CHRISTOFFEL FUNCTIONS AND UNIVERSALITY IN THE BULK FOR MULTIVARIATE ORTHOGONAL POLYNOMIALS

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ABSTRACT. We establish asymptotics for Christoffel functions associated with multivariate orthogonal polynomials. The underlying measures are assumed to be regular on a suitable domain - in particular this is true if they are positive a.e on a compact set that admits analytic parametrization. As a consequence, we obtain asymptotics for Christoffel functions for measures on the ball and simplex, under far more general conditions than previously known. As another consequence, we establish universality type limits in the bulk in a variety of settings.

Orthogonal Polynomials, Random Matrices, Unitary Ensembles, Correlation Functions, Christoffel functions. 42C05, 42C99, 42B05, 60B20

1. INTRODUCTION[1]

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \ldots, \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

The nth *reproducing kernel* is

$$K_n(x, t) = \sum_{j=0}^{n-1} p_j(x) p_j(t)$$

and the nth *Christoffel function* is

$$(1.1) \qquad \lambda_n(\mu, x) = \frac{1}{K_n(x, x)} = \inf_{\deg(P)<n} \frac{\int P(t)^2 d\mu(t)}{P^2(x)}.$$

Asymptotics for Christoffel functions play a crucial role in analysis of orthogonal polynomials, and in weighted approximation [12]. The most general asymptotics for the case where the support supp[μ] is compact, require μ to be *regular in the sense of Stahl, Totik, and Ullman,* or just *regular.* This requires [15, p. 66] that

$$(1.2) \qquad \lim_{n \to \infty} \left(\sup_{\deg(P) \le n} \frac{|P(x)|^2}{\int |P|^2 d\mu} \right)^{1/n} \le 1,$$

q.e. in supp$[\mu]$. Here q.e. (quasi-everywhere) means except on a set of logarithmic capacity 0. This type of regularity should not be confused with the notion of a regular Borel measure. When the complement of supp$[\mu]$ is regular in the sense of the Dirichlet problem, (yet another notion of regularity!) [15, p. 68], one may replace (1.2) by

$$(1.3) \qquad \lim_{n\to\infty} \left(\sup_{\deg(P)\leq n} \frac{\|P\|_{L_\infty(\mathrm{supp}[\mu])}}{\int |P|^2 \, d\mu} \right)^{1/n} \leq 1.$$

If supp$[\mu]$ consists of finitely many intervals, a sufficient condition for regularity, called the Erdős-Turán criterion, is that $\mu' > 0$ a.e. in supp$[\mu]$. See the comprehensive monograph [15].

When μ is regular, and in some subinterval I of the support, we have

$$(1.4) \qquad \int_I \log \mu' > -\infty,$$

Totik [16] proved that for a.e. $x \in I$,

$$\lim_{n\to\infty} n\lambda_n(\mu, x) = \frac{\mu'(x)}{\omega(x)}.$$

Here ω is the equilibrium density for supp$[\mu]$, in the sense of potential theory. In the special case that the support of μ is an interval, this result had been established earlier by Maté, Nevai, and Totik [10].

One application of asymptotics for Christoffel functions is to universality limits. These arise in analysis of random matrices associated with unitary ensembles, and for compactly supported μ, may be reduced to the limit

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{a\mu'(x)}{n\omega(x)}, x + \frac{b\mu'(x)}{n\omega(x)} \right)}{K_n(x,x)} = \frac{\sin \pi(a-b)}{\pi(a-b)},$$

uniformly for a, b in compact subsets of the real line. For the case of compactly supported μ, that are regular, and satisfy (1.4), Totik [17] showed that universality holds for a.e. $x \in I$. The second author showed, that universality holds in measure without assuming regularity [9]. We emphasize that there is a vast literature on this topic, and varying measures are of more interest to physicists than fixed measures with compact support. See [4], [5], [13], [14].

In this paper, we shall analyze asymptotics of Christoffel functions for multivariate orthogonal polynomials, and apply these to universality type limits. Let $d \geq 2$, and Π_n^d denote the space of polynomials in d variables of degree at most n. Let N_n^d denote its dimension, so

$$N_n^d = \binom{n+d}{n}.$$

Let μ be a positive measure on $\mathbb{R}^d$ with compact support such that $\left\{ \mathbf{x} \in \mathbb{R}^d : \mu'(\mathbf{x}) > 0 \right\}$ has non-empty interior. This ensures that

$$\int P^2 d\mu > 0$$

for every non-trivial polynomial P.

We let $K_n\left(\mu, \mathbf{x}, \mathbf{y}\right)$ denote the reproducing kernel for μ and Π_n^d, so that for all $P \in \Pi_n^d$, and all $\mathbf{x} \in \mathbb{R}^d$,

$$P\left(\mathbf{x}\right) = \int K_n\left(\mu, \mathbf{y}, \mathbf{x}\right) d\mu\left(\mathbf{y}\right).$$

Note that this notation is different to the one dimensional case, where we assumed exactness for polynomials of degree $\leq n - 1$, in accordance with the standard univariate notation. We adopt this difference to be consistent with the most common multivariate convention.

One of the convenient features of the reproducing kernel is that it is independent of how we order the monomials, and generate orthonormal polynomials. The nth Christoffel function for μ is

$$\lambda_n\left(\mu, \mathbf{x}\right) = \frac{1}{K_n\left(\mu, \mathbf{x}, \mathbf{x}\right)}.$$

It admits the extremal property

(1.5) $$\lambda_n\left(\mu, \mathbf{x}\right) = \inf_{P \in \Pi_n^d} \frac{\int P\left(\mathbf{t}\right)^2 d\mu\left(\mathbf{t}\right)}{P^2\left(\mathbf{x}\right)}.$$

When μ is absolutely continuous with respect to d dimensional Lebesgue measure, and $\mu' = W$, we shall write $\lambda_n\left(W, \mathbf{x}\right)$.

Asymptotics for these multivariate Christoffel functions have been established in a number of papers [1], [2], [3], [19], [20], [23], for Jacobi weights, and weights that satisfy some structural restriction, such as being radially or centrally symmetric. For our purposes, the most general result is due to Bos, Della Vecchia and Mastroianni [2]. They showed that for a centrally symmetric weight $W\left(\mathbf{x}\right)$ on the d dimensional ball, for which $W\left(\mathbf{x}\right)\sqrt{1 - \|\mathbf{x}\|^2}$ satisfies a centrally symmetric Lipschitz condition of some positive order on the unit ball in $\mathbb{R}^d$,

$$\lim_{n \to \infty} \binom{n + d}{d} \lambda_n\left(W, \mathbf{x}\right) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \sqrt{1 - \|\mathbf{x}\|^2} W\left(\mathbf{x}\right), \ \ \|\mathbf{x}\| < 1.$$

Here, and in the sequel, the norm is the Euclidean norm. Xu [20] established one-sided asymptotics under more general conditions.

By using "needle polynomials", we shall generalize the above quoted results to a much larger class of measures. We first extend the notion of regularity to d dimensions. A compactly supported measure μ on $\mathbb{R}^d$ is said to be *regular*, if

(1.6) $$\lim_{n \to \infty} \left(\sup_{P \in \Pi_n^d} \frac{\|P\|_{L_\infty\left(\text{supp}[\mu]\right)}^2}{\int |P|^2 d\mu} \right)^{1/n} = 1.$$

This is often called the Bernstein-Markov condition [3], but we prefer the term regularity.

Our most general ratio asymptotic is:

Theorem 1.1
Let μ, ν be positive measures, whose support is a compact set $\mathcal{K} \subset \mathbb{R}^d$, and both are regular. Let $D \subset D_1 \subset \mathcal{K}$, where D is compact and D_1 is open. Assume that ν and μ are mutually absolutely continuous in D, and the Radon-Nikodym derivative

$\frac{d\nu}{d\mu}$ is positive and continuous in D, while uniformly in $\overline{D_1}$,

$$(1.7) \qquad \lim_{\varepsilon \to 0+} \left(\limsup_{n \to \infty} \frac{\lambda_{[n(1-\varepsilon)]}(\mu, \mathbf{x})}{\lambda_n(\mu, \mathbf{x})} \right) = 1.$$

Then uniformly for $\mathbf{x} \in D$, and $\mathbf{y} \in B\left(\mathbf{x}, \frac{1}{\sqrt{n}}\right)$, we have

$$(1.8) \qquad \lim_{n \to \infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} = \frac{d\nu}{d\mu}(\mathbf{x}).$$

Remark

(a) In the statement, $[s]$ denotes the greatest integer $\leq s$, while $B(\mathbf{x}, r)$ is the $d-$dimensional open ball with center $\mathbf{x}$, radius r.

(b) Note that (1.7) is satisfied if for some $\beta > 0$ and positive continuous function F,

$$(1.9) \qquad \lim_{n \to \infty} n^\beta \lambda_n(\mu, \mathbf{x}) = F(\mathbf{x}),$$

uniformly in $\overline{D_1}$.

(c) In Section 7, we shall present a version of Theorem 1.1 in which we replace continuity of $\frac{d\nu}{d\mu}$ by a Lebesgue point condition.

A sufficient condition for the regularity (1.6) involves a compact set $\mathcal{K}$ with *analytic parametrization*. This means that for any $\mathbf{x} \in \mathcal{K}$, there exists a curve $\gamma(t) \in \mathbb{R}^d, t \in [0,1]$, analytic and bounded in an open set $\Omega \subset \mathbb{C}$ that contains $[0,1]$, and such that $\gamma(0) = \mathbf{x}$, while for all $0 < t < 1$,

$$B(\gamma(t), \phi(t)) \subset \mathcal{K}.$$

Here Ω and the bound on γ depend only on $\mathcal{K}$, while ϕ is a positive continuous function tending to 0 as $t \to 0$, and that also depends only on $\mathcal{K}$. In particular, any polygon or convex set with non-empty interior has analytic parametrization. In fact local convexity also suffices, see [6] for details.

Theorem 1.2

Assume that $\mathcal{K} \subset \mathbb{R}^d$ is a compact set with analytic parametrization, and that μ, ν are positive measures on $\mathcal{K}$ such that $\mu', \nu' > 0$ a.e. on $\mathcal{K}$. Let $D \subset D_1 \subset \mathcal{K}$, where D is compact and D_1 is open. Assume that the Radon-Nikodym derivative $\frac{\nu'}{\mu'}$ is positive and continuous in D, while uniformly in $\overline{D_1}$, (1.7) holds. Then uniformly for $\mathbf{x} \in D$, and $\mathbf{y} \in B\left(\mathbf{x}, \frac{1}{\sqrt{n}}\right)$, we have (1.8).

The proof of Theorem 1.2 also shows that we may replace the condition that $\mathcal{K}$ has analytic parametrization by a condition that it admits a Remez inequality.

As a consequence, we can deduce asymptotics for Christoffel functions associated with regular measures on the ball and the simplex:

Theorem 1.3

Let $\bar{B} = \overline{B(\mathbf{0}, 1)} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\}$. Let ν be a regular measure on $\bar{B}$, and assume that D is a compact subset of the interior of $\bar{B}$, such that ν' is positive

and continuous in D. Then uniformly for $\mathbf{x} \in D$, and $\mathbf{y} \in B\left(\mathbf{x}, \frac{1}{\sqrt{n}}\right)$,

$$(1.10) \qquad \lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu, \mathbf{y}) = \frac{\nu'(\mathbf{x})}{W_0^{\mathrm{ball}}(\mathbf{x})},$$

where

$$(1.11) \qquad W_0^{\mathrm{ball}}(\mathbf{x}) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} \left(1 - \|\mathbf{x}\|^2\right)^{-1/2}.$$

Theorem 1.4
Let

$$\Sigma^d = \left\{ \mathbf{x} \in \mathbb{R}^d : x_1, x_2, ..., x_d \geq 0; 1 - \sum_{j=1}^{d} x_j \geq 0 \right\}$$

denote the $d-$dimensional simplex. Let ν be a regular measure on Σ^d, and assume that D is a compact subset of the interior of Σ^d, such that ν' is positive and continuous in D. Then uniformly for $\mathbf{x} \in D$, and $\mathbf{y} \in B\left(\mathbf{x}, \frac{1}{\sqrt{n}}\right)$,

$$(1.12) \qquad \lim_{n\to\infty} \binom{n+d}{d} \lambda_n(\nu, \mathbf{y}) = \frac{\nu'(\mathbf{x})}{W_0^{\mathrm{simplex}}(\mathbf{x})},$$

where

$$(1.13) \qquad W_0^{\mathrm{simplex}}(\mathbf{x}) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}} x_1^{-1/2} x_2^{-1/2} ... x_d^{-1/2} x_{d+1}^{-1/2},$$

and

$$(1.14) \qquad x_{d+1} = 1 - \sum_{j=1}^{d} x_j.$$

In the second last line, of course $\mathbf{x} = (x_1, x_2, ..., x_d)$.

We next turn to universality limits. We start with a general ratio result:

Theorem 1.5
Let μ, ν be positive measures and $\mathcal{K}, D, D_1$ be sets satisfying the hypotheses of Theorem 1.1. Assume moreover, that μ and ν are absolutely continuous in D_1, while μ' and ν' are bounded above and below by positive constants in D_1. Then uniformly for $\mathbf{x} \in D$, and $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^d$,

$$(1.15) \qquad \lim_{n\to\infty} \frac{K_n\left(\mu, \mathbf{x}+\frac{\mathbf{u}}{n}, \mathbf{x}+\frac{\mathbf{v}}{n}\right) - \frac{d\nu}{d\mu}(\mathbf{x}) K_n\left(\nu, \mathbf{x}+\frac{\mathbf{u}}{n}, \mathbf{x}+\frac{\mathbf{v}}{n}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 0.$$

As a consequence, we can prove:

Theorem 1.6
Let μ, ν be positive measures and $\mathcal{K}, D, D_1$ be sets satisfying the hypotheses of Theorem 1.2. Assume moreover, that μ and ν are absolutely continuous in D_1, while μ' and ν' are bounded above and below by positive constants in D_1. Assume that for some function $F : \mathbb{R}^2 \to \mathbb{R}$, uniformly for $\mathbf{x} \in D$, and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$(1.16) \qquad \lim_{n\to\infty} \frac{K_n\left(\mu, \mathbf{x}+\frac{\mathbf{u}}{n}, \mathbf{x}+\frac{\mathbf{v}}{n}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = F(\mathbf{u}, \mathbf{v}).$$

Then uniformly for $\mathbf{x} \in D$, *and* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$(1.17) \qquad \lim_{n \to \infty} \frac{K_n \left(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n} \right)}{K_n \left(\nu, \mathbf{x}, \mathbf{x} \right)} = F \left(\mathbf{u}, \mathbf{v} \right).$$

It is straightforward to compute universality limits for the Chebyshev weight on the ball and simplex, from known representations due to Y. Xu, for the reproducing kernel for the Chebyshev weight on the ball and simplex. Using these, and Theorem 1.5, we can obtain general universality results on the ball and simplex. Somewhat surprisingly, the concrete formulations in the bulk involve the Bessel function

$$J_\alpha \left(z \right) = \left(\frac{z}{2} \right)^\alpha \sum_{j=0}^\infty \frac{\left(-\frac{1}{4} z^2 \right)^j}{j! \Gamma \left(j + \alpha + 1 \right)},$$

rather than the usual sinc kernel. Of course, on the real line the Bessel function and its associated Bessel kernel arise primarily at the edge of the spectrum. We shall find it convenient to also use

$$(1.18) \qquad J_\alpha^* \left(z \right) = z^{-\alpha} J_\alpha \left(z \right) = \left(\frac{1}{2} \right)^\alpha \sum_{j=0}^\infty \frac{\left(-\frac{1}{4} z^2 \right)^j}{j! \Gamma \left(j + \alpha + 1 \right)}.$$

It has the advantage of being entire, and in particular, non-zero at 0, with

$$(1.19) \qquad J_\alpha^* \left(0 \right) = \frac{1}{2^\alpha \Gamma \left(\alpha + 1 \right)}.$$

In the literature, $j_\alpha \left(x \right)$ is sometimes used to denote $J_\alpha^* \left(x \right) / J_\alpha^* \left(0 \right)$, but we shall not use this.

Theorem 1.7
Let μ *be a regular measure on* $\bar{B}$, *and assume that* D *is a compact subset of the interior of* $\bar{B}$, *such that* μ' *is positive and continuous in* D. *Then uniformly for* $\mathbf{x} \in D$, *and* $\mathbf{u}, \mathbf{v}$ *in compact subsets of* $\mathbb{R}^d$,

$$(1.20) \qquad \lim_{n \to \infty} \frac{K_n \left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n} \right)}{K_n \left(\mu, \mathbf{x}, \mathbf{x} \right)} = \frac{J_{d/2}^* \left(\sqrt{G \left(\mathbf{x}, \mathbf{u}, \mathbf{v} \right)} \right)}{J_{d/2}^* \left(0 \right)},$$

where, if $\cdot$ *denotes the standard Euclidean inner product,*

$$(1.21) \qquad G \left(\mathbf{x}, \mathbf{u}, \mathbf{v} \right) = \| \mathbf{u} - \mathbf{v} \|^2 + \frac{\left(\mathbf{x} \cdot \left(\mathbf{u} - \mathbf{v} \right) \right)^2}{1 - \| \mathbf{x} \|^2}.$$

For the simplex, we have

Theorem 1.8
Let μ *be a regular measure on* Σ^d, *and assume that* D *is a compact subset of the interior of* Σ^d, *such that* μ' *is positive and continuous in* D. *Then uniformly for* $\mathbf{x} \in D$, *and* $\mathbf{u}, \mathbf{v}$ *in compact subsets of* $\mathbb{R}^d$,

$$(1.22) \qquad \lim_{n \to \infty} \frac{K_n \left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n} \right)}{K_n \left(\mu, \mathbf{x}, \mathbf{x} \right)} = \frac{J_{d/2}^* \left(\sqrt{H \left(\mathbf{x}, \mathbf{u}, \mathbf{v} \right)} \right)}{J_{d/2}^* \left(0 \right)},$$

where, with the notation (1.14),

$$(1.23) \qquad H(\mathbf{x}, \mathbf{u}, \mathbf{v}) = \sum_{j=1}^{d+1} \frac{(u_j - v_j)^2}{x_j}.$$

Of course, in the last line, x_j is the jth component of $\mathbf{x}$, and so on, while u_{d+1}, v_{d+1} are given by (1.14) for the vectors $\mathbf{u}, \mathbf{v}$.

This paper is organised as follows: in Section 2, we prove Theorem 1.1. We prove Theorem 1.2 in Section 3, and Theorems 1.3 and 1.4 in Section 4. Theorem 1.5 and Theorem 1.6 are proved in Section 5, and Theorems 1.7 and 1.8 are proved in Section 6. We prove an extension of Theorem 1.1 in Section 7.

Throughout, $c, C, C_1, C_2, \ldots$ denote positive constants independent of n, and vectors $\mathbf{t}, \mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$, as well as polynomials p. The same constant does not necessarily denote the same constant in different occurrences.

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2. PROOF OF THEOREM 1.1

We shall use the "needle" polynomials constructed by Kroó and Swetits [7]. We could also have used the fast decreasing polynomials of Ivanov and Totik.

Lemma 2.1

Let $n \geq 1$, $\delta \in (0, 1)$, and $\mathbf{x} \in \bar{B}$. There exists $q_n \in \Pi_n^d$ such that
(i) $q_n(\mathbf{x}) = 1$;
(ii)

$$0 \leq q_n \leq 1 \text{ in } \bar{B};$$

(iii)

$$|q_n(\mathbf{y})| \leq e^{-cn\delta}, \ \mathbf{y} \in B \backslash B(\mathbf{x}, \delta).$$

Here c is an absolute constant.

Remark

We emphasize that q_n depends on $\mathbf{x}$ and δ. The theory of fast decreasing polynomials implies that one cannot choose q_n independent of δ.

Proof

We follow the construction of Lemma 3 and Corollary 2 in [7, pp. 92-93]. Consider the polynomial

$$r_m(t) = \left(\frac{T_m(1 + \delta^2 - t^2)}{T_m(1 + \delta^2)} \right)^2 \in \Pi_{4m}^1,$$

where T_m is the usual Chebyshev polynomial. Here $r_m(0) = 1$. For $t \in [-1, 1]$, we have $0 \leq 1 + \delta^2 - t^2 \leq 1 + \delta^2$, and T_n is increasing on $[1, \infty)$, so in $[-1, 1]$

$$0 \leq r_m \leq 1.$$

Finally, for $|t| \in [\delta, 1]$, we have $1 + \delta^2 - t^2 \leq 1$, so

$$r_m(t) \leq \frac{1}{\left(T_m(1 + \delta^2) \right)^2} \leq e^{-Cm\delta},$$

an easy consequence of the identity

$$T_m(t) = \frac{1}{2}\left(\left(t + \sqrt{t^2 - 1}\right)^m + \left(t + \sqrt{t^2 - 1}\right)^{-m}\right).$$

Now we set $m = [n/4]$, and

$$q_n(\mathbf{y}) = r_m\left(\frac{\|\mathbf{y} - \mathbf{x}\|}{4}\right).$$

Then clearly $q_n \in \Pi_n^d$. The properties (i) and (ii) follow directly from those of r_m, while if $\mathbf{y} \in B$ and $\|\mathbf{y} - \mathbf{x}\| \geq 4\delta$,

$$q_n(\mathbf{y}) \leq e^{-Cm\delta} \leq e^{-c_1 n\delta}.$$

Now simply replace δ by $\frac{\delta}{4}$. ∎

Proof of Theorem 1.1

As the measure μ is regular, with support $\mathcal{K}$, there exists a sequence $\{\delta_n\}$ with limit 0 such that for $n \geq 1$,

(2.1)
$$\sup_{P \in \Pi_n^d} \frac{\|P\|_{L_\infty(\mathrm{supp}[\mu])}^2}{\int |P|^2 \, d\mu} \leq e^{n\delta_n^2}.$$

We may assume that

(2.2)
$$\lim_{n \to \infty} n\delta_n^2 = \infty.$$

Since $h = \frac{d\nu}{d\mu}$ is uniformly continuous on D,

$$\varepsilon_n = \sup\{|h(\mathbf{x}) - h(\mathbf{y})| : \mathbf{x} \in D, \mathbf{y} \in \mathcal{K}, \|\mathbf{x} - \mathbf{y}\| \leq \delta_n\}$$
(2.3)
$$\to 0, \; n \to \infty.$$

Let us set $m = m(n) = n - \left[\frac{2\delta_n n}{c}\right] - 1$, where c is the absolute constant in Lemma 2.1. We may assume, by a translation and dilation of the support, that $\mathcal{K} \subset B$. Now choose any $\mathbf{x}_0 \in D$ and any $\mathbf{y} \in B\left(\mathbf{x}_0, \frac{\delta_n}{2}\right)$. Choose $p_m \in \Pi_n^d$ that is extremal for $\lambda_m(\mu, \mathbf{y})$, so that

$$\lambda_m(\mu, \mathbf{y}) = \int p_m^2 \, d\mu \text{ and } p_m(\mathbf{y}) = 1.$$

Choose q_{n-m} as in Lemma 2.1, with the properties $q_{n-m}(\mathbf{y}) = 1$; $0 \leq q_{n-m} \leq 1$ in B; and

$$|q_{n-m}(\mathbf{x})| \leq e^{-c(n-m)\frac{\delta_n}{2}}, \; \mathbf{x} \in B \backslash B\left(\mathbf{y}, \frac{\delta_n}{2}\right).$$

Set

$$S_n = p_m q_{n-m} \in \Pi_n^d.$$

We have $S_n(\mathbf{y}) = 1$, and so the extremal property of λ_n, followed by the properties of q_{n-m}, give

$$\lambda_n(\nu, \mathbf{y})$$

$$\leq \int_{\mathcal{K}} S_n^2 d\nu$$

$$\leq \int_{B(\mathbf{x}_0, \delta_n)} p_m^2 h d\mu + e^{-c(n-m)\delta_n} \|p_m\|_{L_\infty(\mathcal{K})}^2 \int_{\mathcal{K} \setminus B(\mathbf{x}_0, \delta_n)} d\nu$$

$$\leq (h(\mathbf{x}_0) + \varepsilon_n) \int_{B(\mathbf{x}_0, \delta_n)} p_m^2 d\mu + e^{-c(n-m)\delta_n} e^{n\delta_n^2} \left(\int_{\mathcal{K}} p_m^2 d\mu \right) \left(\int_{\mathcal{K}} d\nu \right),$$

by (2.1) and (2.3). Using our choice of m, we continue this as

$$\lambda_n(\nu, \mathbf{y}) \leq \left(\int_{\mathcal{K}} p_m^2 d\mu \right) \left(h(\mathbf{x}_0) + \varepsilon_n + e^{-2n\delta_n^2 + n\delta_n^2} \int_{\mathcal{K}} d\nu \right)$$

$$= \lambda_m(\mu, \mathbf{y}) \left(h(\mathbf{x}_0) + \varepsilon_n + e^{-n\delta_n^2} \int_{\mathcal{K}} d\nu \right).$$

Since δ_n and ε_n are independent of $\mathbf{x}_0 \in D, \mathbf{y} \in B\left(\mathbf{x}_0, \frac{\delta_n}{2}\right)$, we have

$$\frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \leq \frac{\lambda_m(\mu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \left(h(\mathbf{x}_0) + \varepsilon_n + e^{-n\delta_n^2} \int_{\mathcal{K}} d\nu \right)$$

$$\leq h(\mathbf{x}_0) + o(1),$$

uniformly for $\mathbf{x}_0 \in D, \mathbf{y} \in B\left(\mathbf{x}_0, \frac{\delta_n}{2}\right)$, because $\frac{m}{n} = 1 + o(1)$, and by our hypothesis (1.7). Thus we have shown uniformly for such $\mathbf{x}_0, \mathbf{y}$,

$$(2.4) \qquad \limsup_{n \to \infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \leq h(\mathbf{x}_0).$$

Since (2.2) holds, $B\left(\mathbf{x}_0, \frac{\delta_n}{2}\right) \supset B\left(\mathbf{x}_0, \frac{1}{\sqrt{n}}\right)$, for large enough n. For the converse inequality, we note that with $m_1 = m_1(n) = n + \left[\frac{2\delta_n n}{c}\right]$, we obtain by swapping the roles of μ and ν in the above,

$$\lambda_{m_1}(\mu, \mathbf{y}) \leq \lambda_n(\nu, \mathbf{y}) \left(h^{-1}(\mathbf{x}_0) + o(1) + e^{-n\delta_n^2} \int_{\mathcal{K}} d\mu \right),$$

and hence

$$\frac{\lambda_{m_1}(\mu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \leq \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \left(h^{-1}(\mathbf{x}_0) + o(1) + e^{-n\delta_n^2} \int_{\mathcal{K}} d\mu \right).$$

Here the left-hand side is $1 + o(1)$ by our hypothesis (1.7), and as $\frac{m_1}{n} = 1 + o(1)$, so

$$1 \leq \liminf_{n \to \infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} h^{-1}(\mathbf{x}_0),$$

which together with (2.4), gives the result. ∎

3. Proof of Theorem 1.2

Recall that we defined the notion of analytic parametrization before Theorem 1.2. It suffices to prove the following theorem, and then apply Theorem 1.1:

Theorem 3.1
Assume that $\mathcal{K}$ is a compact set with analytic parametrization, and that μ is a

positive measure on $\mathcal{K}$ such that $\mu' > 0$ a.e. on K. Then μ is regular.
Proof
Let $\boldsymbol{m}$ denote Lebesgue measure on $\mathbb{R}^d$. The first fact we need is a Remez inequality, proved in [6, Thm. 6, p. 31]. There exists a positive continuous function ϕ^* defined on $[0, \infty)$, such that

(3.1) $$\lim_{\varepsilon \to 0+} \phi^* (\varepsilon) = 0$$

and if $n \geq 1, P \in \Pi_n^d$, and $\mathcal{F} \subset \mathcal{K}$ with $\boldsymbol{m} (\mathcal{F}) \leq \varepsilon$, then

(3.2) $$\|P\|_{L_\infty(\mathcal{K})} \leq e^{n\phi^*(\varepsilon)} \|P\|_{L_\infty(\mathcal{K} \setminus \mathcal{F})}.$$

Next, for $\delta > 0$, we set

$$\psi (\delta) = \inf \{\mu (L) : L \subset \mathcal{K}, \boldsymbol{m} (L) \geq \delta\}.$$

Since $\mu' > 0$ a.e. on $\mathcal{K}$, $\psi (\delta) > 0$ for $\delta > 0$, and ψ is monotonically increasing, with limit 0 at 0.

We proceed to prove regularity. Let $\varepsilon \in (0, \min\{1, \boldsymbol{m}(\mathcal{K})\})$ and $P \in \Pi_n^d$ be a non-constant polynomial. Choose $a = a(\varepsilon) \in (0, 1)$ such that

$$Q_a = \left\{\mathbf{x} \in K : |P(\mathbf{x})| > a \|P\|_{L_\infty(\mathcal{K})}\right\}$$

has $\boldsymbol{m}(Q_a) = \varepsilon$. Note that $\boldsymbol{m}(Q_a)$ is a strictly decreasing continuous function of a, with limit 0 at 1 and limit $\boldsymbol{m}(K)$ at 0, so we can choose such an a. Now we have

$$|P| \leq a \|P\|_{L_\infty(\mathcal{K})} \text{ in } \mathcal{K} \setminus Q_a.$$

Moreover, since $\boldsymbol{m}(Q_a) = \varepsilon$, the Remez inequality (3.2) gives

$$\|P\|_{L_\infty(\mathcal{K})} \leq e^{n\phi^*(\varepsilon)} \|P\|_{L_\infty(\mathcal{K} \setminus Q_a)}.$$

Combining these two inequalities, yields

$$a e^{n\phi^*(\varepsilon)} \geq 1.$$

In addition, as $\boldsymbol{m}(Q_a) = \varepsilon$,

$$\mu(Q_a) \geq \psi(\varepsilon).$$

These last two inequalities gives

$$\int_{\mathcal{K}} |P|^2 \, d\mu \geq \int_{Q_a} |P|^2 \, d\mu \geq a^2 \|P\|_{L_\infty(\mathcal{K})}^2 \mu(Q_a) \geq e^{-2n\phi^*(\varepsilon)} \|P\|_{L_\infty(\mathcal{K})}^2 \psi(\varepsilon).$$

As $\varepsilon > 0$, ψ and ϕ^* are independent of P, we obtain

$$\sup_{P \in \Pi_n^d} \frac{\|P\|_{L_\infty(\mathcal{K})}^2}{\int_{\mathcal{K}} |P|^2 \, d\mu} \leq \frac{1}{\psi(\varepsilon)} e^{2n\phi^*(\varepsilon)}.$$

Taking nth roots and lim sup's, gives

$$\limsup_{n \to \infty} \left(\sup_{P \in \Pi_n^d} \frac{\|P\|_{L_\infty(\mathcal{K})}^2}{\int_{\mathcal{K}} |P|^2 \, d\mu}\right)^{1/n} \leq e^{2\phi^*(\varepsilon)}.$$

Finally, as ε is arbitrary, and (3.1) holds, we have the result. ∎

4. **Proof of Theorems 1.3 and 1.4**

It is easily seen that the ball and simplex admit analytic parametrization, and we can take the curve γ to be just a straight line segment. All we need is asymptotics for the Christoffel functions for the Chebyshev weight on the ball or simplex, and these have been established by Bos and Xu.

Proof of Theorem 1.3

Bos [1, p. 100] and Xu [20, Theorem 4.1, p. 266] proved that for $\|\mathbf{x}\| < 1$,

$$\lim_{n \to \infty} \binom{n+d}{d} \lambda_n \left(W_0^{\mathrm{ball}}, \mathbf{x} \right) = 1.$$

Bos states the uniform convergence in compact sets, while Xu omits it from his statement, but it is clear from his proof. Our normalization of W_0^{ball} is that of Xu. Then Theorem 1.1, with $d\mu(\mathbf{x}) = W_0^{\mathrm{ball}}(\mathbf{x}) \, d\mathbf{x}$ gives the result. Note that this μ satisfies (1.7) uniformly in compact subsets of B. ∎

Proof of Theorem 1.4

Xu [23, Cor. 2.4, p. 127] proved that for $\mathbf{x}$ in the interior of Σ^d,

$$\lim_{n \to \infty} \binom{n+d}{d} \lambda_n \left(W_0^{\mathrm{simplex}}, \mathbf{x} \right) = 1.$$

Again, the uniform convergence in compact sets, is obvious from the proof. We note a minor misprint in [23, p. 123] in the definition of W_0^{simplex}: the normalization constant should be replaced by its reciprocal. Again, we can apply Theorem 1.1. ∎

5. Proof of Theorems 1.5 and 1.6

The method follows that in [8]. We begin with

Lemma 5.1

Assume that μ, μ^* are measures with compact support $\mathcal{K} \subset \mathbb{R}^d$, and for some $\rho > 0$,

(5.1)
$$d\mu \leq \rho \, d\mu^* \text{ in } \mathcal{K}.$$

Then for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\left| K_n(\mu, \mathbf{x}, \mathbf{y}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{y}) \right| / K_n(\mu, \mathbf{x}, \mathbf{x})$$

(5.2)
$$\leq \left(\frac{K_n(\mu, \mathbf{y}, \mathbf{y})}{K_n(\mu, \mathbf{x}, \mathbf{x})} \right)^{1/2} \left[1 - \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{\rho K_n(\mu, \mathbf{x}, \mathbf{x})} \right]^{1/2}.$$

Proof

This is essentially the same as in [8], but we include the details because of the different setting. Now

$$\int_{\mathcal{K}} \left(K_n(\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{t}) \right)^2 d\mu(\mathbf{t})$$

$$= \int_{\mathcal{K}} K_n^2(\mu, \mathbf{x}, \mathbf{t}) \, d\mu(\mathbf{t}) - \frac{2}{\rho} \int_{\mathcal{K}} K_n(\mu, \mathbf{x}, \mathbf{t}) K_n(\mu^*, \mathbf{x}, \mathbf{t}) \, d\mu(\mathbf{t}) + \frac{1}{\rho^2} \int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) \, d\mu(\mathbf{t})$$

$$= K_n(\mu, \mathbf{x}, \mathbf{x}) - \frac{2}{\rho} K_n(\mu^*, \mathbf{x}, \mathbf{x}) + \frac{1}{\rho^2} \int_{\mathcal{K}} K_n^2(\mu^*, \mathbf{x}, \mathbf{t}) \, d\mu(\mathbf{t}),$$

by the reproducing kernel property. As $d\mu \leq \rho \, d\mu^*$, we also have

$$\int_{\mathcal{K}} K_n^2 (\mu^*, \mathbf{x}, \mathbf{t}) \, d\mu(\mathbf{t}) \leq \rho \int_{\mathcal{K}} K_n^2 (\mu^*, \mathbf{x}, \mathbf{t}) \, d\mu^*(\mathbf{t}) = \rho K_n (\mu^*, \mathbf{x}, \mathbf{x}).$$

So

$$\int_{\mathcal{K}} \left(K_n (\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n (\mu^*, \mathbf{x}, \mathbf{t}) \right)^2 d\mu(\mathbf{t})$$

(5.3)
$$\leq \quad K_n (\mu, \mathbf{x}, \mathbf{x}) - \frac{1}{\rho} K_n (\mu^*, \mathbf{x}, \mathbf{x}).$$

Next for any polynomial $P \in \Pi_n^d$, we have the Christoffel function estimate

(5.4)
$$|P(\mathbf{y})| \leq K_n (\mu, \mathbf{y}, \mathbf{y})^{1/2} \left(\int_{\mathcal{K}} P^2 d\mu \right)^{1/2}.$$

Applying this to $P(\mathbf{t}) = K_n (\mu, \mathbf{x}, \mathbf{t}) - \frac{1}{\rho} K_n (\mu^*, \mathbf{x}, \mathbf{t})$ and using (5.3) gives, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$\left| K_n (\mu, \mathbf{x}, \mathbf{y}) - \frac{1}{\rho} K_n (\mu^*, \mathbf{x}, \mathbf{y}) \right|$$

$$\leq \quad K_n (\mu, \mathbf{y}, \mathbf{y})^{1/2} \left[K_n (\mu, \mathbf{x}, \mathbf{x}) - \frac{1}{\rho} K_n (\mu^*, \mathbf{x}, \mathbf{x}) \right]^{1/2}.$$

∎

Next, we establish an elementary bound on Christoffel functions:

Lemma 5.2
Let μ be a measure with compact support $\mathcal{K}$, and let $B(\mathbf{x}_0, \delta)$ be a ball inside that support. Assume that μ is absolutely continuous in $B(\mathbf{x}_0, \delta)$, satisfying a.e. there

$$0 < C_1 \leq \mu' \leq C_2 < \infty.$$

Then, given $0 < \eta < \delta$, there exist $C_3, C_4 > 0$, such that for $n \geq 1$,

(5.5)
$$C_3 \leq \binom{n+d}{d} \lambda_n (\mu, \mathbf{x}) \leq C_4 \text{ in } B(\mathbf{x}_0, \eta).$$

Proof
The lower bound follows from Theorem 1.3 and monotonicity of Christoffel functions in the measure. Indeed, let $\nu_1' = 1$ in $B(\mathbf{x}_0, \delta)$ and 0 elsewhere. By a scaled form of Theorem 1.3,

$$\lim_{n \to \infty} \binom{n+d}{d} \lambda_n (\nu_1, \mathbf{x}) = C \sqrt{1 - \left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{\delta} \right)^2}, \quad \mathbf{x} \in B(\mathbf{x}_0, \delta),$$

where C depends only on d and δ. The convergence is uniform in compact subsets of $B(\mathbf{x}_0, \delta)$. Since $\lambda_n (\mu, \mathbf{x}) \geq C_1 \lambda_n (\nu_1, \mathbf{x})$, the lower bound in (5.5) follows. Next, choose $r > 0$ so large that $B(\mathbf{x}_0, r)$ contains $\mathcal{K}$, and let $d\nu_2 = d\mu_{|\mathcal{K} \setminus B(\mathbf{x}_0, \delta)} + C_2 d\mathbf{x}_{|B(\mathbf{x}_0, r)}$. Thus $d\nu_2$ is the sum of μ restricted to $\mathcal{K} \setminus B(\mathbf{x}_0, \delta)$, and a multiple of the Legendre weight for the ball $B(\mathbf{x}_0, r)$. Since $d\nu_2 \geq C_2 d\mathbf{x}_{|B(\mathbf{x}_0, r)}$, and the

latter is regular on $B(\mathbf{x}_0, r)$, it is easily seen that ν_2 is also regular. Moreover, ν_2' is positive and continuous on $B(\mathbf{x}_0, \delta)$, so Theorem 1.3 gives uniformly on $B(\mathbf{x}_0, \eta)$,

$$\lim_{n \to \infty} \binom{n+d}{d} \lambda_n(\nu_2, \mathbf{x}) = C \sqrt{1 - \left(\frac{\|\mathbf{x} - \mathbf{x}_0\|}{r} \right)^2},$$

for some C depending on δ. But also $\mu \leq \nu_2$, so $\lambda_n(\mu, \cdot) \leq \lambda_n(\nu_2, \cdot)$, and the upper bound in (5.5) follows. ∎

We turn to the

Proof of Theorem 1.5

Let $\mathbf{x}_0 \in D$, $\varepsilon \in (0,1)$ and choose $\delta > 0$ such that $h = \frac{d\nu}{d\mu}$ (which is positive and continuous on D) satisfies

(5.6) $$1 - \varepsilon \leq h(\mathbf{y})/h(\mathbf{x}) \leq 1 + \varepsilon \text{ for } \mathbf{x}, \mathbf{y} \in B(\mathbf{x}_0, \delta).$$

Set

$$\rho = h(\mathbf{x}_0)(1 + \varepsilon).$$

We shall apply Lemma 5.1 twice. Define a measure μ^* by $d\mu^* = d\mu$ in $B(\mathbf{x}_0, \delta)$, and

$$d\mu^* = \max\left\{1, \frac{1}{\rho}\right\}(d\mu + d\nu) \text{ in } \mathcal{K} \backslash B(\mathbf{x}_0, \delta).$$

Step 1: μ and μ^*

Since $\mu^* \geq \mu$, we have the inequality (5.2). Moreover, since μ is regular and $\mu^* \geq \mu$, μ^* is also regular. Next, from Theorem 1.1,

$$\lim_{n \to \infty} \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 1$$

uniformly in $B(\mathbf{x}_0, \eta)$, for any $\eta < \delta$, while from Lemma 5.2, for $\mathbf{x}, \mathbf{y} \in B(\mathbf{x}_0, \eta)$,

$$\frac{K_n(\mu, \mathbf{y}, \mathbf{y})}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C.$$

It is only here that we need the extra hypothesis in Theorem 1.5 that μ' is bounded above and below by positive constants on D_1. Then Lemma 5.1, with $\rho = 1$ there, shows that for $\mathbf{x} \in B(\mathbf{x}_0, \eta)$, and uniformly for $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

(5.7) $$\lim_{n \to \infty} \frac{K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) - K_n\left(\mu^*, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right)}{K_n(\mu, \mathbf{x}, \mathbf{x})} = 0,$$

Step 2: ν and μ^*

Now $d\nu \leq \rho \, d\mu^*$ in $\mathcal{K} \backslash B(\mathbf{x}_0, \delta)$. Also, in $B(\mathbf{x}_0, \delta)$,

$$d\nu = \frac{d\nu}{d\mu} d\mu \leq \rho \, d\mu = \rho \, d\mu^*.$$

So in $\mathcal{K}$, $d\nu \leq \rho \, d\mu^*$. By Theorem 1.1, and (5.6)

$$\lim_{n \to \infty} \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{\rho K_n(\nu, \mathbf{x}, \mathbf{x})} = \frac{1}{\rho} \frac{d\nu}{d\mu^*}(\mathbf{x}) = \frac{1}{\rho} \frac{d\nu}{d\mu}(\mathbf{x}) \geq \frac{1 - \varepsilon}{1 + \varepsilon},$$

uniformly in $B(\mathbf{x}_0, \eta)$, for any $\eta < \delta$. Note that (1.7) holds for ν and $\mathbf{x}$ in $B(x_0, \delta)$ - indeed, this follows from the ratio limit in Theorem 1.1. Furthermore, from Lemma 5.2, for $\mathbf{x}, \mathbf{y} \in B(\mathbf{x}_0, \eta)$,

$$\frac{K_n(\nu, \mathbf{y}, \mathbf{y})}{K_n(\nu, \mathbf{x}, \mathbf{x})} \leq C.$$

Again, here we need the extra hypothesis in Theorem 1.5 that ν' is bounded above and below by positive constants on D_1. Then Lemma 5.1 gives

$$\frac{\left| K_n\left(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) - \frac{1}{\rho} K_n\left(\mu^*, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) \right|}{K_n(\nu, \mathbf{x}, \mathbf{x})}$$

$$(5.8) \qquad \leq \quad C \left[1 - \frac{K_n(\mu^*, \mathbf{x}, \mathbf{x})}{\rho K_n(\nu, \mathbf{x}, \mathbf{x})} \right]^{1/2} \leq C_1 (2\varepsilon)^{1/2},$$

for $n \geq n_0$, $\mathbf{x} \in B(\mathbf{x}_0, \eta)$, and $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^d$. Since C and C_1 are independent of $\mathbf{u}, \mathbf{v}, \mathbf{x}, n$, we obtain from this and (5.7), and the bound on the Christoffel functions in Lemma 5.2,

$$\frac{\left| \rho K_n\left(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) - K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) \right|}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C_1 \varepsilon^{1/2},$$

and hence for large enough n, and uniformly for $\mathbf{x} \in B(\mathbf{x}_0, \eta)$, and $\mathbf{u}, \mathbf{v}$ in compact subsets of $\mathbb{R}^n$,

$$\frac{\left| \frac{d\nu}{d\mu}(\mathbf{x}_0) K_n\left(\nu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) - K_n\left(\mu, \mathbf{x} + \frac{\mathbf{u}}{n}, \mathbf{x} + \frac{\mathbf{v}}{n}\right) \right|}{K_n(\mu, \mathbf{x}, \mathbf{x})} \leq C_1 \varepsilon^{1/2},$$

where C_1 is independent of $\mathbf{u}, \mathbf{v}, \mathbf{x}, n$. Then using (5.6) again, we obtain (1.15). The uniformity in $B(\mathbf{x}_0, \eta)$ was also established above. As D is compact, uniformity in D follows. ∎

Proof of Theorem 1.6
This follows directly from Theorem 1.5 and Theorem 3.1. ∎

6. Proof of Theorems 1.7 and 1.8

We need only compute the universality limit for the Chebyshev weight on the ball or simplex, and then apply Theorem 1.5. We make essential use of known representations for the reproducing kernel. These involve the standard Jacobi polynomial $P_n^{(\alpha,\beta)}$ of degree n, that satisfies the orthogonality relation

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x) x^j (1-x)^\alpha (1+x)^\beta \, dx = 0, \quad 0 \leq j \leq n-1,$$

and normalized by

$$(6.1) \qquad\qquad P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Lemma 6.1
(a) *Let*

$$c_n = \frac{\Gamma\left(\frac{d+2}{2}\right)}{\Gamma(d+1)} \frac{\Gamma(n+d)}{\Gamma\left(n+\frac{d}{2}\right)}.$$

Then

$$K_n\left(W_0^{\text{ball}}, \mathbf{x}, \mathbf{y}\right) = c_n\{P_n^{(d/2,d/2-1)}\left(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right)$$

$$(6.2) \qquad\qquad +P_n^{(d/2,d/2-1)}\left(\mathbf{x}\cdot\mathbf{y}-\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\right)\}.$$

(b) Let

$$d_n = \frac{\Gamma\left(\frac{d}{2}+1\right)}{\Gamma\left(d+1\right)}\frac{\Gamma\left(n+d+1\right)}{\Gamma\left(n+\frac{d}{2}+1\right)}$$

Then

$$(6.3) \qquad K_n\left(W_0^{\text{simplex}}, \mathbf{x}, \mathbf{y}\right) = \frac{d_n}{2^{2d+2}} \sum_{\varepsilon_i=\pm 1} P_{2n}^{(d/2,d/2)}\left(\sum_{j=1}^{d+1}\sqrt{x_j y_j}\varepsilon_j\right),$$

where

$$(6.4) \qquad\qquad x_{d+1} = 1 - \sum_{j=1}^{d} x_j;\ y_{d+1} = 1 - \sum_{j=1}^{d} y_j.$$

Proof
(a) See [22, eqn. (3.8), Thm 3.3, p. 2449].
(b) See [21, Theorem 2.3, p. 3032]. We have also taken account of his convention of replacing integrals by sums, in the confluent case "$\alpha_i = 0$" when the Jacobi parameters reduce to those of the Chebyshev weight. Note that Xu uses the ultraspherical polynomial $C_n^{(\lambda)}$, defined [18, eqn. (4.7.1), p. 80] by

$$C_n^{(\lambda)}(x) = d_{n,\lambda}P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x),$$

where in our case $\lambda = \frac{d+1}{2}$, and

$$d_{n,\lambda} = \frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma\left(2\lambda\right)}\frac{\Gamma\left(n+2\lambda\right)}{\Gamma\left(n+\lambda+\frac{1}{2}\right)}.$$

∎

Next, we turn to asymptotics of Jacobi polynomials:

Lemma 6.2
(a) Let $\alpha > 0, \beta > -1$. Uniformly for s in bounded subsets of $[0,\infty)$, we have

$$(6.5) \qquad\qquad \lim_{n\to\infty} n^{-\alpha}P_n^{(\alpha,\beta)}\left(1-\frac{s}{2n^2}\right) = 2^\alpha J_\alpha^*\left(\sqrt{s}\right).$$

(b) Let $\alpha, \beta \geq -\frac{1}{2}$.

$$(6.6) \qquad \sup_{n\geq 1} n^{1/2} \sup_{x\in(-1,1)} \left|P_n^{(\alpha,\beta)}(x)\right|(1-x)^{\frac{\alpha+1/2}{2}}(1+x)^{\frac{\beta+1/2}{2}} \leq C < \infty.$$

Proof
(a) This is Mehler-Heine's asymptotic formula [18, Thm. 8.1.1, p. 192].
(b) See for example, [11, Lemma 29, p.170], and use the fact that if ρ_n is the constant such that $\rho_n P_n^{(\alpha,b)}$ is an orthonormal polynomial, then ρ_n grows like $Cn^{1/2}(1+o(1))$, for some positive C. See, for example, [18, p. 68] ∎

Next, we compute asymptotics for the arguments in the kernel in (6.2) and (6.3):

Lemma 6.3
(a) Assume that $\boldsymbol{\xi} \in B$, and

$$(6.7) \qquad \mathbf{x} = \boldsymbol{\xi} + \frac{1}{n}\mathbf{u} \text{ and } \mathbf{y} = \boldsymbol{\xi} + \frac{1}{n}\mathbf{v}.$$

Then

$$(6.8) \qquad \mathbf{x} \cdot \mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2} = 1 - \frac{G}{2n^2} + O\left(\frac{1}{n^3}\right),$$

where $G = G(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})$ is defined by (1.21).
(b) Let $\boldsymbol{\xi}$ lie in the interior of Σ^d, and $\mathbf{x}, \mathbf{y}$ be given by (6.9). Then

$$(6.9) \qquad \sum_{j=1}^{d+1} \sqrt{x_j y_j} = 1 - \frac{H}{8n^2} + O\left(\frac{1}{n^3}\right),$$

where $H = H(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})$ is given by (1.23).
Proof
(a) We see that

$$\mathbf{x} \cdot \mathbf{y} = \|\boldsymbol{\xi}\|^2 + \frac{1}{n}\boldsymbol{\xi} \cdot (\mathbf{u} + \mathbf{v}) + \frac{1}{n^2}\mathbf{u} \cdot \mathbf{v}.$$

Also

$$\sqrt{1 - \|\mathbf{x}\|^2} = \sqrt{1 - \|\boldsymbol{\xi}\|^2 - \frac{2}{n}\boldsymbol{\xi} \cdot \mathbf{u} - \frac{1}{n^2}\|\mathbf{u}\|^2}.$$

A straightforward computation, using the Maclaurin series $\sqrt{1 + t} = 1 + \frac{t}{2} - \frac{t^2}{8} + O\left(t^3\right)$, gives

$$\begin{aligned}
&\sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2} \\
&= 1 - \|\boldsymbol{\xi}\|^2 - \frac{1}{n}\boldsymbol{\xi} \cdot (\mathbf{u} + \mathbf{v}) - \frac{1}{2n^2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) \\
&\quad + \frac{2(\boldsymbol{\xi} \cdot \mathbf{u})(\boldsymbol{\xi} \cdot \mathbf{v})}{n^2\left(1 - \|\boldsymbol{\xi}\|^2\right)} - \frac{(\boldsymbol{\xi} \cdot (\mathbf{u} + \mathbf{v}))^2}{2n^2\left(1 - \|\boldsymbol{\xi}\|^2\right)} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{x} \cdot \mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{x}\|^2} = \\
&= 1 + \frac{1}{n^2}\mathbf{u} \cdot \mathbf{v} - \frac{1}{2n^2}\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) - \frac{1}{2n^2\left(1 - \|\boldsymbol{\xi}\|^2\right)}(\boldsymbol{\xi} \cdot (\mathbf{u} - \mathbf{v}))^2 + O\left(\frac{1}{n^3}\right) \\
&= 1 - \frac{G}{2n^2} + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

(b) For $1 \leq j \leq d$,

$$\begin{aligned}
\sqrt{x_j y_j} &= \sqrt{\left(\xi_j + \frac{u_j}{n}\right)\left(\xi_j + \frac{v_j}{n}\right)} \\
(6.10) \qquad &= \xi_j + \frac{1}{2n}(u_j + v_j) + \frac{1}{2n^2\xi_j}u_j v_j - \frac{1}{8n^2\xi_j}(u_j + v_j)^2 + O\left(\frac{1}{n^3}\right).
\end{aligned}$$

Also,

$$\sqrt{x_{d+1}y_{d+1}}$$

$$= \sqrt{\left|\left(1 - \sum_{j=1}^{d} x_j\right)\left(1 - \sum_{j=1}^{d} y_j\right)\right|}$$

$$= 1 - \sum_{j=1}^{d} \xi_j - \frac{1}{2n}\sum_{j=1}^{d}(u_j + v_j) + \frac{1}{2n^2\left(1 - \sum_{j=1}^{d}\xi_j\right)}\left(\sum_{j=1}^{d} u_j\right)\left(\sum_{j=1}^{d} v_j\right)$$

$$- \frac{1}{8n^2\left(1 - \sum_{j=1}^{d}\xi_j\right)}\left(\sum_{j=1}^{d}(u_j + v_j)\right)^2 + O\left(\frac{1}{n^3}\right).$$

(6.11)

Combining (6.10) and (6.11) gives

$$\sum_{j=1}^{d+1}\sqrt{x_j y_j}$$

$$= 1 + \frac{1}{2n^2}\sum_{j=1}^{d}\frac{u_j v_j}{\xi_j} - \frac{1}{8n^2}\sum_{j=1}^{d}\frac{(u_j + v_j)^2}{\xi_j}$$

$$+ \frac{1}{2n^2\xi_{d+1}}\left(\sum_{j=1}^{d} u_j\right)\left(\sum_{j=1}^{d} v_j\right) - \frac{1}{8n^2\xi_{d+1}}\left(\sum_{j=1}^{d}(u_j + v_j)\right)^2 + O\left(\frac{1}{n^3}\right)$$

$$= 1 - \frac{H}{8n^2} + O\left(\frac{1}{n^3}\right).$$

∎

Proof of Theorem 1.7

As remarked before, we need only establish the limit (1.20) for the Chebyshev weight W_0^{ball}, and then can apply Theorem 1.5. Let $\mathbf{x}, \mathbf{y}$ be given by (6.7). Now the dominant term in the right-hand side in (6.2) is the term with argument $\mathbf{x}\cdot\mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}$. Since $\mathbf{x}\cdot\mathbf{y} - \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}$ remains in a compact subset of $(-1, 1)$ as $n \to \infty$, the bound (6.6) shows that it is of essentially smaller size than the other term. We then have, using (6.8),

$$n^{-d/2}K_n\left(W_0^{\text{ball}}, \mathbf{x}, \mathbf{y}\right) = c_n n^{-d/2}\{P_n^{(d/2, d/2-1)}\left(\mathbf{x}\cdot\mathbf{y} + \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}\right)$$

$$+ P_n^{(d/2, d/2-1)}\left(\mathbf{x}\cdot\mathbf{y} - \sqrt{1 - \|\mathbf{x}\|^2}\sqrt{1 - \|\mathbf{y}\|^2}\right)\}$$

$$= c_n\left\{n^{-d/2}P_n^{(d/2, d/2-1)}\left(1 - \frac{G(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})}{2n^2} + O\left(\frac{1}{n^3}\right)\right) + o(1)\right\}$$

$$= c_n\left\{2^{d/2}J_{d/2}^*\left(\sqrt{G(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})}\right) + o(1)\right\},$$

by (6.5). Using this with $\mathbf{u} = \mathbf{v} = \mathbf{0}$, gives

$$n^{-d/2} K_n \left(W_0^{\text{ball}}, \boldsymbol{\xi}, \boldsymbol{\xi} \right) = c_n \left\{ 2^{d/2} J_{d/2}^* (0) + o(1) \right\}.$$

Then (1.20) follows on changing $\boldsymbol{\xi}$ to $\mathbf{x}$. ∎

Proof of Theorem 1.8

The dominant terms in the right-hand side in (6.3) are the two terms with all $\varepsilon_j = 1$, or with all $\varepsilon_j = -1$. For all other choices of $\{\varepsilon_j\}$, the argument $\sum_{j=1}^{d+1} \sqrt{x_j y_j} \varepsilon_j$ remains in a compact subset of $(-1, 1)$ as $n \to \infty$, so the bound (6.6) shows that the corresponding terms in (6.3) are of essentially smaller size than the above two terms. Because of the evenness of $P_{2n}^{(d/2, d/2)}$, we have, using (6.9),

$$(2n)^{-d/2} K_n \left(W_0^{\text{simplex}}, \mathbf{x}, \mathbf{y} \right)$$

$$= \frac{d_n}{2^{2d+1}} (2n)^{-d/2} P_{2n}^{(d/2, d/2)} \left(\sum_{j=1}^{d+1} \sqrt{x_j y_j} \right) (1 + o(1))$$

$$= \frac{d_n}{2^{2d+1}} (2n)^{-d/2} P_{2n}^{(d/2, d/2)} \left(1 - \frac{H(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})}{2(2n)^2} + O\left(\frac{1}{n^3}\right) \right) (1 + o(1))$$

$$= \frac{d_n}{2^{2d+1}} 2^{d/2} J_{d/2}^* \left(\sqrt{H(\boldsymbol{\xi}, \mathbf{u}, \mathbf{v})} \right) (1 + o(1)),$$

by (6.5). Using this with $\mathbf{u} = \mathbf{v} = \mathbf{0}$, gives

$$(2n)^{-d/2} K_n \left(W_0^{\text{simplex}}, \mathbf{x}, \mathbf{y} \right) = \frac{d_n}{2^{2d+1}} 2^{d/2} J_{d/2}^* (0) (1 + o(1)).$$

Then (1.22) for the Chebyshev weight W_0^{simplex} follows. ∎

7. An Extension of Theorem 1.1

We can weaken the continuity of $\frac{d\nu}{d\mu}$ in Theorem 1.1 to a Lebesgue point condition. Recall that $\mathbf{x}_0$ is a Lebesgue point of a function h of d variables if

$$\lim_{r \to 0+} \frac{\int_{B(\mathbf{x}_0, r)} |h(\mathbf{x}_0) - h(\mathbf{t})| \, d\mathbf{t}}{\boldsymbol{m}(B(\mathbf{x}_0, r))} = 0,$$

where $\boldsymbol{m}$ denotes Lebesgue measure on $\mathbb{R}^d$.

Theorem 7.1

Let μ, ν be positive measures, whose support is a compact set $\mathcal{K} \subset \mathbb{R}^d$, and both are regular. Let $\mathbf{x}_0 \in \mathcal{K}$ and assume that μ, ν are absolutely continuous in some ball $B(\mathbf{x}_0, \delta) \subset \mathcal{K}$, while μ', ν' are bounded above and below a.e. by positive constants there, and (1.7) holds uniformly in $B(\mathbf{x}_0, \delta) \subset \mathcal{K}$. Assume that $\mathbf{x}_0$ is a Lebesgue point of $\frac{\nu'}{\mu'}$. Then, given $r > 0$, we have uniformly for $\mathbf{y} \in B\left(\mathbf{x}_0, \frac{r}{n}\right)$,

$$(7.1) \qquad \lim_{n \to \infty} \frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} = \frac{d\nu}{d\mu}(\mathbf{x}_0).$$

Proof

We may assume that $\mathcal{K} \subset B = B(\mathbf{0}, 1)$. Fix $r > 0$ and let $\mathbf{y} \in B\left(\mathbf{x}_0, \frac{r}{n}\right)$. Let

$\tau \geq 2r$ and $\varepsilon \in (0,1)$. Let $\ell_n = \left[\frac{\varepsilon n}{2}\right]$ and $m = m(n) = n - 2\ell_n$. Choose $p_m \in \Pi_m^d$ that is extremal for $\lambda_m(\mu, \mathbf{y})$, so that

$$\lambda_m(\mu, \mathbf{y}) = \int p_m^2 d\mu \text{ and } p_m(\mathbf{y}) = 1.$$

Choose $q_{\ell_n}^{(k)}$, $k = 1, 2$, as in Lemma 2.1, with the properties $q_{\ell_n}^{(k)}(\mathbf{y}) = 1; 0 \leq q_{\ell_n}^{(k)} \leq 1$ in B; and

$$(7.2) \qquad \left| q_{\ell_n}^{(1)}(\mathbf{x}) \right| \quad \leq \quad e^{-c\ell_n \frac{\tau}{n}} \leq e^{-C\varepsilon\tau}, \mathbf{x} \in B \backslash B\left(\mathbf{y}, \frac{\tau}{n}\right);$$

$$(7.3) \qquad \left| q_{\ell_n}^{(2)}(\mathbf{x}) \right| \quad \leq \quad e^{-c\ell_n \delta} \leq e^{-C\varepsilon n}, \mathbf{x} \in B \backslash B(\mathbf{y}, \delta).$$

As above, let $h = \frac{d\nu}{d\mu}$ and

$$S_n = p_m q_{\ell_n}^{(1)} q_{\ell_n}^{(2)} \in \Pi_n^d.$$

We have $S_n(\mathbf{y}) = 1$, and so

$$\lambda_n(\nu, \mathbf{y})$$
$$\leq \quad \int_{\mathcal{K}} S_n^2 d\nu$$
$$\leq \quad h(\mathbf{x}_0) \int_{B\left(\mathbf{y}, \frac{\tau}{n}\right)} p_m^2 d\mu + \|p_m\|_{L_\infty\left(B\left(\mathbf{y}, \frac{\tau}{n}\right)\right)}^2 \int_{B\left(\mathbf{y}, \frac{\tau}{n}\right)} |h(\mathbf{x}_0) - h(t)| \, d\mu(t)$$
$$\quad + e^{-C\varepsilon\tau} \int_{B(\mathbf{x}_0, 2\delta) \backslash B\left(\mathbf{y}, \frac{\tau}{n}\right)} p_m^2 h \, d\mu + \|p_m\|_{L_\infty(\mathcal{K})}^2 e^{-C\varepsilon n} \int_{\mathcal{K} \backslash B(\mathbf{x}_0, 2\delta)} d\nu$$
$$= \quad : T_1 + T_2 + T_3 + T_4,$$

by (7.2), (7.3). Firstly,

$$T_1 \leq h(\mathbf{x}_0) \lambda_m(\mu, \mathbf{y}).$$

Next, Lemma 5.2 gives

$$\|p_m\|_{L_\infty\left(B\left(\mathbf{y}, \frac{\tau}{n}\right)\right)}^2 \quad \leq \quad \left\|\lambda_m^{-1}(\mu, \cdot)\right\|_{L_\infty\left(B\left(\mathbf{y}, \frac{\tau}{n}\right)\right)} \int p_m^2 d\mu$$
$$\leq \quad Cn^d \lambda_m(\mu, \mathbf{y}),$$

so as $\tau > r$, and $\|\mathbf{y} - \mathbf{x}_0\| < \frac{r}{n}$,

$$T_2 \leq Cn^d \lambda_m(\mu, \mathbf{y}) \|\mu'\|_{L_\infty(B(\mathbf{x}_0, \delta))} \int_{B\left(\mathbf{x}_0, \frac{2\tau}{n}\right)} |h(x_0) - h(t)| \, dt = o(\lambda_m(\mu, \mathbf{y})),$$

as $\mathbf{x}_0$ is a Lebesgue point of h. Next, by (7.2),

$$T_3 \quad \leq \quad e^{-C\varepsilon\tau} \|h\|_{L_\infty(B(\mathbf{x}_0, \delta))} \int_{B(\mathbf{x}_0, \delta) \backslash B\left(\mathbf{y}, \frac{\tau}{n}\right)} p_m^2 \, d\mu$$
$$\leq \quad C_1 e^{-C\varepsilon\tau} \lambda_m(\mu, \mathbf{y}).$$

Finally, using the regularity of μ,

$$T_4 \leq (1 + o(1))^n \left(\int p_m^2 \, d\mu\right) e^{-C\varepsilon n} \int_{\mathcal{K} \backslash B(\mathbf{x}_0, \delta_n)} d\nu = o(\lambda_m(\mu, \mathbf{y})).$$

Combining all the above estimates gives

$$\frac{\lambda_n(\nu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \leq \frac{\lambda_m(\mu, \mathbf{y})}{\lambda_n(\mu, \mathbf{y})} \left(h(\mathbf{x}_0) + C_1 e^{-C\varepsilon\tau} + o(1)\right).$$

Here τ is independent of ε, and may be chosen as large as we please. We deduce that

$$\limsup_{n\to\infty} \frac{\lambda_n\left(\nu, \mathbf{y}\right)}{\lambda_n\left(\mu, \mathbf{y}\right)} \leq h\left(\mathbf{x}_0\right) \limsup_{n\to\infty} \frac{\lambda_{n-2\left[\frac{\varepsilon n}{2}\right]}\left(\mu, \mathbf{y}\right)}{\lambda_n\left(\mu, \mathbf{y}\right)}.$$

The proof may now be completed as in Theorem 1.1. ■

We note that the absolute continuity of μ and boundedness below of μ', was needed only for the upper bound for $\left\|\lambda_m^{-1}\left(\mu, \cdot\right)\right\|_{L_\infty\left(B\left(\mathbf{y}, \frac{\tau}{n}\right)\right)}$. If we assumed a suitable bound for this, we could allow μ to have a singular part.

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