ON WEIGHTED MEAN CONVERGENCE OF LAGRANGE INTERPOLATION FOR GENERAL ARRAYS

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ABSTRACT. For $n \geq 1$, let $\{x_{jn}\}_{j=1}^{n}$ be n distinct points and let $L_n[\cdot]$ denote the corresponding Lagrange Interpolation operator. Let $W : \mathbb{R} \to [0, \infty)$. What conditions on the array $\{x_{jn}\}_{1 \leq j \leq n,\, n \geq 1}$ ensure the existence of $p > 0$ such that

$$\lim_{n \to \infty} \| (f - L_n[f]) \, W \phi^b \|_{L_p(\mathbb{R})} = 0$$

for every continuous $f : \mathbb{R} \to \mathbb{R}$ with suitably restricted growth, and some "weighting factor" ϕ^b? We obtain a necessary and sufficient condition for such a p to exist. The result is the weighted analogue of our earlier work for interpolation arrays contained in a compact set.

1. The Result

While there are very many results on mean convergence of Lagrange interpolation, the vast majority of these results deal with interpolation at zeros of orthogonal polynomials and their close cousins - at least in terms of sufficient conditions for mean convergence - see [3], [5], [6], [9]. In a recent paper [2], the author used distribution functions to treat general interpolation arrays contained in a compact set. Here we consider the non-compact case, and use decreasing rearrangements of functions, as well as a well known inequality of Hardy and Littlewood.

Throughout, we consider an array X of interpolation points $X = \{x_{jn}\}_{1 \leq j \leq n,\, n \geq 1}$ where

$$-\infty < x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n} < \infty.$$

We denote by $L_n[\,\cdot\,]$ the associated Lagrange interpolation operator, so that for $f : \mathbb{R} \to \mathbb{R}$, we have

$$L_n[f](x) = \sum_{j=1}^{n} f(x_{jn}) \, \ell_{jn}(x),$$

where the fundamental polynomials $\{\ell_{kn}\}_{k=1}^{n}$ satisfy

$$\ell_{kn}(x_{jn}) = \delta_{jk}.$$

We also let π_n denote a polynomial of degree n (without any specific normalisation) whose zeros are $\{x_{jn}\}_{j=1}^{n}$. In [2] we proved:

Theorem 1
Let $K \subset \mathbb{R}$ be compact, and let $v \in L_q(K)$ for some $q > 0$. Let the array X of

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interpolation points lie in K. The following are equivalent:
(I) *There exists $p>0$ such that for every continuous $f:K\to\mathbb{R}$, we have*

$$\lim_{n\to\infty}\parallel\left(f-L_n\left[f\right]\right)v\parallel_{L_p(K)}=0. \tag{1}$$

(II) *There exists $r>0$ such that*

$$\sup_{n\geq 1}\parallel\pi_n v\parallel_{L_r(K)}\left(\sum_{j=1}^{n}\frac{1}{\left|\pi_n'\right|\left(x_{jn}\right)}\right)<\infty. \tag{2}$$

The essential feature is that a single condition, namely (2), is sufficient for mean convergence of Lagrange interpolation in L_p for at least one $p>0$. This should be compared to results surveyed in [3], [5], [6], [9], where amongst other things, the interpolation points are assumed to be zeros of orthogonal polynomials associated with weights satisfying a number of conditions. The price one pays for the simplicity of (2) is that invariably $p<1$ or even $p<\frac{1}{2}$, and p and r are different in (I) and (II).

In extending this results to the case where the array of interpolation points is unbounded, it is instructive to recall a special result for the Freud weights

$$W_\beta\left(x\right):=\exp\left(-\frac{1}{2}\left|x\right|^\beta\right),x\in\mathbb{R},\beta>1.$$

Theorem 2
For $n\geq 1$, let $\left\{x_{jn}\right\}_{j=1}^{n}$ denote the zeros of the orthonormal polynomial for the weight W_β^2. Let $1<p<\infty$, $\Delta\in\mathbb{R}$, and let

$$\tau:=\tau\left(p\right):=\frac{1}{p}-1+\begin{cases}0, & p\leq 4\\ \frac{\beta}{6}\left(1-\frac{p}{4}\right), & p>4\end{cases}.$$

Then for

$$\lim_{n\to\infty}\parallel\left(f\left(x\right)-L_n\left[f\right]\left(x\right)\right)W_\beta\left(x\right)\left(1+\left|x\right|\right)^{-\Delta}\parallel_{L_p(\mathbb{R})}=0,$$

to hold for every continuous function $f:\mathbb{R}\to\mathbb{R}$ satisfying

$$\lim_{|x|\to\infty}\left|f\left(x\right)\right|W_\beta\left(x\right)\left(1+\left|x\right|\right)=0,$$

it is necessary and sufficient that

$$\Delta>\tau.$$

The technical nature of the formulation is fairly typical. (It is the case $\alpha=1$ of Theorem 1.1 in [4]). But from the point of view of the present paper, it is the need to include powers of $(1+|x|)$ to get anything positive at all that is important.

We shall allow far more general weights W and weighting factors $\phi\left(x\right)$ that generalize $1+|x|$. We shall use the convention

$$\parallel g\parallel_{L_\infty(\mathbb{R})}:=\sup\left\{\left|g\left(x\right)\right|:x\in\mathbb{R}\right\},$$

instead of essential sup.

Our first result concerns boundedness of the Lagrange operators:

Theorem 3

Let $W:\mathbb{R}\to[0,\infty)$ be measurable and such that $W(x_{jn})>0\ \forall j,n$. Let $\phi:\mathbb{R}\to[0,\infty)$ be continuous, and such that $W\phi^a$ has limit 0 at $\pm\infty\ \forall a\in\mathbb{R}$, and with

$$\phi(x)\geq 1+|x|\,, x\in\mathbb{R}. \tag{3}$$

Then the following are equivalent:
(I) There exist $b,c\in\mathbb{R}$ and $p,C>0$ such that for every function $f:\mathbb{R}\to\mathbb{R}$ and $n\geq 1$,

$$\| L_n[f]\,W\phi^b \|_{L_p(\mathbb{R})}\leq C\,\| fW\phi^c \|_{L_\infty(\mathbb{R})}\,. \tag{4}$$

(II) There exist $\beta,\gamma\in\mathbb{R}$ and $r>0$ such that

$$\sup_{n\geq 1}\| \pi_n W\phi^\beta \|_{L_r(\mathbb{R})}\sum_{j=1}^{n}\frac{1}{|\pi_n' W\phi^\gamma|(x_{jn})}<\infty. \tag{5}$$

We emphasize that b,c,p are not the same as the corresponding parameters β,γ,r. The simplest choice of ϕ would be

$$\phi(x)=1+|x|\,.$$

It would typically be a slowly growing function, whereas W would typically be a rapidly decaying function. The restriction that $W(x_{jn})>0\ \forall j,n$ ensures that we do not have division by 0 in the sum in (5).

The passage from boundedness of $\{L_n\}_{n=1}^\infty$ to convergence is not immediate, as it depends on density of polynomials in an appropriate weighted space. Let $u:\mathbb{R}\to[0,\infty)$, be measurable, and let $supp\,(u)$ denote its support. We let $\mathcal{C}_u$ denote the space of all measurable functions $f:\mathbb{R}\to\mathbb{R}$ with the following properties:
(A) f vanishes outside $supp\,(u)$.
(B) fu is continuous in $\mathbb{R}$.
(C) If $a=\pm\infty$ or a is a limit point of $\mathbb{R}\backslash supp\,(u)$,

$$\lim_{x\to a}(fu)(x)=0.$$

(D)

$$\| fu \|_{L_\infty(\mathbb{R})}<\infty.$$

It is not difficult to see that $\mathcal{C}_u$ is a Banach space. Indeed, if $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in $\mathcal{C}_u$, then it is clear that $f_n u$ has a continuous limit g as $n\to\infty$. One may define the limit of $\{f_n\}_{n=1}^\infty$ as $f:=g/u$ when $u\neq 0$ and as 0 in $\mathbb{R}\backslash\text{supp}(u)$. The only possible ambiguity is at limit points of $\mathbb{R}\backslash supp\,(u)$, and there we may define f to be 0.

One difficulty with (A) of this definition, is that polynomials, or even constant functions, will not belong to $\mathcal{C}_u$ if $supp\,(u)\neq\mathbb{R}$. So we talk of polynomials restricted to $supp\,(u)$, that is, set to 0 outside $supp\,(u)$.

Theorem 4
Let W and ϕ be as in Theorem 3. Assume that the polynomials restricted to $supp(W)$ are dense in $\mathcal{C}_{W\phi^a}$ for each $a\in\mathbb{R}$. The following are equivalent:
(I) There exist $b,c\in\mathbb{R}$ and $p>0$ such that for every $f\in\mathcal{C}_{W\phi^c}$,

$$\lim_{n\to\infty} \| (f - L_n[f]) W\phi^b \|_{L_p(\mathbb{R})} = 0. \tag{6}$$

(II) There exist $\beta, \gamma \in \mathbb{R}$ and $r > 0$ such that (5) holds.

Of course our hypothesis on the density of the polynomials places restrictions on W. If

$$W(x) = \exp\left(-|x|^\beta\right), x \in \mathbb{R},$$

then it is true iff $\beta \geq 1$. Additional restrictions on W, such as its behaviour at limits points of $\mathbb{R}\backslash supp(W)$, arise from the way we defined $\mathcal{C}_W$. In particular, if the polynomials, restricted to $supp(W)$ lie in $\mathcal{C}_W$, then (C) forces W to vanish at such limit points.

2. Proof of the Theorems

We begin by recalling some standard facts about distribution functions and decreasing rearrangements. Given measurable $g : \mathbb{R} \to \mathbb{R}$, its *distribution function* is

$$m_g(\lambda) := meas(\{x : |g(x)| > \lambda\}), \lambda \geq 0.$$

Here *meas* denotes linear Lebesgue measure. The *decreasing rearrangement* of g is

$$g^*(t) := \inf\{\lambda : m_g(\lambda) \leq t\} = \sup\{\lambda : m_g(\lambda) > t\}, t \geq 0.$$

For $0 < p < \infty$, we have

$$(|g|^p)^* = (g^*)^p. \tag{7}$$

Moreover, if $h : \mathbb{R} \to \mathbb{R}$ is measurable,

$$|g| \leq |h| \text{ a.e.} \Rightarrow g^* \leq h^*. \tag{8}$$

For all this, see [1, p. 41]. We shall also use an inequality of Hardy and Littlewood, [1, p. 44]

$$\int_{-\infty}^{\infty} |gh| \leq \int_0^\infty g^* h^*. \tag{9}$$

Theorem 3 will follow from two lemmas, that offer more information about the relationship between the parameters b, c, p and β, γ, r. Throughout, we assume that W and ϕ are as in Theorem 3.

Lemma 2.1
Let $b, c \in \mathbb{R}$ and $p > 0$, and assume that

$$2p(1 + c) > 1 > 2p. \tag{10}$$

Let $f : \mathbb{R} \to \mathbb{R}$ and assume that $fW\phi^c$ is bounded on $\mathbb{R}$. Then for $n \geq 1$ and for some C_0 depending only on c, p,

$$\begin{aligned} &\| L_n[f] W\phi^b \|_{L_p(\mathbb{R})} / \| fW\phi^c \|_{L_\infty(\mathbb{R})} \\ &\leq C_0 \sup_n \| \pi_n W\phi^{b+c} \|_{L_{2p}(\mathbb{R})} \sum_{j=1}^n \frac{1}{|\pi_n' W\phi^c|(x_{jn})}. \end{aligned} \tag{11}$$

Proof
We assume that the sup in the right-hand side of (11) is finite. We may also suppose that $\| fW\phi^c \|_{L_\infty(\mathbb{R})} = 1$. Now we can write

$$L_n[f](x) = \pi_n(x) \sum_{j=1}^{n} \frac{f(x_{jn})}{\pi_n'(x_{jn})(x - x_{jn})} =: \pi_n(x) g_n(x).$$

Then

$$\| L_n[f] W\phi^b \|_{L_p(\mathbb{R})} \leq \| \pi_n W\phi^{b+c} \|_{L_{2p}(\mathbb{R})} \| g_n \phi^{-c} \|_{L_{2p}(\mathbb{R})}. \tag{12}$$

To estimate the norm involving g_n, we use a well known lemma of Loomis (see [1, pp. 127-129], [2, p. 223]): for $\lambda > 0$,

$$m_{g_n}(\lambda) \leq \frac{8}{\lambda} \sum_{j=1}^{n} \left| \frac{f}{\pi_n'}(x_{jn}) \right| \leq \frac{8}{\lambda} \sum_{j=1}^{n} \frac{1}{|\pi_n' W\phi^c|(x_{jn})} =: \frac{8}{\lambda}\Omega_n.$$

Then for $t > 0$,

$$g_n^*(t) = \sup\{\lambda : m_{g_n}(\lambda) > t\} \leq \sup\left\{\lambda : \frac{8}{\lambda}\Omega_n > t\right\} = \frac{8\Omega_n}{t}. \tag{13}$$

Next, by (9) and (8),

$$\begin{aligned} \| \; g_n\phi^{-c} \|_{L_{2p}(\mathbb{R})}^{2p} &= \int_{-\infty}^{\infty} \left| g_n\phi^{-c} \right|^{2p} \\ &\leq \int_0^\infty \left(|g_n|^{2p}\right)^* \left(\phi^{-c2p}\right)^* = \int_0^\infty (g_n^*)^{2p} \left(\left(\phi^{-1}\right)^*\right)^{2pc}. \end{aligned} \tag{14}$$

Here we have used the fact that $c > 0$, which follows from (10). Let

$$\psi(x) := (1 + |x|)^{-1}, x \in \mathbb{R}.$$

By (3) and (8), followed by a straightforward calculation,

$$\left(\phi^{-1}\right)^*(t) \leq \psi^*(t) = \psi\left(\frac{t}{2}\right), t \geq 0.$$

Then (14) and (13) give

$$\| g_n\phi^{-c} \|_{L_{2p}(\mathbb{R})}^{2p} \leq \int_0^\infty \left(\frac{8\Omega_n}{t}\right)^{2p} \psi\left(\frac{t}{2}\right)^{2pc} dt = (8\Omega_n)^{2p} \int_0^\infty t^{-2p}\left(1 + \frac{t}{2}\right)^{-2pc} dt.$$

Here the integral converges because of our hypothesis (10). Then we obtain from (12),

$$\| L_n[f] W\phi^b \|_{L_p(\mathbb{R})} \leq C_1 \| \pi_n W\phi^{b+c} \|_{L_{2p}(\mathbb{R})} \sum_{j=1}^{n} \frac{1}{|\pi_n' W\phi^c|(x_{jn})},$$

with C_1 depending only on c, p. ■

Next, we turn to the converse:

Lemma 2.2
Let $p > 0$ and $b, c \in \mathbb{R}$. *Assume that for every* $n \geq 1$ *and measurable* $f : \mathbb{R} \to \mathbb{R}$, *and some* C *depending on* f,

$$\| L_n[f] W\phi^b \|_{L_p(\mathbb{R})} \leq C \| fW\phi^c \|_{L_\infty(\mathbb{R})}. \tag{15}$$

Then

$$\sup_n \| \pi_n W\phi^{b-1} \|_{L_p(\mathbb{R})} \sum_{j=1}^{n} \frac{1}{\left|\pi_n' W\phi^{c+1}\right|(x_{jn})} < \infty. \tag{16}$$

Proof

We use Shi's ideas [8] in a modified form. Let Y be the space of all measurable $h : \mathbb{R} \to \mathbb{R}$ that vanish outside $supp\,(W)$ with

$$\| h \|_Y := \| hW\phi^b \|_{L_p(\mathbb{R})} < \infty.$$

If $p \geq 1$, then Y is a Banach space, and if $p < 1$, it is a topological vector space. Our hypothesis implies that for each $f \in \mathcal{C}_{W\phi^c}$ (which is a Banach space),

$$\sup_n \| L_n [f] \|_Y = \sup_n \| L_n [f]\, W\phi^b \|_{L_p(\mathbb{R})} < \infty.$$

Then the uniform boundedness principle shows that there exists $C_0 > 0$ such that

$$\| L_n [f]\, W\phi^b \|_{L_p(\mathbb{R})} = \| L_n [f] \|_Y \leq C_0 \| fW\phi^c \|_{L_\infty(\mathbb{R})}, \tag{17}$$

where C_0 is independent of n and $f \in \mathcal{C}_{W\phi^c}$. Note that there is a suitable version of the uniform boundedness principle that may be applied even if $p < 1$. See, for example, [7, p. 44, Thm. 2.6]. Next, for a given n, choose $f : \mathbb{R} \to \mathbb{R}$ such that

$$\left(fW\phi^{c+1}\right)(x_{kn}) = sign\,(\pi_n' (x_{kn}))\,, \quad 1 \leq k \leq n$$

and

$$\| fW\phi^{c+1} \|_{L_\infty(\mathbb{R})} = 1$$

(for example, we could choose $fW\phi^{c+1}$ to be a piecewise linear function). Let

$$g(x) := xf(x)\,, x \in \mathbb{R}.$$

Of course, as $\phi(x) \geq |x|$, and $\phi(x) \geq 1$, also

$$\begin{aligned} \| \quad gW\phi^c \|_{L_\infty(\mathbb{R})} &\leq \| fW\phi^{c+1} \|_{L_\infty(\mathbb{R})} = 1; \\ \| \quad fW\phi^c \|_{L_\infty(\mathbb{R})} &\leq \| fW\phi^{c+1} \|_{L_\infty(\mathbb{R})} = 1. \end{aligned}$$

Let

$$S_n(x) := |\pi_n(x)| \sum_{k=1}^{n} \frac{1}{\left|\left(\pi_n' W\phi^{c+1}\right)(x_{kn})\right|}$$

and let $\sigma_n(x) := sign\,(\pi_n(x))$. We see that

$$\begin{aligned} S_n(x) &= \sigma_n(x)\pi_n(x) \sum_{k=1}^{n} \frac{f(x_{kn})}{\pi_n'(x_{kn})} = \sigma_n(x) \sum_{k=1}^{n} f(x_{kn})\,(x - x_{kn})\, \ell_{kn}(x) \\ &= \sigma_n(x)\,(xL_n[f](x) - L_n[g](x))\,. \end{aligned}$$

Then (17) and (3) give

$$\begin{aligned} \| S_n W\phi^{b-1} \|_{L_p(\mathbb{R})} &\leq 2^{1/p} \left(\| x \left(L_n[f]\, W\phi^{b-1}\right)(x) \|_{L_p(\mathbb{R})} + \| L_n[g]W\phi^{b-1} \|_{L_p(\mathbb{R})} \right) \\ &\leq 2^{1/p} \left(\| L_n[f]\, W\phi^b \|_{L_p(\mathbb{R})} + \| L_n[g]W\phi^b \|_{L_p(\mathbb{R})} \right) \\ &\leq 2^{1/p} C_0 \left(\| fW\phi^c \|_{L_\infty(\mathbb{R})} + \| gW\phi^c \|_{L_\infty(\mathbb{R})} \right) \\ &\leq 2^{1+1/p} C_0 \| fW\phi^{c+1} \|_{L_\infty(\mathbb{R})}\,. \end{aligned}$$

So we have (16). ■

We turn to

The Proof of Theorem 3
(I) ⇒(II)
It follows from Lemma 2.2 that (4) holds with

$$r := p; \beta := b - 1; \gamma := c + 1.$$

(II)⇒(I)
We claim that we may assume that $r < 1$ in (5). Indeed, by Hölder's inequality, if $s < r$, and $\alpha > 0$,

$$\| \pi_n W \phi^{\beta-\alpha} \|_{L_s(\mathbb{R})} \leq \| \pi_n W \phi^{\beta} \|_{L_r(\mathbb{R})} \left(\int_{\mathbb{R}} \phi^{-\frac{\alpha r s}{r-s}} \right)^{\frac{r-s}{rs}},$$

and the second integral on the right-hand side converges if

$$\frac{\alpha r s}{r - s} > 1.$$

It is also depends only on r, s, α, ϕ. Then it follows that if (5) holds for a given r and some β, then it holds for any smaller s, and appropriately smaller β. Next, as $\phi \geq 1$, it follows that if (5) holds with a given γ, then it holds for any larger γ. Thus we may assume that

$$r(1 + \gamma) > 1 > r.$$

Let us now choose $p := r/2$, $c := \gamma$, and $b \in \mathbb{R}$ such that

$$b + c = \beta.$$

Then (10) is satisfied, so (5) and Lemma 2.1 give (4). ■

Finally, we give

The Proof of Theorem 4
(I)⇒(II)
Let $f \in \mathcal{C}_{W\phi^c}$, and

$$\varepsilon_n := \| (f - L_n [f]) W \phi^b \|_{L_p(\mathbb{R})}, n \geq 1.$$

Our hypothesis implies that

$$\lim_{n \to \infty} \varepsilon_n = 0.$$

Then for $n \geq 1$,

$$\begin{aligned} \| \quad & L_n [f] W \phi^b \|_{L_p(\mathbb{R})} \leq 2^{1/p} \| f W \phi^b \|_{L_p(\mathbb{R})} + 2^{1/p} \varepsilon_n \\ \leq \quad & 2^{1/p} \| f W \phi^c \|_{L_\infty(\mathbb{R})} \| \phi^{b-c} \|_{L_p(\mathbb{R})} + 2^{1/p} \varepsilon_n. \end{aligned}$$

We may assume that b in (6) is so small that $(b - c) p < -1$, and then (3) and this last inequality give

$$\sup_n \| L_n [f] W \phi^b \|_{L_p(\mathbb{R})} < \infty.$$

Then (the proof of) Lemma 2.2 gives (5) with $r := p; \beta := b-1; \gamma := c+1$.
(II)⇒(I)
Let $f \in \mathcal{C}_{W\phi^c}$. For P a polynomial of degree $\leq m$ and $n > m$, we have

$$\begin{aligned} &\| \quad (f - L_n[f]) W\phi^b \|_{L_p(\mathbb{R})} \\ \leq &\quad 2^{1/p} \left(\| (f-P) W\phi^b \|_{L_p(\mathbb{R})} + \| L_n[P-f] W\phi^b \|_{L_p(\mathbb{R})} \right) \\ \leq &\quad 2^{1/p} \left(\| (f-P) W\phi^c \|_{L_\infty(\mathbb{R})} \| \phi^{b-c} \|_{L_p(\mathbb{R})} + C_0 \| (f-P) W\phi^c \|_{L_\infty(\mathbb{R})} \right), \end{aligned}$$

by Theorem 3, with the appropriate choice of b, c, p. Here if $(b-c)p < -1$, as we may assume (for if (4) holds for a given b, it holds for any smaller b), then we may continue this as

$$\| (f - L_n[f]) W\phi^b \|_{L_p(\mathbb{R})} \leq C_1 \| (f-P) W\phi^c \|_{L_\infty(\mathbb{R})},$$

with C_1 independent of f, n, m, P. The assumed density of the polynomials then shows that this may be made arbitrarily small if the degree m of P is large enough. ■

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