# CHRISTOFFEL FUNCTIONS AND UNIVERSALITY ON THE BOUNDARY OF THE BALL

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Abstract. We establish asymptotics for Christoffel functions, and universality limits, associated with multivariate orthogonal polynomials, on the boundary of the unit ball in $\mathbb{R}^d$.

Orthogonal Polynomials, Universality Limits, Christoffel functions. 42C05, 42C99, 42B05, 60B20

## 1. Introduction[1]

Let $d \geq 2$, and $\Pi_n^d$ denote the space of polynomials in $d$ variables of degree at most $n$. Let $N_n^d$ denote its dimension, so

$$N_n^d = \binom{n+d}{n}.$$

Let $\mu$ be a positive measure on $\mathbb{R}^d$ with compact support such that $\left\{\mathbf{x} \in \mathbb{R}^d : \mu'(\mathbf{x}) > 0\right\}$ has non-empty interior. This ensures that

$$\int P^2 d\mu > 0$$

for every non-trivial polynomial $P$.

We let $K_n(\mu, \mathbf{x}, \mathbf{y})$ denote the reproducing kernel for $\mu$ and $\Pi_n^d$, so that for all $P \in \Pi_n^d$, and all $\mathbf{x} \in \mathbb{R}^d$,

$$P(\mathbf{x}) = \int K_n(\mu, \mathbf{y}, \mathbf{x}) P(\mathbf{y}) \, d\mu(\mathbf{y}).$$

The $n$th Christoffel function for $\mu$ is

$$\lambda_n(\mu, \mathbf{x}) = \frac{1}{K_n(\mu, \mathbf{x}, \mathbf{x})}.$$

It admits the extremal property

$$\lambda_n(\mu, \mathbf{x}) = \inf_{P \in \Pi_n^d} \frac{\int P(\mathbf{t})^2 \, d\mu(\mathbf{t})}{P^2(\mathbf{x})}. \tag{1.1}$$

When $\mu$ is absolutely continuous with respect to $d$ dimensional Lebesgue measure, and $\mu' = W$, we shall write $\lambda_n(W, \mathbf{x})$.

Asymptotics for these multivariate Christoffel functions have been established in a number of papers [1], [2], [3], [11], [12], [15], for Jacobi weights, and weights that satisfy some structural restriction, such as being radially or centrally symmetric.

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Xu [12] established one-sided asymptotics under more general conditions. In all these results, explicit formulae for the reproducing kernel, due mostly to Xu, play a crucial role.

In a recent paper [5], we extended the range of these asymptotics to the class of regular measures: A compactly supported measure $\mu$ on $\mathbb{R}^d$ is said to be *regular*, if

$$\lim_{n\to\infty}\left(\sup_{P\in\Pi_n^d}\frac{\|P\|^2_{L_\infty(\text{supp}[\mu])}}{\int|P|^2\,d\mu}\right)^{1/n}=1. \tag{1.2}$$

This is alternatively called the *Bernstein-Markov condition* [3]. When supp$[\mu]$ is a convex region such as a ball, a sufficient condition for regularity is that $\mu'>0$ a.e. in that convex region.

We established asymptotics for ratios of Christoffel functions for regular measures $\mu,\nu$, with the same support, and that are mutually absolutely continuous in an open subset of the support, with $\frac{d\nu}{d\mu}$ continuous in some compact subset of that open set. As a consequence, for the ball, and simplex, we obtained both asymptotics for Christoffel functions and universality limits. The latter involve the normalized Bessel function

$$J^*_\alpha(z)=z^{-\alpha}J_\alpha(z)=\left(\frac{1}{2}\right)^\alpha\sum_{j=0}^{\infty}\frac{\left(-\frac{1}{4}z^2\right)^j}{j!\Gamma(j+\alpha+1)}. \tag{1.3}$$

It has the advantage over $J_\alpha$, of being entire. For the unit ball, we proved:

**Theorem A**
*Let* $\bar{B}=\overline{B(\mathbf{0},1)}=\left\{\mathbf{x}\in\mathbb{R}^d:\|\mathbf{x}\|\le1\right\}$. *Let* $\mu$ *be a regular measure on* $\bar{B}$, *and assume that* $D$ *is a compact subset of the interior of* $\bar{B}$, *such that* $\mu'$ *is positive and continuous in* $D$.
*(a) Uniformly for* $\mathbf{x}\in D$, *and* $\mathbf{y}_n\in B\left(\mathbf{x},\frac{1}{\sqrt{n}}\right)$, $n\ge1$,

$$\lim_{n\to\infty}\binom{n+d}{d}\lambda_n(\mu,\mathbf{y}_n)=\frac{\mu'(\mathbf{x})}{W_0(\mathbf{x})},$$

*where*

$$W_0(\mathbf{x})=\frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}\left(1-\|\mathbf{x}\|^2\right)^{-1/2}.$$

*(b) Uniformly for* $\mathbf{x}\in D$, *and* $\mathbf{u},\mathbf{v}$ *in compact subsets of* $\mathbb{R}^d$,

$$\lim_{n\to\infty}\frac{K_n\left(\mu,\mathbf{x}+\frac{\mathbf{u}}{n},\mathbf{x}+\frac{\mathbf{v}}{n}\right)}{K_n(\mu,\mathbf{x},\mathbf{x})}=\frac{J^*_{d/2}\left(\sqrt{G(\mathbf{x},\mathbf{u},\mathbf{v})}\right)}{J^*_{d/2}(0)},$$

*where if* $\cdot$ *denotes the standard Euclidean inner product,*

$$G(\mathbf{x},\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|^2+\frac{(\mathbf{x}\cdot(\mathbf{u}-\mathbf{v}))^2}{1-\|\mathbf{x}\|^2}.$$

In the above, and in the sequel, $B(\mathbf{x},r)$ denotes the Euclidean ball center $\mathbf{x}$, radius $r$, so that

$$B(\mathbf{x},r)=\left\{\mathbf{y}\in\mathbb{R}^d:\|\mathbf{y}-\mathbf{x}\|<r\right\},$$

while $\bar{B}$ denotes the closed unit ball $\overline{B(\mathbf{0},1)}$.

Practically all the above mentioned papers deal with asymptotics in the interior of the support. In the cases where the boundary is considered, asymptotics are restricted either to the Chebyshev weight, or to less precise forms of the asymptotic.

It is the purpose of this paper, to consider the boundary of the ball. This is a more complicated case, especially when one allows weights that vanish or are infinite on the boundary, such as ultraspherical weights. For $\rho \geq 0$, define the ultraspherical weight

$$W_\rho(\mathbf{x}) = \omega_\rho \left(1 - \|\mathbf{x}\|^2\right)^{\rho - 1/2}, \mathbf{x} \in B(\mathbf{0}, 1). \tag{1.4}$$

Here $\omega_\rho$ is a positive constant chosen so that $\int W_\rho = 1$. It is known [12, p.259] that

$$\omega_\rho = \frac{\Gamma\left(\rho + \frac{d+1}{2}\right)}{\pi^{d/2}\Gamma\left(\rho + \frac{1}{2}\right)}. \tag{1.5}$$

For Christoffel functions, we'll prove:

**Theorem 1.1**

*Let $\mu$ be a regular measure on $\bar{B} = \overline{B(\mathbf{0}, 1)}$. Let $D_1$ be an open set in $\mathbb{R}^d$ such that $D = D_1 \cap \left\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\right\}$ is non-empty. Assume that in $D_1 \cap \bar{B}$, $\mu$ is absolutely continuous, and satisfies there, for some $\rho \geq 0$,*

$$\mu'(\mathbf{x}) = h(\mathbf{x}) W_\rho(\mathbf{x}), \tag{1.6}$$

*where $h$ is positive and uniformly continuous on $D$, as a function in $D_1 \cap \bar{B}$. Let $\{\mathbf{x}_n\}$ be a sequence in $D_1 \cap \bar{B}$ such that for some $s \geq 0$,*

$$\lim_{n \to \infty} n^2 \left(1 - \|\mathbf{x}_n\|^2\right) = s. \tag{1.7}$$

*Let*

$$\alpha = \rho + \frac{d}{2}. \tag{1.8}$$

*(a) If $\rho > 0$,*

$$\begin{aligned} &\lim_{n \to \infty} n^{-2\alpha} K_n(\mu, \mathbf{x}_n, \mathbf{x}_n) h(\mathbf{x}_n) \\ = \;& 2^{\alpha+1} \frac{\Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} \frac{\int_0^\pi J_\alpha^*\left(2\sqrt{s}\left|\sin\frac{\psi}{2}\right|\right)(\sin\psi)^{2\rho - 1}\, d\psi}{\int_0^\pi (\sin\psi)^{2\rho - 1}\, d\psi}. \end{aligned} \tag{1.9}$$

*The limit holds uniformly for $s$ in bounded subsets of $[0, \infty)$. In particular, if (1.7) holds with $s = 0$,*

$$\lim_{n \to \infty} n^{-2\alpha} K_n(\mu, \mathbf{x}_n, \mathbf{x}_n) h(\mathbf{x}_n) = \frac{2}{\Gamma(2\alpha + 1)}. \tag{1.10}$$

*(b) If $\rho = 0$,*

$$\begin{aligned} &\lim_{n \to \infty} n^{-2\alpha} K_n(\mu, \mathbf{x}_n, \mathbf{x}_n) h(\mathbf{x}_n) \\ = \;& \frac{1}{\Gamma(2\alpha + 1)} \left\{1 + \frac{J_\alpha^*(2\sqrt{s})}{J_\alpha^*(0)}\right\}. \end{aligned} \tag{1.11}$$

*In particular, if (1.7) holds with $s = 0$, then (1.10) holds.*

**Remarks**

(a) Note that $h$ does not have to possess any structural property such as radial invariance. It only needs to be positive and continuous.
(b) By $h$ being positive and uniformly continuous on $D$, as a function in $D_1 \cap \bar{B}$, we mean the following: given $\varepsilon > 0$, there exists $\delta > 0$ such that for $\mathbf{x} \in D$ and $\mathbf{y} \in D_1 \cap \bar{B}$ with $\|\mathbf{y} - \mathbf{x}\| < \delta$, we have $|h(\mathbf{y}) - h(\mathbf{x})| < \varepsilon$.
(c) As noted above, we make essential use of explicit formulae for the reproducing kernel due to Xu.

As regards universality limits, we'll prove:

**Theorem 1.2**
*Let $\mu$ satisfy the hypotheses of Theorem 1.1, with $\rho > 0$.*
*(a) Let $\mathbf{x} \in D$, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$ with $\mathbf{x} \cdot \mathbf{u} < 0$ and $\mathbf{x} \cdot \mathbf{v} < 0$. Then*

$$(1.12) \quad \lim_{n\to\infty} \frac{K_n\left(\mu, \mathbf{x} + \frac{1}{n^2}\mathbf{u}, \mathbf{x} + \frac{1}{n^2}\mathbf{v}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = \frac{\int_0^\pi J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, \psi)}\right)(\sin\psi)^{2\rho-1} d\psi}{J_\alpha^*(0) \int_0^\pi (\sin\psi)^{2\rho-1} d\psi},$$

*where*

$$(1.13) \quad G(\mathbf{u}, \mathbf{v}, \psi) = \left(|\mathbf{x} \cdot \mathbf{u}|^{1/2} - |\mathbf{x} \cdot \mathbf{v}|^{1/2}\right)^2 + 4|\mathbf{x} \cdot \mathbf{u}|^{1/2} |\mathbf{x} \cdot \mathbf{v}|^{1/2} \left(\sin\frac{\psi}{2}\right)^2.$$

(1.14)

*(b) For $n \geq 1$, let $\|\mathbf{a}_n\| = 1 = \|\mathbf{b}_n\|$,* with $\|\mathbf{a}_n - \mathbf{b}_n\| = O\left(\frac{1}{n}\right)$. *Then as $n \to \infty$,*

$$(1.15) \quad \frac{K_n(\mu, \mathbf{a}_n, \mathbf{b}_n)}{K_n(\mu, \mathbf{a}_n, \mathbf{a}_n)} = \frac{J_\alpha^*(n\|\mathbf{a}_n - \mathbf{b}_n\|)}{J_\alpha^*(0)} + o(1).$$

For $\rho = 0$, we prove:

**Theorem 1.3**
*Let $\mu$ satisfy the hypotheses of Theorem 1.1, with $\rho = 0$.*
*(a) Let $\mathbf{x} \in D$, and $\mathbf{x} \cdot \mathbf{u} < 0$ and $\mathbf{x} \cdot \mathbf{v} < 0$. Then*
(1.16)

$$\lim_{n\to\infty} \frac{K_n\left(\mu, \mathbf{x} + \frac{1}{n^2}\mathbf{u}, \mathbf{x} + \frac{1}{n^2}\mathbf{v}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = \frac{1}{2} \frac{J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, 0)}\right) + J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, \pi)}\right)}{J_\alpha^*(0)},$$

*(b) For $n \geq 1$, let $\|\mathbf{a}_n\| = 1 = \|\mathbf{b}_n\|$, with $\|\mathbf{a}_n - \mathbf{b}_n\| = O\left(\frac{1}{n}\right)$. Then as $n \to \infty$, (1.15) holds.*

Observe in Theorems 1.2 and 1.3, that when we move off the unit sphere, we need an increment of $O\left(\frac{1}{n^2}\right)$, while if we stay on the unit sphere, the correct increment is $O\left(\frac{1}{n}\right)$.

This paper is organised as follows: in Section 2, we analyze Christoffel functions for ultraspherical weights. In section 3, we prove Theorem 1.1. In Section 4, we establish universality for ultraspherical weights. In Section 5, we prove Theorems 1.2 and 1.3.

Throughout, $c, C, C_1, C_2, \ldots$ denote positive constants independent of $n$, and vectors $\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$, as well as polynomials $p$. The same constant does not necessarily denote the same constant in different occurrences.

## 2. Ultraspherical Weights

We begin with the explicit formula for the reproducing kernel due to Xu. This involves the Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree $n$, that satisfies the orthogonality relation

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(x)\, x^j (1-x)^\alpha (1+x)^\beta\, dx = 0,\ 0 \leq j \leq n-1,$$

normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{2.1}$$

**Theorem 2.1**
*(a) Let $\rho > 0$ and*

$$\alpha = \rho + \frac{d}{2}; \beta = \rho + \frac{d}{2} - 1. \tag{2.2}$$

*Let*

$$c_{n,\rho} = \frac{2\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \frac{\Gamma(n+2\alpha)}{\Gamma(n+\alpha)} / \int_0^\pi (\sin\psi)^{2\rho-1}\, d\psi. \tag{2.3}$$

*Then for $\mathbf{x}, \mathbf{y} \in \bar{B}$,*

$$K_n(W_\rho, \mathbf{x}, \mathbf{y}) = c_{n,\rho} \int_0^\pi P_n^{(\alpha,\beta)}\left(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\cos\psi\right)(\sin\psi)^{2\rho-1}\, d\psi.$$

(2.4)

*(b) For $\rho = 0$, let*

$$\alpha = \frac{d}{2}; \beta = \frac{d}{2} - 1. \tag{2.5}$$

*Let*

$$c_{n,0} = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \frac{\Gamma(n+2\alpha)}{\Gamma(n+\alpha)}. \tag{2.6}$$

*Then*

$$\begin{aligned} K_n(W_0, \mathbf{x}, \mathbf{y}) &= c_{n,0}\{P_n^{(\alpha,\beta)}\left(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right) \\ &\quad + P_n^{(\alpha,\beta)}\left(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right)\}. \end{aligned} \tag{2.7}$$

**Proof**
See [14, Thm 3.3, pp. 2448-2449]. ■

Next, we recall the Mehler-Heine asymptotic formula for Jacobi polynomials:

**Lemma 2.2**
*Let $\alpha > 0, \beta > -1$. Uniformly for $s$ in bounded subsets of $[0,\infty)$, we have*

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{s}{2n^2}\right) = 2^\alpha J_\alpha^*\left(\sqrt{s}\right). \tag{2.8}$$

**Proof**

See [10, Thm. 8.1.1, p. 192]. ■

We turn to the special case of Theorem 1.1 for ultraspherical weights. Note that if $s = 0$, this has been partly done by Xu:

**Theorem 2.3**

*Let $\{\mathbf{x}_n\}$ be a sequence in $\bar{B}$ such that for some $s \geq 0$,*

$$\lim_{n\to\infty} n^2\left(1 - \|\mathbf{x}_n\|^2\right) = s. \tag{2.9}$$

*(a) If $\rho > 0$, and $\alpha$ is as in (2.2),*

$$\begin{aligned} &\lim_{n\to\infty} n^{-2\alpha} K_n\left(W_\rho, \mathbf{x}_n, \mathbf{x}_n\right) \\ &= 2^{\alpha+1} \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \frac{\int_0^\pi J_\alpha^*\left(2\sqrt{s}\left|\sin\frac{\psi}{2}\right|\right)(\sin\psi)^{2\rho-1}\,d\psi}{\int_0^\pi (\sin\psi)^{2\rho-1}\,d\psi}. \end{aligned} \tag{2.10}$$

*The limit holds uniformly for $s$ in bounded subsets of $[0, \infty)$. In particular, if $s = 0$, then*

$$\lim_{n\to\infty} n^{-2\alpha} K_n\left(W_\rho, \mathbf{x}_n, \mathbf{x}_n\right) = \frac{2}{\Gamma(2\alpha+1)}. \tag{2.11}$$

*(b) If $\rho = 0$, and $\alpha$ is as in (2.5),*

$$\lim_{n\to\infty} n^{-2\alpha} K_n\left(W_\rho, \mathbf{x}_n, \mathbf{x}_n\right) = \frac{1}{\Gamma(2\alpha+1)}\left\{1 + \frac{J_\alpha^*(2\sqrt{s})}{J_\alpha^*(0)}\right\}. \tag{2.12}$$

*In particular, if $s = 0$, then (2.11) holds.*

**Proof**

(a) Write

$$1 - \|\mathbf{x}_n\|^2 = \frac{s_n}{n^2},\ n \geq 1, \tag{2.13}$$

where

$$\lim_{n\to\infty} s_n = s.$$

From Theorem 2.1(a),

$$\begin{aligned} K_n\left(W_\rho, \mathbf{x}_n, \mathbf{x}_n\right) &= c_{n,\rho} \int_0^\pi P_n^{(\alpha,\beta)}\left(\|\mathbf{x}_n\|^2 + \left(1 - \|\mathbf{x}_n\|^2\right)\cos\psi\right)(\sin\psi)^{2\rho-1}\,d\psi \\ &= c_{n,\rho} \int_0^\pi P_n^{(\alpha,\beta)}\left(1 - \frac{2s_n}{n^2}\sin^2\frac{\psi}{2}\right)(\sin\psi)^{2\rho-1}\,d\psi, \end{aligned} \tag{2.14}$$

by (2.13). Here, uniformly for $\psi \in [0, \pi]$, Lemma 2.2 gives

$$n^{-\alpha} P_n^{(\alpha,\beta)}\left(1 - \frac{2s_n}{n^2}\sin^2\frac{\psi}{2}\right) = 2^\alpha J_\alpha^*\left(2\sqrt{s}\left|\sin\frac{\psi}{2}\right|\right) + o(1). \tag{2.15}$$

Moreover, using the fact that as $x \to \infty$,

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = x^{a-b}(1 + o(1)),$$

we see from (2.3) that

$$c_{n,\rho}=n^{\alpha}\left(1+o\left(1\right)\right)\frac{2\Gamma\left(\alpha+1\right)}{\Gamma\left(2\alpha+1\right)}/\int_{0}^{\pi}\left(\sin\psi\right)^{2\rho-1}d\psi.$$

Substituting this and (2.15) into (2.14), gives (2.10). In the special case when $s=0$, we have

$$\frac{\int_{0}^{\pi}J_{\alpha}^{*}\left(2\sqrt{s}\left|\sin\frac{\psi}{2}\right|\right)\left(\sin\psi\right)^{2\rho-1}d\psi}{\int_{0}^{\pi}\left(\sin\psi\right)^{2\rho-1}d\psi}=J_{\alpha}^{*}\left(0\right)=\frac{1}{2^{\alpha}\Gamma\left(\alpha+1\right)},$$

and (2.10) simplifies to (2.11).
(b) Here Theorem 2.1(b) gives

$$\begin{aligned}K_{n}\left(W_{0},\mathbf{x}_{n},\mathbf{x}_{n}\right) &= c_{n,0}\{P_{n}^{(\alpha,\beta)}\left(1\right)+P_{n}^{(\alpha,\beta)}\left(2\left\|\mathbf{x}_{n}\right\|^{2}-1\right)\}\\ &= c_{n,0}\{P_{n}^{(\alpha,\beta)}\left(1\right)+P_{n}^{(\alpha,\beta)}\left(1-\frac{2s_{n}}{n^{2}}\right)\}.\end{aligned}$$

The result follows from Lemma 2.2 in an easier fashion than (a). ■

## 3. Proof of Theorem 1.1

We use "needle" polynomials from [5], based on univariate needle polynomials from [6]:

**Lemma 3.1**
*Let $n\geq 1$, $\delta\in(0,1)$, and $\mathbf{x}\in\bar{B}$. There exists $q_{n}\in\Pi_{n}^{d}$ such that*
*(i)* $q_{n}\left(\mathbf{x}\right)=1$;
*(ii)*

$$0\leq q_{n}<1\text{ in }\bar{B};$$

*(iii)*

$$\left|q_{n}\left(\mathbf{y}\right)\right|\leq e^{-cn\delta},\ \mathbf{y}\in B\backslash B\left(\mathbf{x},\delta\right).$$

*Here $c$ is an absolute constant.*
**Remark**
We emphasize that $q_{n}$ depends on $\mathbf{x}$ and $\delta$.
**Proof**
See Lemma 2.1 in [5]. ■

**Proof of Theorem 1.1(a), (b)**
The proof is very similar to that of Theorem 1.1 in [5]. As the measure $\mu$ is regular, with support $\bar{B}$, there exists a sequence $\{\delta_{n}\}$ with limit 0 such that for $n\geq 1$,

$$\sup_{P\in\Pi_{n}^{d}}\frac{\left\|P\right\|_{L_{\infty}(\bar{B})}^{2}}{\int\left|P\right|^{2}d\mu}\leq e^{n\delta_{n}^{2}}. \tag{3.1}$$

We may assume that

$$\lim_{n\rightarrow\infty}n\delta_{n}^{2}=\infty. \tag{3.2}$$

Since $h$ is uniformly continuous on $D$ as a function in $D_{1}\cap\bar{B}$,

$$\begin{aligned}\varepsilon_n &= \sup\{|h(\mathbf{x})-h(\mathbf{y})| : \mathbf{x},\mathbf{y}\in D_1\cap B \text{ with } \|\mathbf{x}-\mathbf{y}\|\le\delta_n \text{ and } \operatorname{dist}(\mathbf{x},D)\le\delta_n\}\\ &\to 0,\ n\to\infty.\end{aligned}$$

(3.3)

Let us set $m=m(n)=n-\left[\frac{2\delta_n n}{c}\right]-1$, where $c$ is the absolute constant in Lemma 3.1. Choose $p_m\in\Pi_n^d$ that is extremal for $\lambda_m(W_\rho,\mathbf{x}_n)$, so that

$$\lambda_m(W_\rho,\mathbf{x}_n)=\int p_m^2W_\rho \text{ and } p_m(\mathbf{x}_n)=1.$$

Choose $q_{n-m}$ as in Lemma 3.1, with the properties $q_{n-m}(\mathbf{x}_n)=1$; $0\le q_{n-m}\le 1$ in $B$; and

$$|q_{n-m}(\mathbf{x})|\le e^{-c(n-m)\frac{\delta_n}{2}},\ \mathbf{x}\in B\backslash B\left(\mathbf{x}_n,\frac{\delta_n}{2}\right).$$

Set

$$S_n=p_mq_{n-m}\in\Pi_n^d.$$

We have $S_n(\mathbf{x}_n)=1$, and so the extremal property of $\lambda_n$, followed by the properties of $q_{n-m}$, give

$$\begin{aligned}&\lambda_n(\mu,\mathbf{x}_n)\\ \le\ &\int_{\bar B}S_n^2d\mu\\ \le\ &\int_{B(\mathbf{x}_n,\delta_n)\cap\bar B}p_m^2hW_\rho+e^{-c(n-m)\delta_n}\|p_m\|^2_{L_\infty(\bar B)}\int_{\bar B\backslash B(\mathbf{x}_n,\delta_n)}d\mu\\ \le\ &(h(\mathbf{x}_n)+\varepsilon_n)\int_{B(\mathbf{x}_n,\delta_n)\cap\bar B}p_m^2W_\rho+e^{-c(n-m)\delta_n}e^{n\delta_n^2}\left(\int_{\bar B}p_m^2W_\rho\right)\left(\int_{\bar B}d\mu\right),\end{aligned}$$

by (3.1) and (3.3). Using our choice of $m$, we continue this as

$$\begin{aligned}\lambda_n(\mu,\mathbf{x}_n)&\le\left(\int_{\bar B}p_m^2W_\rho\right)\left(h(\mathbf{x}_n)+\varepsilon_n+e^{-2n\delta_n^2+n\delta_n^2}\int_{\bar B}d\mu\right)\\ &=\lambda_m(W_\rho,\mathbf{x}_n)\left(h(\mathbf{x}_n)+\varepsilon_n+e^{-n\delta_n^2}\int_{\bar B}d\mu\right).\end{aligned}$$

Since $\delta_n$ and $\varepsilon_n$ are independent of $\mathbf{x}_n$, we have

$$\begin{aligned}\frac{\lambda_n(\mu,\mathbf{x}_n)}{\lambda_n(W_\rho,\mathbf{x}_n)}&\le\frac{\lambda_m(W_\rho,\mathbf{x}_n)}{\lambda_n(W_\rho,\mathbf{x}_n)}\left(h(\mathbf{x}_n)+\varepsilon_n+e^{-n\delta_n^2}\int_{\bar B}d\mu\right)\\ &\le h(\mathbf{x}_n)+o(1),\end{aligned}$$

because $\frac{m}{n}=1+o(1)$, and we have the asymptotic in Theorem 2.3, holding uniformly for $s$ in compact subsets of $[0,\infty)$. Thus

$$\limsup_{n\to\infty}\frac{\lambda_n(\mu,\mathbf{x}_n)}{\lambda_n(W_\rho,\mathbf{x}_n)h(\mathbf{x}_n)}\le 1. \tag{3.4}$$

For the converse inequality, we note that with $m_1=m_1(n)=n+\left[\frac{2\delta_n n}{c}\right]$, we obtain by swapping the roles of $W_\rho$ and $\mu$ in the above,

$$\lambda_{m_1}(W_\rho,\mathbf{x}_n)\le\lambda_n(\mu,\mathbf{x}_n)\left(h^{-1}(\mathbf{x}_n)+o(1)+e^{-n\delta_n^2}\int_{\bar B}W_\rho\right),$$

and hence

$$\frac{\lambda_{m_1}(W_\rho, \mathbf{x}_n)}{\lambda_n(W_\rho, \mathbf{x}_n)} \leq \frac{\lambda_n(\mu, \mathbf{x}_n)}{\lambda_n(W_\rho, \mathbf{x}_n)} \left( h^{-1}(\mathbf{x}_n) + o(1) + e^{-n\delta_n^2} \int_{\bar{B}} W_\rho \right).$$

Here the left-hand side is $1 + o(1)$ by Theorem 2.3, and as $\frac{m_1}{n} = 1 + o(1)$, so

$$1 \leq \liminf_{n\to\infty} \frac{\lambda_n(\mu, \mathbf{x}_n)}{\lambda_n(W_\rho, \mathbf{x}_n)} h^{-1}(\mathbf{x}_n).$$

Together with (3.4), this gives

$$\lim_{n\to\infty} \frac{\lambda_n(\mu, \mathbf{x}_n)}{\lambda_n(W_\rho, \mathbf{x}_n) h(\mathbf{x}_n)} = 1.$$

Now apply Theorem 2.3. ■

## 4. Universality for Ultraspherical Weights

In this section, we obtain universality results for ultraspherical weights, as a special case of Theorem 1.2. We have to distinguish between the cases where we stay on the sphere (where the perturbation may have size $O\left(\frac{1}{n}\right)$) and where we move inside (where it needs to have size $O\left(\frac{1}{n^2}\right)$). We also distinguish between $W_\rho$ for $\rho > 0$ and $\rho = 0$. Let $G$ be given by (1.13), so that

$$G(\mathbf{u}, \mathbf{v}, \psi) = \left( |\mathbf{x} \cdot \mathbf{u}|^{1/2} - |\mathbf{x} \cdot \mathbf{v}|^{1/2} \right)^2 + 4 |\mathbf{x} \cdot \mathbf{u}|^{1/2} |\mathbf{x} \cdot \mathbf{v}|^{1/2} \left( \sin \frac{\psi}{2} \right)^2.$$

**Theorem 4.1**
*Fix $\rho > 0$ and let $\alpha, \beta$ be defined by (2.2).*
*(a) Let $\|\mathbf{x}\| = 1$, $\mathbf{x} \cdot \mathbf{u} < 0$ and $\mathbf{x} \cdot \mathbf{v} < 0$. Then*

$$\lim_{n\to\infty} \frac{K_n\left(W_\rho, \mathbf{x} + \frac{1}{n^2}\mathbf{u}, \mathbf{x} + \frac{1}{n^2}\mathbf{v}\right)}{K_n(W_\rho, \mathbf{x}, \mathbf{x})} = \frac{\int_0^\pi J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, \psi)}\right) (\sin\psi)^{2\rho - 1} d\psi}{J_\alpha^*(0) \int_0^\pi (\sin\psi)^{2\rho-1} d\psi}. \tag{4.1}$$

*(b) For $n \geq 1$, let $\|\mathbf{a}_n\| = 1 = \|\mathbf{b}_n\|$,* with *$\|\mathbf{a}_n - \mathbf{b}_n\| = O\left(\frac{1}{n}\right)$. Then as $n \to \infty$,*

$$\frac{K_n(W_\rho, \mathbf{a}_n, \mathbf{b}_n)}{K_n(W_\rho, \mathbf{a}_n, \mathbf{a}_n)} = \frac{J_\alpha^*(n\|\mathbf{a}_n - \mathbf{b}_n\|)}{J_\alpha^*(0)} + o(1). \tag{4.2}$$

For $\rho = 0$, we prove:

**Theorem 4.2**
*Let $\alpha, \beta$ be defined by (2.5).*
*(a) Let $\|\mathbf{x}\| = 1$, $\mathbf{x} \cdot \mathbf{u} < 0$ and $\mathbf{x} \cdot \mathbf{v} < 0$. Then*
(4.3)

$$\lim_{n\to\infty} \frac{K_n\left(W_0, \mathbf{x} + \frac{1}{n^2}\mathbf{u}, \mathbf{x} + \frac{1}{n^2}\mathbf{v}\right)}{K_n(W_0, \mathbf{x}, \mathbf{x})} = \frac{1}{2} \frac{J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, 0)}\right) + J_\alpha^*\left(\sqrt{2G(\mathbf{u}, \mathbf{v}, \pi)}\right)}{J_\alpha^*(0)}.$$

*(b) For $n \geq 1$, let $\|\mathbf{a}_n\| = 1 = \|\mathbf{b}_n\|$,* with *$\|\mathbf{a}_n - \mathbf{b}_n\| = O\left(\frac{1}{n}\right)$. Then as $n \to \infty$,*

$$\frac{K_n(W_0, \mathbf{a}_n, \mathbf{b}_n)}{K_n(W_0, \mathbf{a}_n, \mathbf{a}_n)} = \frac{J_\alpha^*(n\|\mathbf{a}_n - \mathbf{b}_n\|)}{J_\alpha^*(0)} + o(1). \tag{4.4}$$

We begin with an elementary lemma. In the sequel, we abbreviate $G(\mathbf{u},\mathbf{v},\psi)$ as $G(\psi)$.

**Lemma 4.3** *Assume that* $\|\mathbf{x}\|=1$, *and*

$$\mathbf{z}=\mathbf{x}+\frac{1}{n^2}\mathbf{u} \text{ and } \mathbf{y}=\mathbf{x}+\frac{1}{n^2}\mathbf{v}, \tag{4.5}$$

*where* $\mathbf{x}\cdot\mathbf{u}<0$ *and* $\mathbf{x}\cdot\mathbf{u}<\mathbf{0}$. *Then uniformly for* $\psi\in[0,\pi]$,

$$\begin{aligned}&\mathbf{z}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\cos\psi\\ =\;& 1-\frac{1}{n^2}G(\psi)+O\left(\frac{1}{n^4}\right).\end{aligned} \tag{4.6}$$

**Proof**
Now

$$\|\mathbf{z}\|^2=1+\frac{2}{n^2}\mathbf{x}\cdot\mathbf{u}+\frac{1}{n^4}\|\mathbf{u}\|^2$$

so

$$\sqrt{1-\|\mathbf{z}\|^2}=\frac{1}{n}\sqrt{-2\mathbf{x}\cdot\mathbf{u}-\frac{1}{n^2}\|\mathbf{u}\|^2}=\frac{1}{n}\sqrt{2|\mathbf{x}\cdot\mathbf{u}|}+O\left(\frac{1}{n^3}\right).$$

Similarly,

$$\sqrt{1-\|\mathbf{y}\|^2}=\frac{1}{n}\sqrt{2|\mathbf{x}\cdot\mathbf{v}|}+O\left(\frac{1}{n^3}\right).$$

Next,

$$\mathbf{z}\cdot\mathbf{y}=1+\frac{1}{n^2}\mathbf{x}\cdot(\mathbf{u}+\mathbf{v})+\frac{1}{n^4}\mathbf{u}\cdot\mathbf{v}.$$

Then

$$\begin{aligned}&\mathbf{z}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\cos\psi\\
=\;&\mathbf{z}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\left\{1-2\sin^2\frac{\psi}{2}\right\}\\
=\;&1+\frac{1}{n^2}\left\{\mathbf{x}\cdot(\mathbf{u}+\mathbf{v})+2\sqrt{|\mathbf{x}\cdot\mathbf{u}|\,|\mathbf{x}\cdot\mathbf{v}|}-4\sqrt{|\mathbf{x}\cdot\mathbf{u}|\,|\mathbf{x}\cdot\mathbf{v}|}\sin^2\frac{\psi}{2}\right\}+O\left(\frac{1}{n^4}\right)\\
=\;&1-\frac{1}{n^2}\left\{\left(|\mathbf{x}\cdot\mathbf{u}|^{1/2}-|\mathbf{x}\cdot\mathbf{v}|^{1/2}\right)^2+4\sqrt{|\mathbf{x}\cdot\mathbf{u}|\,|\mathbf{x}\cdot\mathbf{v}|}\sin^2\frac{\psi}{2}\right\}+O\left(\frac{1}{n^4}\right)\\
=\;&1-\frac{G(\psi)}{n^2}+O\left(\frac{1}{n^4}\right).\end{aligned}$$

■

**Proof of Theorem 4.1(a)**
Let $\mathbf{z},\mathbf{y}$ be given by (4.5). From (2.4), and (4.6),

$$\begin{aligned}n^{-\alpha}K_n(W_\rho,\mathbf{z},\mathbf{y}) &= c_{n,\rho}\int_0^\pi n^{-\alpha}P_n^{(\alpha,\beta)}\left(\mathbf{z}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\cos\psi\right)(\sin\psi)^{2\rho-1}\,d\psi\\
&= c_{n,\rho}\int_0^\pi n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{G(\psi)}{n^2}+O\left(\frac{1}{n^4}\right)\right)(\sin\psi)^{2\rho-1}\,d\psi\\
&= c_{n,\rho}\left\{2^\alpha\int_0^\pi J_\alpha^*\left(\sqrt{2G(\psi)}\right)(\sin\psi)^{2\rho-1}\,d\psi+o(1)\right\},\end{aligned}$$

by Lemma 2.2. In particular, when $\mathbf{u}=\mathbf{v}=\mathbf{0}$, so that $\mathbf{z}=\mathbf{y}=\mathbf{x}$, and $G(\psi)=0$, we obtain

$$n^{-\alpha}K_n(W_\rho,\mathbf{x},\mathbf{x})=c_{n,\rho}\left\{2^\alpha J_\alpha^*(0)\int_0^\pi(\sin\psi)^{2\rho-1}\,d\psi+o(1)\right\}. \tag{4.7}$$

These last two limits give the result. ■

**Proof of Theorem 4.1(b)**
As $\|\mathbf{a}_n\|=\|\mathbf{b}_n\|=1$,

$$\mathbf{a}_n\cdot\mathbf{b}_n=1-\frac{1}{2n^2}\left(n\|\mathbf{a}_n-\mathbf{b}_n\|\right)^2. \tag{4.8}$$

Then (2.4) shows that

$$\begin{aligned}
n^{-\alpha}K_n(W_\rho,\mathbf{a}_n,\mathbf{b}_n) &= c_{n,\rho}\int_0^\pi n^{-\alpha}P_n^{(\alpha,\beta)}\left(\mathbf{a}_n\cdot\mathbf{b}_n+\sqrt{1-\|\mathbf{a}_n\|^2}\sqrt{1-\|\mathbf{b}_n\|^2}\cos\psi\right)(\sin\psi)^{2\rho-1}\,d\psi\\
&= c_{n,\rho}\int_0^\pi n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{1}{2n^2}\left(n\|\mathbf{a}_n-\mathbf{b}_n\|\right)^2\right)(\sin\psi)^{2\rho-1}\,d\psi\\
&= c_{n,\rho}\left\{2^\alpha J_\alpha^*\left(n\|\mathbf{a}_n-\mathbf{b}_n\|\right)\int_0^\pi(\sin\psi)^{2\rho-1}\,d\psi+o(1)\right\},
\end{aligned}$$

by Lemma 2.2. Using this and its special case with $\mathbf{b}_n=\mathbf{a}_n$ gives the result.

**Proof of Theorem 4.2(a)**
Let $\mathbf{z},\mathbf{y}$ be given by (4.5). By (2.7) and (4.6), with $\psi=0,\pi$,

$$\begin{aligned}
n^{-\alpha}K_n(W_0,\mathbf{z},\mathbf{y}) &= c_{n,0}n^{-\alpha}\{P_n^{(\alpha,\beta)}\left(\mathbf{z}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right)\\
&\quad+P_n^{(\alpha,\beta)}\left(\mathbf{z}\cdot\mathbf{y}-\sqrt{1-\|\mathbf{z}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right)\}\\
&= c_{n,0}\{n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{G(0)}{n^2}+O\left(\frac{1}{n^4}\right)\right)\\
&\quad+n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{G(\pi)}{n^2}+O\left(\frac{1}{n^4}\right)\right)\}\\
&= c_{n,0}2^\alpha\left\{J_\alpha^*\left(\sqrt{2G(0)}\right)+J_\alpha^*\left(\sqrt{2G(\pi)}\right)+o(1)\right\},
\end{aligned}$$

by Lemma 2.2. Also, as $\|\mathbf{x}\|=1$, (2.7) gives

$$\begin{aligned}
n^{-\alpha}K_n(W_0,\mathbf{x},\mathbf{x}) &= 2c_{n,0}n^{-\alpha}P_n^{(\alpha,\beta)}(1)\\
&= c_{n,0}\left\{2^{\alpha+1}J_\alpha^*(0)+o(1)\right\},
\end{aligned}$$

by Lemma 2.2 again. Combining the last two limits, gives the result. ■

**Proof of Theorem 4.2(b)**

Here we still have (4.8), so (2.7) gives

$$\begin{aligned} K_n\left(W_0, \mathbf{a}_n, \mathbf{b}_n\right) &= c_{n,0}\{P_n^{(\alpha,\beta)}\left(\mathbf{a}_n\cdot\mathbf{b}_n+\sqrt{1-\|\mathbf{a}_n\|^2}\sqrt{1-\|\mathbf{b}_n\|^2}\right) \\ &\quad +P_n^{(\alpha,\beta)}\left(\mathbf{a}_n\cdot\mathbf{b}_n-\sqrt{1-\|\mathbf{a}_n\|^2}\sqrt{1-\|\mathbf{b}_n\|^2}\right)\} \\ &= 2c_{n,0}P_n^{(\alpha,\beta)}\left(\mathbf{a}_n\cdot\mathbf{b}_n\right) \\ &= c_{n,0}\left\{2^{\alpha+1}J_\alpha^*\left(n\left\|\mathbf{a}_n-\mathbf{b}_n\right\|\right)+o\left(1\right)\right\}, \end{aligned}$$

while

$$K_n\left(W_0, \mathbf{a}_n, \mathbf{a}_n\right) = c_{n,0}\left\{2^{\alpha+1}J_\alpha^*\left(0\right)+o\left(1\right)\right\}.$$

■

## 5. Proof of Theorems 1.2 and 1.3

The method follows that in [7]. We begin with

**Lemma 5.1**
*Assume that $\mu, \mu^*$ are measures with support $\bar{B} \subset \mathbb{R}^d$, and for some $\Delta > 0$,*

$$d\mu \leq \Delta\ d\mu^* \text{ in } \mathcal{K}. \tag{5.1}$$

*Then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,*

$$\begin{aligned} &\left|K_n\left(\mu, \mathbf{x}, \mathbf{y}\right)-\frac{1}{\Delta}K_n\left(\mu^*, \mathbf{x}, \mathbf{y}\right)\right|/K_n\left(\mu, \mathbf{x}, \mathbf{x}\right) \\ \leq\ &\left(\frac{K_n\left(\mu, \mathbf{y}, \mathbf{y}\right)}{K_n\left(\mu, \mathbf{x}, \mathbf{x}\right)}\right)^{1/2}\left[1-\frac{K_n\left(\mu^*, \mathbf{x}, \mathbf{x}\right)}{\Delta K_n\left(\mu, \mathbf{x}, \mathbf{x}\right)}\right]^{1/2}. \end{aligned} \tag{5.2}$$

**Proof**
See [5, Lemma 5.1]. ■

Now we can follow the method of [5], [7].

**Proof of Theorem 1.2(a)**
Let $\varepsilon \in (0,1)$ and choose $\delta > 0$ such that $h$ (which is positive and uniformly continuous on $D$) satisfies

$$(1+\varepsilon)^{-1} \leq h\left(\mathbf{y}\right)/h\left(\mathbf{z}\right) \leq 1+\varepsilon \text{ for } \mathbf{z}, \mathbf{y} \in B\left(\mathbf{x}, \delta\right) \cap \bar{B}, \tag{5.3}$$

whenever dist($\mathbf{x}$,$D$) $\leq \boldsymbol{\delta}$. Choose $\mathbf{x} \in D$. Set,

$$\tau = h\left(\mathbf{x}\right)^{-1}\left(1+\varepsilon\right).$$

We shall apply Lemma 5.1 twice. Define a measure $\mu^*$ by $d\mu^* = d\mu$ in $B\left(\mathbf{x}, \delta\right) \cap \bar{B}$, and

$$d\mu^* = \max\left\{1, \frac{1}{\tau}\right\}\left(W_\rho dm + d\mu\right) \text{ in } \bar{B}\backslash B\left(\mathbf{x}, \delta\right),$$

where, recall, $dm$ is Lebesgue measure.
**Step 1: $\mu$ and $\mu^*$**
Since $\mu^* \geq \mu$, we have the inequality (5.2) with $\Delta = 1$. Moreover, since $\mu$ is regular

and $\mu^* \geq \mu$, so $\mu^*$ is also regular. Next, from (1.9) of Theorem 1.1, as $\mu = \mu^*$ in $B(\mathbf{x}, \delta) \cap \bar{B}$,

$$\lim_{n \to \infty} \frac{K_n(\mu^*, \mathbf{x}_n, \mathbf{x}_n)}{K_n(\mu, \mathbf{x}_n, \mathbf{x}_n)} = 1$$

for any sequence $\{\mathbf{x}_n\}$ in $B\left(\mathbf{x}, \frac{\delta}{2}\right)$. In particular, this is the case if

$$\mathbf{x}_n = \mathbf{x} + \frac{\mathbf{u}}{n^2} \text{ or } \mathbf{x}_n = \mathbf{x} + \frac{\mathbf{v}}{n^2}, n \geq 1. \tag{5.4}$$

Moreover, Theorem 1.1(a), and the continuity in $s$, and positivity of the right-hand side of (1.9), show that with such $\{\mathbf{x}_n\}$,

$$\frac{K_n(\mu, \mathbf{x}_n, \mathbf{x}_n)}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C. \tag{5.5}$$

Then Lemma 5.1, with $\Delta = 1$ there, shows that for $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

$$\lim_{n \to \infty} \frac{K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right) - K_n\left(\mu^*, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 0. \tag{5.6}$$

**Step 2: $W_\rho dm$ and $\mu^*$**

Now $W_\rho dm \leq \tau \, d\mu^*$ in $\bar{B} \backslash B(\mathbf{x}, \delta)$. Also, in $\bar{B} \cap B(\mathbf{x}, \delta)$, (5.3) and our choice of $\tau$ show that

$$W_\rho dm = h^{-1} d\mu \leq \tau \, d\mu \leq \tau \, d\mu^*. \tag{5.7}$$

So in $\bar{B}$, $W_\rho dm \leq \tau \, d\mu^*$. By Theorem 1.1, and (5.6), with $\mathbf{x}_n = \mathbf{x} + \frac{\mathbf{u}}{n^2}$, $n \geq 1$, and by continuity of $h$,

$$\lim_{n \to \infty} \frac{K_n(\mu^*, \mathbf{x}_n, \mathbf{x}_n)}{\tau K_n(W_\rho, \mathbf{x}_n, \mathbf{x}_n)} = \frac{1}{\tau h(\mathbf{x})} = (1 + \varepsilon)^{-1}.$$

Furthermore, we have

$$\frac{K_n(W_\rho, \mathbf{x}_n, \mathbf{x}_n)}{K_n(W_\rho, \mathbf{x}, \mathbf{x})} \leq C. \tag{5.8}$$

Then Lemma 5.1 with $\Delta = \tau$ gives

$$\frac{\left|K_n\left(W_\rho, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right) - \frac{1}{\tau} K_n\left(\mu^*, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right)\right|}{K_n(W_\rho, \mathbf{x}, \mathbf{x})}$$

$$\leq C \left[1 - \frac{K_n(\mu^*, \mathbf{x}_n, \mathbf{x}_n)}{\tau K_n(W_\rho, \mathbf{x}_n, \mathbf{x}_n)}\right]^{1/2} \leq C_1 (2\varepsilon)^{1/2}, \tag{5.9}$$

for $n \geq n_0$, and $\mathbf{u}, \mathbf{v}$ in a given compact subset of $\mathbb{R}^d$. Since $C$ is independent of $\mathbf{u}, \mathbf{v}, n$, we obtain from this and (5.6), and the bound on the Christoffel functions in Lemma 5.2,

$$\frac{\left|\tau K_n\left(W_\rho, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right) - K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right)\right|}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C_1 \varepsilon^{1/2},$$

and hence for large enough $n$, and $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

$$\frac{\left|h(\mathbf{x})^{-1} K_n\left(W_\rho, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right) - K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n^2}, \mathbf{x} + \frac{\mathbf{v}}{n^2}\right)\right|}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C_1 \varepsilon^{1/2},$$

where $C_1$ is independent of $\mathbf{u}, \mathbf{v}, \mathbf{x}, n$. Then Theorem 4.1(a) and Theorem 1.1 now give the result. ■

**Proof of Theorem 1.2(b)**
This follows similarly from Theorem 4.1(b). ■

**Proof of Theorem 1.3**
This follows similarly from Theorem 4.2. ■

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