definition remark

Mathematica Evidence that Ramanujan kills Baker-Gammel-Wills

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Abstract

A 1961 conjecture of Baker, Gammel and Wills asserts that if a function f is meromorphic in the unit ball, and analytic at 0, then a subsequence of its diagonal Pade approximants converges uniformly in compact subsets omitting poles. Inasmuch as the denominators of the Padé approximants are complex orthogonal polynomials, and the convergence of sequences of Padé approximants is determined largely by the behaviour of their poles, the conjecture deals with distribution of zeros of complex orthogonal polynomials. In this paper, we present numerical evidence from the Mathematica package, that Ramanujan's continued fraction

$$H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2 z|}{|1} + \frac{q^3 z|}{|1} + ...$$

provides a counterexample, provided q is appropriately chosen on the unit circle.

1 Introduction

Let

$$f(z) = \sum_{j-0}^{\infty} a_j z^j \tag{1.1}$$

be a formal power series with complex coefficients $\{a_j\}_{j=0}^{\infty}$. We may expand f in a formal continued fraction, or C-fraction

$$f(z) = c_0 + \cfrac{c_1 z^{\ell_1}}{1 + \cfrac{c_2 z^{\ell_2}}{1 + \cfrac{c_3 z^{\ell_3}}{1 + \ddots}}}.$$

$$= c_0 + \frac{c_1 z^{\ell_1}|}{|1} + \frac{c_2 z^{\ell_2}|}{|1} + \frac{c_3 z^{\ell_3}|}{|1} + \dots, \tag{1.2}$$

where $\{c_j\}_{j=0}^{\infty}$ are complex and $\{\ell_j\}_{j=1}^{\infty}$ are positive integers. One way to develop the formal continued fraction is to write, as a first step,

$$f(z) = c_0 + \frac{c_1 z^{\ell_1}}{1 + zg(z)},$$

where g is also a formal power series. One can solve for the power series coefficients of g by multiplying by $1 + zg(z)$, and equating coefficients of powers of z on both sides. (This is very similar to finding the power series expansion of $1/f$ if $f(0) \neq 0$). The next step is to to g what was done to f. Continuing ad infinitum gives (2).

The value of a power series is defined as the limit of its sequence of partial sums, if that limit exists. In much the same way, we may truncate the c.f. (continued fraction) (2), giving the nth convergent,

$$\frac{\mu_n}{\nu_n}(z) = c_0 + \frac{c_1 z^{\ell_1}|}{|1} + \frac{c_2 z^{\ell_2}|}{|1} + \frac{c_3 z^{\ell_3}|}{|1} + \dots \frac{c_n z^{\ell_n}|}{|1}.$$

The value of the c.f. is defined to be

$$\lim_{n\to\infty} \frac{\mu_n}{\nu_n}(z),$$

if that limit exists.

In many c.f.'s that arise in practice, all $\ell_j = 1$. Thus formally,

$$f(z) = c_0 + \frac{c_1 z|}{|1} + \frac{c_2 z|}{|1} + \frac{c_3 z|}{|1} + ... \tag{1.3}$$

and

$$\frac{\mu_n}{\nu_n}(z) = c_0 + \frac{c_1 z|}{|1} + \frac{c_2 z|}{|1} + \frac{c_3 z|}{|1} + ... \frac{c_n z|}{|1}. \tag{1.4}$$

This is the so-called *normal case*, and in the sequel, we focus on this case. We use the notation $\frac{\mu_n}{\nu_n}$ for the convergent to emphasize that it is a rational function of z. For the normal case μ_n and ν_n are polynomials of degree $\left[\frac{n+1}{2}\right]$ and $\left[\frac{n}{2}\right]$ respectively, where $[x]$ denotes the greatest integer $\leq x$ and they have no common factors. In the sequel, we use the normalization

$$\nu_n(0) = 1.$$

The convergents $\frac{\mu_n}{\nu_n}$ are also *Padé approximants* to f. Recall that if $m, n \geq 0$, the $[m/n]$ *Padé approximant* to f is a rational function $[m/n] = P/Q$, where P, Q have degree $\leq m, n$ respectively, Q is not identically 0, and

$$(fQ - P)(z) = O(z^{m+n+1}).$$

Here the order relation indicates that the coefficients of $1, z, z^2, ..., z^{m+n}$ on the left-hand side vanish. Thus, in the normal case,

$$(f\nu_n - \mu_n)(z) = O\left(z^{\left[\frac{n+1}{2}\right]+\left[\frac{n}{2}\right]+1}\right) = O\left(z^{n+1}\right). \tag{1.5}$$

The denominators ν_n are also *complex orthogonal polynomials*: if f is analytic in a neighbourhood of 0, it follows from (5) that for small enough $r > 0$,

$$\int_{|t|=r} \frac{(f\nu_n)(t)}{t^{j+1}} dt = 0, \left[\frac{n+1}{2}\right] < j \leq n.$$

Then if, for example, $n = 2\ell$, the reverted polynomial

$$\widetilde{\nu}_n(z) := z^{\ell} \nu_n\left(\frac{1}{z}\right) \tag{1.6}$$

satisfies

$$\int_{|z|=\frac{1}{r}} f\left(\frac{1}{z}\right) \widetilde{\nu}_n(z) z^k dz = 0, 0 \leq k < \ell. \tag{1.7}$$

As $\widetilde{\nu}_n$ is a polynomial of degree ℓ, it follows that it is a complex orthogonal polynomial for the weight $f\left(\frac{1}{z}\right)$. This last relation has been exploited to analyze complex orthogonal polynomials from both the algebraic and analytic aspects. The deepest work in this direction is due to Stahl [15], [16]. In turn, (7) has been used to obtain dramatic advances in investigating Padé approximants for functions with branchpoints [17].

There is also a close relation between real orthogonal polynomials and Padé approximants. Let f be a Markov function,

$$f(z) = \int_0^\infty \frac{d\sigma(t)}{1-zt} = \sum_{j=0}^\infty \left[\int_0^\infty t^j d\sigma(t)\right] z^j,$$

where σ is a positive Borel measure for which all the moments $\int_0^\infty t^j d\sigma(t), j \geq 0$, are finite. Then for $n = 2\ell$, ν_n is a polynomial of degree n, whose reverted polynomial $\widetilde{\nu}_n$ satisfies,

$$\int_0^\infty \widetilde{\nu}_n(t)\, t^j t d\sigma(t) = 0, 0 \leq j < n.$$

Thus $\widetilde{\nu}_n$ is an orthogonal polynomial of degree ℓ for the measure $td\sigma(t)$. See [5,p.216ff.]. Again, this relation has enriched both the theory of orthogonal polynomials and Padé approximation.

While Padé approximants and continued fractions have numerous applications, their theoretical properties - not least their convergence theory - is rich and complex [8], [9], [5]. Thus, for example, H. Wallin [20] constructed an entire function whose continued fraction, and in particular whose diagonal Padé sequence $\{[\ell/\ell]\}_{\ell=1}^\infty$ diverges everywhere in $\mathbb{C}\setminus\{0\}$:

$$\limsup_{n\to\infty} |[\mu_n/\nu_n](z)| = \infty, \forall z \in \mathbb{C}\setminus\{0\}.$$

(Note that it is lim sup, not lim: the bad behaviour in Wallin's example occurs only for a subsequence). Conversely, there are power series with zero radius of convergence, for which the continued fraction (2) converges to a function analytic in $\mathbb{C}\setminus(-\infty, 0]$ [2] [5].

One of the driving forces behind Padé approximation has been G.A. Baker of Los Alamos Scientific Laboratory. He observed that pathological behaviour of convergents μ_n/ν_n occur only for a "thin" subsequence of integers n. This encouraged him and his collaborators to formulate [2], [3], [5]:

Baker-Gammel-Wills Conjecture (1961)
Let $f(z)$ be analytic in $|z| < 1$, except for poles, none at 0. There exists an infinite subsequence of positive integers $\mathcal{S}$ such that

$$\lim_{\substack{\ell \to \infty \\ \ell \in \mathcal{S}}} [\ell/\ell](z) = f(z) \tag{1.8}$$

uniformly in compact subsets of $|z| < 1$ omitting poles of f.

Note that we may rewrite (8) as

$$\lim_{\substack{\ell \to \infty \\ \ell \in \mathcal{S}}} \mu_{2\ell}/\nu_{2\ell}(z) = f(z) \tag{1.9}$$

uniformly in compact subsets of $|z| < 1$ omitting poles of f. We shall use the abbreviation BGW for the conjecture.

The conjecture is widely believed to be false in the above form, though possibly true for functions that are meromorphic in the whole plane, that is analytic except for poles. Credence for this modification is provided by a related easier conjecture for the columns of the Padé table, the so-called Baker-Graves-Morris Conjecture [4]. It turned out to be true for functions meromorphic in the whole plane, but false for functions meromorphic in the unit ball [7]. For various perspectives on the conjecture, see [10], [11], [12], [18].

In this paper, we shall present numerical evidence that a counterexample to the conjecture is provided by Ramanujan's continued fraction

$$H_q(z) = 1 + \frac{qz|}{|1} + \frac{q^2z|}{|1} + \frac{q^3z|}{|1} + \frac{q^4z|}{|1} + ... \,. \tag{1.10}$$

In 1920, Ramanujan (and somewhat earlier, Rogers, see [1]) noted that for $|q| < 1$, and $z \in \mathbb{C}$, there is the identity

$$H_q(z) = \frac{G_q(z)}{G_q(qz)} \tag{1.11}$$

where

$$G_q(z) = 1 + \sum_{j=1}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)(1-q^3)...(1-q^j)} z^j. \tag{1.12}$$

For $|q|<1$, the coefficients q^j of the continued fraction decay to 0 as $j \to \infty$, so classical theorems show that the continued fraction converges in the plane to a meromorphic function H_q.

It is for $|q|=1$ that Ramanujan's continued fraction might provide a counterexample to BGW. Obviously q should not be a root of unity for the power series G_q to be defined. But for further analysis, one needs more, such as assumptions on diophantine approximation. Let us write

$$q = e^{2\pi i \tau} \text{ where } \tau \in (0,1)\,. \tag{1.13}$$

For a real number x, we use

$$\|x\| := \min\{|x-j| : j \in \mathbb{Z}\}$$

to denote the distance from x to the nearest integer. Let us set

$$R(q) := R\left(e^{2\pi i\tau}\right) := \liminf_{n\to\infty} \|n\tau\|^{1/n}. \tag{1.14}$$

It is easily seen that the radius of convergence of G_q is $R(q)$ and only a little more difficult that it has a natural boundary on $\{z : |z| = R(q)\}$. Moreover, for almost every $\tau \in (0,1)$, $R(q)=1$ [12], [13]. From the point of view of BGW, $R(q)=1$ is the most interesting case:

Theorem 1

Let q be given by (13) and let $R(q)=1$.
(i) H_q is analytic in $|z|<\frac{1}{4}$ and meromorphic in $|z|<1$ with a natural boundary on the unit circle;
(ii) The identity (11) holds in $|z|<1$;
(iii) As q approaches a root of unity, the number of poles of H_q in $|z|<1$ approaches ∞.

Thus H_q has analytic properties that fall within the ambit of BGW. It turns out that subsequences of $\{\mu_n\}_{n=1}^{\infty}$ and $\{\nu_n\}_{n=1}^{\infty}$ converge separately. To describe this behaviour, one needs the fact that for q not a root of unity, or equivalently, τ irrational, $\{q^n\}_{n=1}^{\infty} = \{e^{2\pi i n\tau}\}_{n=1}^{\infty}$ is dense on the unit circle. Thus given β with $|\beta|=1$, we can find an infinite sequence of positive integers $\mathcal{S}$ with

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{S}}} q^n = \beta. \tag{1.15}$$

One may prove [12], [13]:

Theorem 2
Let $R(q) = 1$. Let $|\beta| = 1$ and let $\mathcal{S}$ be an infinite sequence of positive integers for which (15) holds. Then uniformly in compact subsets of $|z| < 1$,

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{S}}} \mu_n(z) = G_q(z)\overline{G_q(\overline{\beta q z})}; \tag{1.16}$$

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{S}}} \nu_n(z) = G_q(qz)\overline{G_q(\overline{\beta q z})}. \tag{1.17}$$

After division, we obtain

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{S}}} \frac{\mu_n}{\nu_n}(z) = \frac{G_q(z)}{G_q(qz)} = H_q(z)$$

and we have convergence, so how does this pose a problem for the Baker-Gammel-Wills Conjecture? Well, recall that the latter refers to uniform convergence, and because of Hurwitz' Theorem, every zero of $G_q(qz)\overline{G_q(\overline{\beta q z})}$ attracts zeros of ν_n as $n \to \infty, n \in \mathcal{S}$. In turn, as μ_n and ν_n are coprime polynomials, each zero of $G_q(qz)\overline{G_q(\overline{\beta q z})}$ attracts poles of μ_n/ν_n. The zeros of $G_q(qz)$ *are* poles of $H_q(z)$, but the zeros of $\overline{G_q(\overline{\beta q z})}$ need not be. Thus the latter's zeros attract extra poles of μ_n/ν_n, which need not converge to poles of $H_q(z)$.

Since $|\beta| = 1$, and the operations that give $\overline{G_q(\overline{\beta q z})}$ from $G_q(qz)$ involve rotation and reflection (both of which preserve circles center 0), one deduces [12], [13]:

Corollary 3
Let $R(q) = 1$ and $0 < r < 1$. If H_q has k poles counting multiplicity on $|z| = r$, then for large enough n, μ_n/ν_n has $2k$ poles counting multiplicity in any small enough neighbourhood of that circle.

This is already interesting: it is the first known example in which every convergent of large enough order to a function meromorphic in the unit ball has spurious poles, and even double as many poles, as the function from which it is formed. In every other example that this author knows, some subsequence of the convergents behaves pathologically, but another subsequence has the correct number of poles and satisfies BGW.

However, it does not on its own contradict BGW. We haven't ruled out the possibility that for infinitely many n, these $2k$ poles of μ_n/ν_n are all close to the k poles of H_q, and so away from the poles of H_q we still have convergence for a subsequence. This is the possibility we would like to rule out! To do this, we again need to look at the relation between $G_q\left(qz\right)$ and $\overline{G_q(\overline{\beta qz})}$ and realize that if $\forall\,|\beta| = 1$,

$$\{z : G_q\left(qz\right) = 0\} \neq \left\{z : \overline{G_q(\overline{\beta qz})} = 0\right\},$$

then we do have the desired counterexample. We can reformulate this as:

Corollary 4
If the zeros of $G_q(z)$ are not symmetric about any line through the origin, then H_q provides a counterexample to the Baker-Gammel-Wills Conjecture.

The highly oscillatory nature of the arguments of the Maclaurin series coefficients of G_q lend credence to the desired asymmetry (Think of power series with real coefficients, for which zeros occur in conjugate pairs). While a proof is still lacking, one can look for numerical evidence, by plotting the zeros of the partial sums

$$S_{m,q}\left(z\right) = 1 + \sum_{j=1}^{m} \frac{q^{j^2}}{(1-q)(1-q^2)(1-q^3)...(1-q^j)} z^j.$$

Since uniformly in compact subsets of $|z| < 1$,

$$\lim_{m\to\infty} S_{m,q}(z) = G_q(z),$$

the zeros of $S_{m,q}$ well inside the unit ball provide good approximations to those of G_q. In contrast, the zeros of $S_{m,q}$ close to the unit circle (which is the circle of convergence of G_q), do not say anything about zeros of G_q, rather they illustrate a theory of Jentsch, Szegö, and others [6], [14].

The rate of convergence of zeros of $S_{m,q}$ to those of f is geometric inside the unit circle, as evidenced by the following simple proposition.

Proposition 5

Let $R(q) = 1$. Let a be a zero of $G_q(z)$ of multiplicity k inside the unit ball. Then for $m \geq 1$, $S_{m,q}$ has a zero a_m with

$$\limsup_{m \to \infty} |a - a_m|^{1/m} \leq |a|^{1/k} < 1. \tag{1.18}$$

More completely, $S_{m,q}$ has zeros a_m of total multiplicity k all satisfying (18).

Proof

This type of proof is standard, but we provide the details. Firstly, by Hurwitz' theorem, $S_{m,q}$ has zeros of total multiplicity k that approach a as $m \to \infty$.Choose a small $\delta > 0$ such that a is the only zero of G_q inside or on the circle Γ centre a, radius $\delta > 0$. Let $|a| < \rho < 1$. If δ is small enough, the fact that G_q has radius of convergence 1 gives for some C_1 independent of m and for $j = 0, 1$,

$$\max_{z \in \Gamma} \left| S_{m,q}^{(j)}(z) - G_q^{(j)}(z) \right| \leq C_1 \rho^m.$$

Let $1 \leq j \leq k$. Then taking account of the fact that G_q does not vanish on Γ, we obtain for large m and some C_2 independent of m,

$$\left| \int_\Gamma (z-a)^j \left(\frac{S'_{m,q}(z)}{S_{m,q}(z)} - \frac{G'_q(z)}{G_q(z)} \right) dz \right| \leq C_2 \rho^m.$$

The residue theorem (or the principle of the argument, if you prefer) gives for some C_3 independent of m,

$$\max_{1 \leq j \leq k} \left| \sum_{\substack{z \text{ inside } \Gamma \\ \text{with } S_{m,q}(z)=0}} (z-a)^j \right| \leq C_3 \rho^m. \tag{1.19}$$

Now the power sum method in the form due to Atkinson [19,p.24] shows that for arbitrary $\xi_1, \xi_2, ... \xi_k$,

$$\max_{1 \leq j \leq k} \left| \xi_j \right| \leq \max_{1 \leq l \leq k} \left(6 \left| \sum_{j=1}^{j} \xi_j^l \right| \right)^{1/l}.$$

Applying this to (19) gives

$$\max_{\substack{z \text{ inside } \Gamma \\ \text{with } S_{m,q}(z)=0}} |z-a| \leq (C_4 \rho^m)^{1/k},$$

where C_4 is independent of m. Now take mth roots, let $m \to \infty$ and then let $\rho \to |a|$. $\square$

2 Numerical Results

We used Mathematica to generate plots of the zeros. While Mathematica is a “black box”, careful computation of $S_{m,q}$ and consistency between plots for successive values of m, suggests some reliability. It is clear from all the plots below, that the desired asymmetry is present. Thus Mathematica suggests that Ramanujan *does* kill BGW. These plots have also suggested a new approach to proving the asymmetry.

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