\leq C2^{-M\alpha}.\tag{5.8}$$

Moreover, by Lemma 2.1(b), and (5.6),

$$\begin{aligned}&\|P_nW_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}\\&\quad\leq \|P_nW_1\|_{L_p(|x|\geq a_{2n})}\\&\quad\leq \exp\left(-C_1n\right)\|P_nW_1\|_{L_p(\mathbb{R})}\\&\quad\leq \exp\left(-C_1n\right)\left(\|\left(f-P_n\right)W_1\|_{L_p(\mathbb{R})}+\|fW_1\|_{L_p(\mathbb{R})}\right)\\&\quad\leq \exp\left(-C_1n\right)\left(C2^{-M\alpha}+\|fW_1\|_{L_p(\mathbb{R})}\right).\end{aligned}$$

Together with (5.7) and (5.8), this gives

$$\|fW_1\|_{L_p\left(|x|\geq \exp\left(\frac{1}{h}\right)\right)}\leq C2^{-M\alpha}\leq Ch^{\alpha}.$$ □

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