On Mean Convergence of Lagrange Interpolation for General Arrays

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Abstract

For $n \geq 1$, let $\{x_{jn}\}_{j=1}^{n}$ be n distinct points in a compact set $K \subset \mathbb{R}$ and let $L_n[\cdot]$ denote the corresponding Lagrange Interpolation operator. Let v be a suitably restricted function on K. What conditions on the array $\{x_{jn}\}_{1 \leq j \leq n,\, n \geq 1}$ ensure the existence of $p > 0$ such that

$$\lim_{n \to \infty} \parallel (f - L_n[f])\, v \parallel_{L_p(K)} = 0$$

for every continuous $f :: K \to \mathbb{R}$? We show that it is necessary and sufficient that there exists $r > 0$ with

$$\sup_{n \geq 1} \parallel \pi_n v \parallel_{L_r(K)} \sum_{j=1}^{n} \frac{1}{|\pi_n'|\,(x_{jn})} < \infty.$$

Here for $n \geq 1$, π_n is a polynomial of degree n having $\{x_{jn}\}_{j=1}^{n}$ as zeros. The necessity of this condition is due to Ying Guang Shi.

1 The Result

There is a vast literature on mean convergence of Lagrange interpolation, based primarily at zeros of orthogonal polynomials and their close cousins. See [3 – 10] for recent references. Most of the work dealing with mean convergence of Lagrange interpolation for general arrays involves necessary conditions [6], [9], since sufficient conditions are hard to come by. Some sufficient conditions for convergence of general arrays in L_p, $p > 1$, have been given in [3].

In a recent paper, the author showed that distribution functions and Loomis' Lemma may be used to investigate mean convergence of Lagrange interpolation in L_p, $p < 1$ [2]. Indeed those techniques show that investigating convergence of Lagrange interpolation in L_p is inherently easier for $p < 1$ than for $p \geq 1$. Here we show that similar ideas may be used to solve the problem of whether there is convergence in weighted L_p spaces for at least one $p > 0$.

Throughout, we consider an array X of interpolation points $X = \{x_{jn}\}_{1\le j\le n,\, n\ge 1}$ in a compact set $K \subset \mathbb{R}$, with

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

We denote by $L_n[\,\cdot\,]$ the associated Lagrange interpolation operator, so that for $f:\ K \to \mathbb{R}$, we have

$$L_n[f](x) = \sum_{j=1}^{n} f\left(x_{jn}\right)\ell_{jn}(x),$$

where the fundamental polynomials $\{\ell_{kn}\}_{k=1}^{n}$ satisfy

$$\ell_{kn}\left(x_{jn}\right) = \delta_{jk}.$$

We also let π_n denote a polynomial of degree n (without any specific normalisation) whose zeros are $\{x_{jn}\}_{j=1}^{n}$. Our result is:

Theorem 1
Let $K \subset \mathbb{R}$ be compact, and let $v \in L_q(K)$ for some $q > 0$. Let the array X of interpolation points lie in K. The following are equivalent:

(I) *There exists $p > 0$ such that for every continuous $f : K \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \| \left(f - L_n[f]\right) v \|_{L_p(K)} = 0. \tag{1}$$

(II) *There exists $r > 0$ such that*

$$\sup_{n\ge 1} \| \pi_n v \|_{L_r(K)} \left(\sum_{j=1}^{n} \frac{1}{|\pi_n'|\left(x_{jn}\right)} \right) < \infty. \tag{2}$$

Remarks

(a) The new feature is the sufficiency; the necessity is essentially due to Ying Guang Shi [9]. An alternative way to formulate (2) is

$$\sup_{n\ge 1} \| S_n v \|_{L_r(K)} < \infty$$

where

$$S_n(x) := \sum_{j=1}^{n} \left|\left(x - x_{jn}\right)\ell_{jn}(x)\right| = |\pi_n(x)| \sum_{j=1}^{n} \frac{1}{|\pi_n'|\left(x_{jn}\right)}. \tag{3}$$

Indeed, Shi [9] used this in necessary conditions on $[-1, 1]$.

(b) Note that if (2) holds for a given r, it holds for any smaller r. Likewise if (1) holds for some $p > 0$, then it holds for all smaller p. Our proof shows that if (2) holds for a given r, then (1) holds for $p < \min\left\{\frac{1}{2}, \frac{r}{2}, q\right\}$. Conversely if (1) holds for a given p, then (2) holds with $r = p$.

(c) Note that K could, for example, consist of finitely many intervals. What is somewhat restrictive is the formulation of (2). We may insert a weight w in (2), so that it becomes

$$\sup_{n\geq 1} \parallel \pi_n v \parallel_{L_r(K)} \left(\sum_{j=1}^{n} \frac{1}{|\pi_n' w|(x_{jn})} \right) < \infty.$$

The advantage of this is that the requirement on the $\{x_{jn}\}$ is weakened, if $w(x)$ approaches ∞ as $x \to \mathbb{R}\backslash K$. For the proof to work in this more general formulation, we need

(i) w to be positive and continuous in a neighbourhood (in K) of each interpolation point;

(ii) the polynomials to be dense in a weighted Banach space of continuous functions.

Thus, one could assume, for example, that w is positive and continuous in the interior K° of K and that each $x_{jn} \in K^{\circ}$. Moreover, one can assume that the polynomials are dense in

$$C(w) := \left\{ f : K \to \mathbb{R} \text{ s.t. } f \text{ is continuous in } K^{\circ} \text{ and } \parallel fw \parallel_{L_\infty(K)} < \infty \right\}$$

and that

$$\parallel v/w \parallel_{L_p(K)} < \infty.$$

(The density is not trivial, and need not be true if $w(x) \to \infty$ fast enough as $x \to \mathbb{R}\backslash K$). If one wants only boundedness, and not convergence of $\{L_n\}$, then one can weaken these requirements on w.

We turn to:

The Proof of Theorem 1

We let $C(K)$ denote the Banach space of continuous $f: K \to \mathbb{R}$ with norm

$$\parallel f \parallel := \parallel f \parallel_{L_\infty(K)}.$$

We suppose, as we may, that $K \subset [-1, 1]$.

(II) $\Rightarrow$ (I)

We first suppose that $\| f \|_{L_\infty(K)} \leq 1$. Now we can write

$$L_n[f](x) = \pi_n(x) \sum_{j=1}^{n} \frac{f(x_{jn})}{\pi_n'(x_{jn})(x - x_{jn})} =: \pi_n(x) g_n(x).$$

Let $p > 0$. Then

$$\| L_n[f]v \|_{L_p(K)} \leq \| \pi_n v \|_{L_{2p}(K)} \| g_n \|_{L_{2p}(K)} . \tag{4}$$

To estimate the norm of g_n, we use its distribution function

$$m_{g_n}(\lambda) := meas\{x \in K : \ |g_n(x)| > \lambda\}, \quad \lambda > 0.$$

Here *meas* denotes linear Lebesgue measure. A well known lemma of Loomis, that is often used in proving boundedness of the Hilbert transform between appropriate spaces (see [1, pp.127–129] and [2, p.402, Lemma 3]) implies that

$$m_{g_n}(\lambda) \leq \frac{8}{\lambda} \sum_{j=1}^{n} \left| \frac{f}{\pi_n'}(x_{jn}) \right| \leq \frac{8}{\lambda} \sum_{j=1}^{n} \frac{1}{|\pi_n'|(x_{jn})} =: \frac{8}{\lambda} \Omega_n, \quad \lambda > 0.$$

Moreover, there is the trivial bound $m_{g_n}(\lambda) \leq 2$ (the linear measure of $[-1,1] \supseteq K$). We now use the representation of an L_p norm in terms of distribution functions [1, p.43]:

$$\begin{aligned} \| g_n \|_{L_{2p}(K)}^{2p} &= 2p \int_0^\infty \lambda^{2p-1} m_{g_n}(\lambda)\, d\lambda \\ &\leq 2p \int_0^\infty \lambda^{2p-1} \min\left\{2, \frac{8\Omega_n}{\lambda}\right\} d\lambda = 2p\Omega_n^{2p} \int_0^\infty s^{2p-1} \min\left\{2, \frac{8}{s}\right\} ds =: C_p^p \Omega_n^{2p}. \end{aligned}$$

Of course C_p is finite if $p < \frac{1}{2}$, which we now assume. (We note that the last estimate is essentially an inequality relating the weak L_1 norm of g_n and its L_{2p} norm.) Then (4) gives

$$\sup_n \| L_n[f]\, v \|_{L_p(K)} \leq C_p \sup_n \| \pi_n v \|_{L_{2p}(K)} \Omega_n < \infty,$$

by (2), provided $2p \leq r$. It then follows that for every $f \in C(K)$,

$$\sup_n \| L_n[f]\, v \|_{L_p(K)} \leq c \| f \|_{L_\infty(K)},$$

where c is independent of f. Next, let $\varepsilon > 0$. We may find a polynomial P such that

$$\| f - P \|_{L_\infty(K)} < \varepsilon.$$

Indeed, f has a continuous extension from K to $[-1,1]$ and then Weierstrass' Theorem may be applied. Then for large enough n,

$$\begin{aligned} \| (f - L_n[f])\, v \|_{L_p(K)}^p &\leq \| (f - P)\, v \|_{L_p(K)}^p + \left(c \| f - P \|_{L_\infty(K)}\right)^p \\ &\leq \varepsilon^p \left[\| v \|_{L_p(K)}^p + c^p \right], \end{aligned}$$

provided $p \leq q$, so that $\| v \|_{L_p(K)}$ is finite. Then the convergence (1) follows.

(I) $\Rightarrow$ (II)

We follow Shi [9, pp.30–31, Lemma 1]. Assume that we have the convergence (1). Then the uniform boundedness principle gives

$$\| (f - L_n [f]) v \|_{L_p(K)} \leq C \| f \|_{L_\infty(K)},$$

where C is independent of n and f, and consequently, for some possibly different C,

$$\| L_n [f] v \|_{L_p(K)} \leq C \left(\| f \|_{L_\infty(K)} + \| fv \|_{L_p(K)} \right). \tag{5}$$

Of course if $p < 1$, the space

$$\left\{ h : K \to \mathbb{R} \text{ with } \| hv \|_{L_p(K)} < \infty \right\}$$

is not a normed space, but it is a topological vector space, while $C(K)$ is a Banach space, and there is a version of the uniform boundedness principle that may be applied. See, for example, [8, p. 44, Thm. 2.6]. Next, choose f continuous on K such that

$$f(x_{kn}) = sign\left(\pi_n'(x_{kn})\right), \quad 1 \leq k \leq n$$

and $\| f \|_{L_\infty(K)} = 1$ (for example, we could choose f to be a piecewise linear function). We may also assume that the support of f is so small that

$$\| fv \|_{L_p(K)} \leq 1. \tag{6}$$

Let $S_n(x)$ be given by (3) and let $\sigma_n(x) := sign\left(\pi_n(x)\right)$. We see that

$$\begin{aligned} S_n(x) &= \sigma_n(x)\pi_n(x) \sum_{k=1}^{n} \frac{f(x_{kn})}{\pi_n'(x_{kn})} = \sigma_n(x) \sum_{k=1}^{n} f(x_{kn})(x - x_{kn})\,\ell_{kn}(x) \\ &= \sigma_n(x)\left(x L_n[f](x) - L_n[g](x) \right), \end{aligned}$$

where $g(x) := xf(x)$. Then (5) and (6) and the fact that $|g| \leq |f|$ give

$$\begin{aligned} \| S_n v \|_{L_p(K)} &\leq 2^{1/p} \left(\| L_n[f] v \|_{L_p(K)} + \| L_n[g] v \|_{L_p(K)} \right) \\ &\leq 2^{1/p} C \left(\| f \|_{L_\infty(K)} + \| g \|_{L_\infty(K)} + 1 \right) \leq 2^{1/p} 3C. \end{aligned}$$

As C is independent of n, we have (2) with $r = p$. $\square$

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