Asymptotics for Entropy Integrals associated with Exponential Weights

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Abstract

We establish a first order asymptotic for the entropy integrals

$$\int_I p_n^2 \left(\log p_n^2\right) W^2 \text{ and } \int_I p_n^2 \left(\log \left(p_n W\right)^2\right) W^2$$

where $\{p_n\}_{n=0}^{\infty}$ are the orthonormal polynomials associated with the exponential weight W^2.

1 The Result

Let $I = (c, d)$ be a real interval, where

$$-\infty \leq c < 0 < d \leq \infty,$$

and let $Q : I \to [0, \infty)$ be convex. Let

$$W := \exp\left(-Q\right)$$

and assume that all power moments

$$\int_I x^n W^2(x) dx, n = 0, 1, 2, 3, ...$$

are finite. Then we may define orthonormal polynomials

$$p_n(x) = p_n(W^2, x) = \gamma_n x^n + ..., \gamma_n > 0,$$

satisfying

$$\int_I p_n p_m W^2 = \delta_{mn}, m, n = 0, 1, 2, ...$$

In the last twenty years, there has been a remarkable development of quantitative analysis around exponential weights W, and in particular around asymptotics for $p_n(x)$. See [3], [5], [7], [8], [10], [13], [15], [17] for references and reviews.

In this paper, we compute first order asymptotics for the entropy integrals

$$E(p_n) = -\int_I p_n^2 \left(\log p_n^2\right) W^2$$

and

$$E^*(p_n) = -\int_I p_n^2 \left(\log (p_n W)^2\right) W^2,$$

as $n \to \infty$, for the most explicit class of weights in [8]. These entropy integrals arise in a number of contexts, for example quantum mechanics and information theory. An extensive survey of recent developments is given in [6]. For example, if $\lambda > 1$ and one considers the Freud weight

$$W(x) = \exp\left(-|x|^\lambda\right), x \in \mathbb{R},$$

it is known that there is the very precise asymptotic

$$E(p_n) = -\frac{2n+1}{\lambda} + \frac{1}{\lambda}\ln(2n) - C + o(1),$$

where C is an explicit constant. See [1], [18] and also [2], [16].

In this paper, we treat a far more general class of weights than Freud weights, but obtain only first order asymptotics, with a relative error of order n^{-c}, some $c > 0$. To define our classes of weights, we need the notion of a quasi-decreasing/ quasi-increasing function. A function $g : (0, b) \to (0, \infty)$ is said to be *quasi-increasing* if there exists $C > 0$ such that

$$g(x) \le Cg(y), 0 < x \le y < b.$$

In particular, an increasing function is quasi-increasing. Similarly we may define the notion of a *quasi-decreasing* function. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x,

$$C_1 \le f(x)/g(x) \le C_2.$$

Similar notation is used for sequences and sequences of functions. Throughout, $C, C_1, C_2, ...$ denote positive constants independent of n, x and polynomials P of degree at most n.

Definition 1
Let $W = e^{-Q}$ *where* $Q : I \to [0, \infty)$ *satisfies the following properties:*
(a) Q' *is continuous and* $Q(0) = 0$;
(b) Q'' *exists and is positive in* $I \backslash \{0\}$;
(c)

$$\lim_{t \to c+} Q(t) = \lim_{t \to d-} Q(t) = \infty;$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, t \neq 0$$

is quasi-increasing in $(0, d)$, *and quasi-decreasing in* $(c, 0)$, *with*

$$T(t) \geq \Lambda > 1, t \in I \backslash \{0\};$$

(e) There exists $C_1 > 0$ *such that*

$$\frac{Q''(x)}{| Q'(x) |} \leq C_1 \frac{| Q'(x) |}{Q(x)}, \text{ a.e. } x \in I \backslash \{0\}.$$

Then we write $W \in \mathcal{F}\left(C^2\right)$. *If in addition, there exists a compact subinterval* J *of* I *such that for some* $C_2 > 0$,

$$\frac{Q''(x)}{| Q'(x) |} \geq C_2 \frac{| Q'(x) |}{Q(x)}, \text{ a.e. } x \in I \backslash J,$$

then we write $W \in \mathcal{F}\left(C^2+\right)$.

We now motivate this (complicated!) definition with some examples. Let

$$\exp_0(x) := x$$

and for $j \geq 1$, recursively define the jth iterated exponential

$$\exp_j(x) := \exp\left(\exp_{j-1}(x)\right).$$

Let k, ℓ be nonnegative integers.
(I) Let $I = \mathbb{R}$ and for $\alpha, \beta > 1$, let

$$Q(x) = Q_{\ell,k,\alpha,\beta}(x) := \begin{cases} \exp_\ell(x^\alpha) - \exp_\ell(0), & x \in [0, \infty) \\ \exp_k(|x|^\beta) - \exp_k(0), & x \in (-\infty, 0) \end{cases}.$$

In particular,

$$Q_{0,0,\alpha,\beta}(x) = \begin{cases} x^\alpha, & x \in [0, \infty) \\ |x|^\beta, & x \in (-\infty, 0) \end{cases}.$$

(II) Let $I = (-1, 1)$ and for $\alpha, \beta > 0$, let

$$Q(x) = Q^{(\ell,k,\alpha,\beta)}(x) := \begin{cases} \exp_\ell((1 - x^2)^{-\alpha}) - \exp_\ell(1), & x \in [0, 1) \\ \exp_k((1 - x^2)^{-\beta}) - \exp_k(1), & x \in (-1, 0) \end{cases}.$$

In both cases, the subtraction of a constant ensures continuity of Q at 0. It is fairly straightforward to verify that these examples of exponents correspond to $W = \exp(-Q) \in \mathcal{F}\left(C^2+\right)$. See [8] for further orientation.

In analysis of exponential weights, a crucial role is played by the Mhaskar-Rakhmanov-Saff numbers $a_t, t \in \mathbb{R}$. For $t > 0$ and $W \in \mathcal{F}\left(C^2+\right)$, $c < a_{-t} < 0 < a_t < d$ are uniquely defined by the equations

$$t = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{xQ'(x)}{\sqrt{(x-a_{-t})(a_t-x)}}dx;$$

$$0 = \frac{1}{\pi}\int_{a_{-t}}^{a_t} \frac{Q'(x)}{\sqrt{(x-a_{-t})(a_t-x)}}dx.$$

It is a fairly basic result that a_t is an increasing function of $t \in \mathbb{R}$, with

$$\lim_{t\to-\infty} a_t = c; \lim_{t\to\infty} a_t = d.$$

One of the properties of $a_{\pm t}$ is the Mhaskar-Saff identity: for all polynomials P of degree at most n,

$$\| PW \|_{L_\infty(I)} = \| PW \|_{L_\infty[a_{-n},a_n]}.$$

Moreover, $a_{\pm n}$ are essentially the smallest numbers for which this is true [10], [11], [12], [15]. We use the notation

$$\delta_t = \frac{1}{2}\left(a_t + |a_{-t}|\right), t > 0.$$

This is not to be confused with the unit mass at t, or Dirac delta!

Our result is

Theorem 2
Let $W \in \mathcal{F}\left(C^2+\right)$.
(I) *Assume that for each* $\varepsilon > 0$,

$$T(x) = O\left(Q(x)^{\varepsilon}\right), x \to c+ \text{ or } x \to d-. \tag{1}$$

Then there exists $\kappa > 0$ *such that*

$$\begin{aligned} E(p_n) &= -\frac{2}{\pi}\int_{a_{-n}}^{a_n} \frac{Q(x)}{\sqrt{(a_n-x)(x-a_{-n})}}dx\left(1+O\left(n^{-\kappa}\right)\right) && (2)\\ &= -2\int_0^n \log\frac{\delta_n}{\delta_t}dt\left(1+O\left(n^{-\kappa}\right)\right)\\ &= -2\int_0^n \frac{s}{\delta_s}\delta_s'\,ds\ \left(1+O\left(n^{-\kappa}\right)\right). && (3)\end{aligned}$$

(II)

$$E^*(p_n) = -\log\delta_n + O(1).$$

Remarks
(a) For the Freud weight $\exp\left(-|x|^{\lambda}\right)$ on $I=\mathbb{R}$, where $\lambda>1$, we have

$$\delta_s = a_s = C_\lambda s^{1/\lambda}, s>0,$$

where C_λ may be expressed explicitly in terms of the gamma function. In this case, (2) and (3) become

$$\begin{aligned} E\left(p_n\right) &= -\frac{2n}{\lambda}(1+O\left(n^{-\kappa}\right)); \\ E^*\left(p_n\right) &= -\frac{\log n}{\lambda}+O\left(1\right), \end{aligned}$$

for some $\kappa>0$. This is of course weaker then the result quoted above.
(b) We note that the growth condition (1) on T ensures that for each $\varepsilon>0$, the integral on the right of (2) grows faster than $n^{1-\varepsilon}$ as $n\to\infty$. It is needed in our proofs, but the result should hold without it. It is satisfied for all regularly decaying weights on the real line, in particular for $Q_{\ell,k,\alpha,\beta}$ for $k,\ell\geq 0$ and $\alpha,\beta>1$. It is also true for weights that decay sufficiently rapidly near the endpoints of a finite interval. For example, if $I=(-1,1)$ and $\alpha>0$ and

$$Q\left(x\right)=\left(1-x^2\right)^{-\alpha}-1, x\in\left(-1,1\right),$$

then in I,

$$T\left(x\right)\sim\frac{1}{1-x^2}=Q\left(x\right)^{1/\alpha}$$

and (1) is not true for $\varepsilon<\frac{1}{\alpha}$. But if

$$Q\left(x\right)=\exp\left(\left(1-x^2\right)^{-\alpha}\right)-\exp\left(1\right), x\in\left(-1,1\right),$$

then (1) is true. Thus (1) is *essentially a lower growth restriction on Q if I is a finite interva*l. More generally, (1) is true for $Q^{(\ell,k,\alpha,\beta)}$ for $\ell,k\geq 1$ and $\alpha,\beta>0$.
(c) In the case of even weights, it follows from (23) below that

$$E\left(p_n\right)\sim-\int_0^n\frac{ds}{T\left(a_s\right)}.$$

When T is unbounded, it may well happen that $-E\left(p_n\right)=o\left(n\right)$, in contrast to the so-called Freud case where Q is of polynomial growth, and where $T\sim 1$.
(d) Our methods permit one also to obtain first order asymptotics for the more general integrals

$$\int_I |x|^{\gamma}\left|\left(p_nW\right)\left(x\right)\right|^{p}\left|\log\left|p_n\left(x\right)\right|\right|^{q}dx,$$

as $n\to\infty$, provided $\gamma\geq 0, q\geq 0$, and $p<4$. For $p\geq 4$, the dominant contribution to the integral comes from the growth of p_n near $a_{\pm n}$, and our methods fail. The necessary asymptotics to handle these integrals are available only for special weights, such as $\exp\left(-|x|^{\lambda}\right), \lambda > 0$ [7].

(e) The result actually holds for a larger class of weights than that in Definition 1, namely it holds for the class $\mathcal{F}\left(lip\frac{1}{2}+\right)$ in [8]. However, the definition of that class of weights is more implicit, so we spare the reader the details.

We present the main parts of the proof in the next section, deferring technical details till later. In Section 3, we record some technical estimates, and estimate some of the integrals for Theorem 2. The remainder are done in Section 4.

2 The Proof of Theorem 2

We begin with (II), which admits a short proof:

The Proof of Theorem 2(II)

We apply Jensen's inequality with the unit measure $d\nu(x) = (p_n W)^2(x)\,dx$ on I. This gives

$$\begin{aligned} -E^*(p_n) &= 2\int_I \log|p_n W|\,d\nu \\ &\leq 2\log\int_I |p_n W|\,d\nu \\ &= 2\log\int_I |p_n W|^3 . \end{aligned}$$

Similarly, Jensen's inequality gives

$$\begin{aligned} -E^*(p_n) &= -2\int_I \log\frac{1}{|p_n W|}d\nu \\ &\geq -2\log\int_I \frac{1}{|p_n W|}d\nu \\ &= -2\log\int_I |p_n W| . \end{aligned}$$

We record in Lemma 3.2(iii) below, the estimate

$$\int_I |p_n W|^p \sim \delta_n^{1-\frac{p}{2}}, n \geq 1,$$

valid for any fixed $p < 4$. Applying this with $p = 3$ and $p = 1$ in the upper and lower bounds, gives (remarkably!)

$$-E^*(p_n) = -\log\delta_n + O(1) .$$

■

We shall give the main part of the proof of (I) in several steps, and defer all technical details till later. We shall use a parameter $\kappa \in (0,1)$ that is independent of n, x, θ and may be different in different occurrences. However, as it is used finitely many times, and in all uses the statements hold true for smaller κ than the given κ, we can simply keep reducing it.

Proof of Theorem 2(I)
Step 1: Reduce to the main part of $-E(p_n)$
We write

$$\begin{aligned} -E(p_n) &= \int_I (p_n W)^2 \log (p_n W)^2 + 2 \int_I (p_n W)^2 Q \\ &=: -E^*(p_n) + J_2. \end{aligned}$$

Here from what we have just proved, and (12) in Lemma 3.1(i),

$$-E^*(p_n) = O(\log n).$$

Next, we split J_2 into a main part and a tail:

$$\begin{aligned} J_2 &= 2 \int_{a_{-n}}^{a_n} (p_n W)^2 Q + 2 \int_{I \setminus [a_{-n}, a_n]} (p_n W)^2 Q \\ &=: J_{21} + J_{22}. \end{aligned}$$

Despite the rapid growth of Q near c, d, one may still apply the ideas of restricted range inequalities (or infinite-finite range inequalities) to show that

$$J_{22} = O\left(n^{1-\kappa}\right),$$

some $\kappa \in (0, 1)$. This will be done in Lemma 4.1(b). Then in summary, we have

$$-E(p_n) = J_{21} + O\left(n^{1-\kappa}\right). \tag{4}$$

Step 2: Apply asymptotics of orthonormal polynomials
We let

$$\delta_n = \frac{1}{2}\left(a_n + |a_{-n}|\right) \text{ and } \beta_n = \frac{1}{2}\left(a_n + a_{-n}\right)$$

and let L_n denote the linear map of $[a_{-n}, a_n]$ onto $[-1, 1]$, and $L_n^{[-1]}$ denote its inverse, so that

$$L_n(x) = \frac{x - \beta_n}{\delta_n} \text{ and } L_n^{[-1]}(u) = \delta_n u + \beta_n.$$

The asymptotic for the orthonormal polynomials $\{p_n\}$ in [8] may be cast in the form

$$\begin{aligned} &\delta_n^{1/2} (p_n W)\left(L_n^{[-1]}(\cos\theta)\right)\sqrt{\sin\theta} \\ &= \sqrt{\frac{2}{\pi}} \cos\left(\frac{\theta}{2} - \frac{\pi}{4} + \frac{n}{2}\Phi_n(\theta)\right) + O(\varepsilon_n(\theta)), \end{aligned}$$

where the order term is uniform in n and $\theta \in [0, \pi]$ and the error function ε_n satisfies

$$\sup_n \sup_{\theta \in [0,\pi]} |\varepsilon_n(\theta)| < \infty$$

and for some $\kappa \in (0, 1)$, and some $C > 0$,

$$\sup\{|\varepsilon_n(\theta)| : \theta \in [n^{-\kappa}, \pi - n^{-\kappa}]\} \le Cn^{-\kappa}.$$

The function Φ_n may be expressed in terms of a transformed equilibrium measure. For the moment we just note that

$$\Phi_n(\theta) = 2\pi \int_{\cos\theta}^{1} \sigma_n^*(t)\, dt, \theta \in [0, \pi], \tag{5}$$

where σ_n^* is a positive function on $(-1, 1)$, with

$$\int_{-1}^{1} \sigma_n^* = 1.$$

All this will be made more precise in Lemma 3.2. A little trigonometry then shows that that uniformly in n and θ,

$$\begin{aligned} & \delta_n (p_n W)^2 \left(L_n^{[-1]}(\cos\theta)\right) \sin\theta \\ = \quad & \frac{1}{\pi} + \frac{1}{\pi} \sin(\theta + n\Phi_n(\theta)) + O(\varepsilon_n(\theta)) \\ = \quad & \frac{1}{\pi} + \frac{1}{\pi} \sin\theta \cos n\Phi_n(\theta) + \frac{1}{\pi} \cos\theta \sin n\Phi_n(\theta) + O(\varepsilon_n(\theta)). \end{aligned}$$

Then the substitution $x = L_n^{[-1]}(\cos\theta)$ and this last asymptotic shows that

$$\begin{aligned} J_{21} \quad &= \quad 2\int_{a_{-n}}^{a_n} (p_n W)^2 Q \\ &= \quad 2\int_0^\pi \delta_n (p_n W)^2 \left(L_n^{[-1]}(\cos\theta)\right) \sin\theta \; Q\left(L_n^{[-1]}(\cos\theta)\right) d\theta \\ &= \quad \frac{2}{\pi}\int_0^\pi Q\left(L_n^{[-1]}(\cos\theta)\right) d\theta + \frac{2}{\pi}\int_0^\pi Q\left(L_n^{[-1]}(\cos\theta)\right) \sin\theta \cos n\Phi_n(\theta)\, d\theta \\ & \quad\; + \frac{2}{\pi}\int_0^\pi Q\left(L_n^{[-1]}(\cos\theta)\right) \cos\theta \sin n\Phi_n(\theta)\, d\theta \\ & \quad\; + O\left(\int_0^\pi |\varepsilon_n(\theta)|\, Q\left(L_n^{[-1]}(\cos\theta)\right) d\theta\right) \\ &= \quad : J_{21,1} + J_{21,2} + J_{21,3} + O(J_{21,4}). \end{aligned} \tag{6}$$

The main part here is $J_{21,1}$. In Lemma 3.1(v), we shall estimate $J_{21,1}$ below, and show that for each $\varepsilon > 0$, there exists $C > 0$ such that

$$J_{21,1} \ge Cn^{1-\varepsilon}. \tag{7}$$

Moreover, in Lemma 3.2(v), we shall use the estimates for ε_n and upper bounds for Q to show that for some $\kappa \in (0,1)$,

$$J_{21,4} \leq Cn^{1-\kappa}. \tag{8}$$

Step 3: Apply Jackson Theorems to estimate $J_{21,2}$ and $J_{21,3}$
We shall describe the estimation of $J_{21,2}$. That for $J_{21,3}$ is similar. Now by its definition in (5),

$$\Phi_n(0) = 0 \text{ and } \Phi_n(\pi) = 2\pi$$

and

$$\Phi_n'(\theta) = 2\pi\sigma_n^*(\cos\theta)\sin\theta > 0 \text{ in } (0,\pi)$$

so the substitution $\phi = \Phi_n(\theta)$ gives

$$J_{21,2} = \frac{2}{\pi}\int_0^{2\pi} g_n(\phi)\cos n\phi \, d\phi,$$

where

$$g_n(\phi) = Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}(\phi)\right)\right)\sin\Phi_n^{[-1]}(\phi)/\Phi_n'\left(\Phi_n^{[-1]}(\phi)\right)$$

and of course, $\Phi_n^{[-1]}$ denotes the inverse of Φ_n. Now if g_n was independent of n, the Riemann-Lebesgue Lemma would show that

$$J_{21,2} = o(1), n \to \infty.$$

Since g_n is not independent of n, we must proceed differently. The orthogonality of $\cos n\phi$ to trigonometric polynomials S of degree less than n gives

$$|J_{21,2}| \leq \inf_{\deg(S)<n} \frac{2}{\pi}\int_0^{2\pi} |g_n(\phi) - S(\phi)| \, d\phi.$$

Then Jackson type Theorems [4, Theorem 2.3, p. 205] show that

$$|J_{21,2}| \leq C \sup_{|u|\leq 1/n} \int_0^{2\pi} |g_n(\phi+u) - g_n(\phi)| \, d\phi.$$

Using estimates for Q', Φ_n', we shall estimate this sup and show in Lemma 4.3 that for some $\kappa \in (0,1)$,

$$J_{21,2} = O\left(n^{1-\kappa}\right).$$

As we noted, a similar estimate holds for $J_{21,3}$. Then (6), (7), (8) and these last estimates show that for some $\kappa \in (0,1)$,

$$J_{21} = \left(\frac{2}{\pi}\int_0^{\pi} Q\left(L_n^{[-1]}(\cos\theta)\right) d\theta\right)\left(1 + O\left(n^{-\kappa}\right)\right).$$

So (4) gives

$$-E(p_n) = \left(\frac{2}{\pi}\int_0^{\pi} Q\left(L_n^{[-1]}(\cos\theta)\right) d\theta\right)\left(1 + O\left(n^{-\kappa}\right)\right).$$

Now the reverse substitution $x = L_n^{[-1]}(\cos\theta)$ gives the first form of the asymptotic in Theorem 2.

Step 4: The other forms of the asymptotic

We shall use a little potential theory. Let μ_n denote the equilibrium measure of mass n for the external field Q. This is a non-negative measure on $[a_{-n}, a_n]$ with total mass n, that has a number of extremal properties. The one that we shall use involves the potential

$$V^{\mu_n}(x) = \int \log \frac{1}{|x-y|} d\mu_n(y).$$

It is known that

$$V^{\mu_n}(x) + Q(x) = c_n, x \in [a_{-n}, a_n], \tag{9}$$

where

$$c_n = \int_0^n \log \frac{2}{\delta_t} dt.$$

For a statement of this, with this representation of the constant c_n, see [8, Theorem 2.7, p. 46] or [3]. For a detailed discussion of the potential theory, see [15]. Then

$$\begin{aligned} & \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q(x)}{\sqrt{(a_n - x)(x - a_{-n})}} dx \\ = & \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{(c_n - V^{\mu_n})(x)}{\sqrt{(a_n - x)(x - a_{-n})}} dx \\ = & c_n + \int_{a_{-n}}^{a_n} \left[\frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{\log|x-y|}{\sqrt{(a_n - x)(x - a_{-n})}} dx\right] d\mu_n(y) \\ = & c_n + \int_{a_{-n}}^{a_n} \log \frac{\delta_n}{2} d\mu_n(y) \\ = & c_n + n\log\frac{\delta_n}{2} = \int_0^n \log\frac{\delta_n}{\delta_t} dt. \end{aligned}$$

In the third last line we used the fact that $\log|x-y|$ is bounded above in the integral, to allow interchange of the integrals (Fubini's Theorem). Moreover, in the second last line, we used the classical equilibrium potential for an interval $[a, b]$ [15, pp. 45-46]. So we have the second form of the asymptotic. Finally, it was proved in [8, Lemma 2.12, p. 52] that $a_{\pm t}$ are absolutely continuous functions of t, and so the same is true of δ_t. Then

$$\int_0^n \log\frac{\delta_n}{\delta_t} dt = \int_0^n \left[\int_t^n \frac{\delta_s'}{\delta_s} ds\right] dt = \int_0^n \frac{\delta_s'}{\delta_s}\left[\int_0^s dt\right] ds.$$

So we have the last form in (3). Of course δ_s' exists a.e. and is non-negative, so the interchange is justified. ■

3 Technical Estimates

In this section, we record a number of estimates for orthogonal polynomials and related quantities, and also establish some simple consequences of them. Throughout, we assume that $W = \exp(-Q) \in \mathcal{F}\left(C^2+\right)$ and that (1) holds. Moreover, we denote the zeros of the nth orthonormal polynomial p_n by

$$x_{nn} < x_{n-1,n} < x_{n-2,n} < ... < x_{2n} < x_{1n}.$$

Note that for large enough n,

$$a_{-n} < x_{nn} < x_{1n} < a_n,$$

as follows from Lemma 3.2(ii) below.

First some estimates involving Q :

Lemma 3.1
(i) For $n \geq 1$,

$$Q\left(a_{\pm n/2}\right) \sim Q\left(a_{\pm n}\right) \sim n\sqrt{\frac{|a_{\pm n}|}{\delta_n T\left(a_{\pm n}\right)}} \leq Cn; \tag{10}$$

$$Q'\left(a_{\pm n}\right) \sim n\sqrt{\frac{T\left(a_{\pm n}\right)}{|a_{\pm n}|\,\delta_n}}; \tag{11}$$

and for some $\kappa > 0$,

$$\delta_n = O\left(n^{1-\kappa}\right). \tag{12}$$

(ii) From (1) follows that for each $\varepsilon > 0$,

$$T\left(a_{\pm n}\right) = O\left(n^{\varepsilon}\right), n \to \infty. \tag{13}$$

(iii) Uniformly for $n \geq 1$ and $s \in [-1, 1]$,

$$\delta_n \left|Q'\left(L_n^{[-1]}(s)\right)\right| \sqrt{1-s^2} \leq Cn.$$

(iv) For $n \geq 1$,

$$1 - \frac{a_{\pm n/2}}{a_{\pm n}} \sim \frac{1}{T\left(a_{\pm n}\right)}. \tag{14}$$

(v) Given $\varepsilon > 0$, there exists $C > 0$ such that

$$J_{21,1} = \int_{a_{-n}}^{a_n} \frac{Q(x)}{\sqrt{(a_n - x)(x - a_{-n})}} dx \geq Cn^{1-\varepsilon}.$$

Proof
(i) The first two estimates are (3.17) and (3.18) of Lemma 3.4 in [8, p. 69] and part of (3.28) of Lemma 3.5 in [8, p. 72]. The third follows from Lemma 3.5(c)

in [8, p. 72].
(ii) We have from (1) and (i) above,

$$T\left(a_{\pm n}\right)=O\left(Q\left(a_{\pm n}\right)^{\varepsilon}\right)=O\left(n^{\varepsilon}\right), n\rightarrow\infty.$$

(iii) After the substitution $s=L_n\left(x\right)$, the desired inequality becomes

$$\left|Q'\left(x\right)\right|\sqrt{\left(a_n-x\right)\left(x-a_{-n}\right)}\leq Cn, x\in\left[a_{-n},a_n\right].$$

This is a weaker form of (3.40) of Lemma 3.8 in [8, p. 77].
(iv) This is (3.52) of Lemma 3.11 in [8, p. 81].
(v) Now as T is quasi-increasing in $(0,d)$ and quasi-decreasing in $(c,0)$,

$$\begin{aligned}
& \int_{a_{-n}}^{a_n}\frac{Q\left(x\right)}{\sqrt{\left(a_n-x\right)\left(x-a_{-n}\right)}}dx \\
= & \int_{a_{-n}}^{a_n}\frac{xQ'\left(x\right)}{T\left(x\right)\sqrt{\left(a_n-x\right)\left(x-a_{-n}\right)}}dx \\
\geq & \frac{C}{\max\left\{T\left(a_n\right),T\left(a_{-n}\right)\right\}}\int_{a_{-n}}^{a_n}\frac{xQ'\left(x\right)}{\sqrt{\left(a_n-x\right)\left(x-a_{-n}\right)}}dx \\
= & C\frac{n}{\max\left\{T\left(a_n\right),T\left(a_{-n}\right)\right\}},
\end{aligned}$$

by definition of $a_{\pm n}$. Then (ii) gives the result. ■

Next, we record estimates involving the orthonormal polynomials and their asymptotics. Some of this was already discussed in the previous section, though here we shall give more details. Recall that there is an equilibrium measure μ_n with total mass n and support on $[a_{-n},a_n]$ and satisfying (9). Now μ_n is absolutely continuous and has density σ_n, so that for $\mathcal{E}\subset\left[a_{-n},a_n\right]$,

$$\mu_n\left(\mathcal{E}\right)=\int_{\mathcal{E}}\sigma_n\left(t\right)dt$$

where $\sigma_n>0$ in $\left(a_{-n},a_n\right)$ and

$$\int_{a_{-n}}^{a_n}\sigma_n\left(t\right)dt=n.$$

It is often preferable to work on the fixed interval $[-1,1]$, rather than on $[a_{-n},a_n]$, which varies with n. Accordingly, we define

$$\sigma_n^*\left(x\right)=\frac{\delta_n}{n}\sigma_n\left(L_n^{[-1]}\left(x\right)\right), x\in\left(-1,1\right),$$

so that $\sigma_n^*>0$ in $\left(-1,1\right)$ and

$$\int_{-1}^{1}\sigma_n^*\left(x\right)dx=1.$$

As in the proof of Theorem 2, we let Φ_n be the function defined by (5). Moreover, we use the same "J" notation for some integrals as in the proof of Theorem 2.

Lemma 3.2
(i) For each $\varepsilon > 0$,

$$\| p_n W \|_{L_\infty(I)} \sim n^{1/6} \delta_n^{-1/3} \max \left\{ \frac{T(a_n)}{a_n}, \frac{T(a_{-n})}{|a_{-n}|} \right\}^{1/6} = O\left(n^{1/6+\varepsilon}\right). \tag{15}$$

(ii) There exists n_0 *such that for* $n \geq n_0$,

$$1 - \frac{x_{1n}}{a_n} \sim \eta_n \text{ and } 1 - \frac{x_{nn}}{a_{-n}} \sim \eta_{-n}, \tag{16}$$

where

$$\eta_{\pm n} = \left\{ nT(a_{\pm n}) \sqrt{\frac{|a_{\pm n}|}{\delta_n}} \right\}^{-2/3}. \tag{17}$$

(iii) Let $0 < p < 4$. *Without assuming (1), we have for* $n \geq 1$,

$$\| p_n W \|_{L_p(I)} \sim \delta_n^{\frac{1}{p}-\frac{1}{2}}.$$

(iv) Uniformly for $n \geq 1$ *and* $\theta \in [0, \pi]$,

$$\begin{aligned} & \delta_n^{1/2} (p_n W) \left(L_n^{[-1]}(\cos\theta) \right) \sqrt{\sin\theta} \\ = & \sqrt{\frac{2}{\pi}} \cos\left(\frac{\theta}{2} - \frac{\pi}{4} + \frac{n}{2}\Phi_n(\theta) \right) + \varepsilon_n(\theta), \end{aligned} \tag{18}$$

where the error function ε_n *satisfies*

$$\sup_n \sup_{\theta \in [0,\pi]} |\varepsilon_n(\theta)| < \infty \tag{19}$$

and for some $\kappa \in (0, 1)$, *and some* $C > 0$,

$$\sup\{|\varepsilon_n(\theta)| : \theta \in [n^{-\kappa}, \pi - n^{-\kappa}]\} \leq Cn^{-\kappa}. \tag{20}$$

(v) For some $\kappa > 0$,

$$J_{21,4} = \int_0^\pi |\varepsilon_n(\theta)| \, Q\left(L_n^{[-1]}(\cos\theta) \right) d\theta \leq Cn^{1-\kappa}.$$

Proof
(i) The $\sim$ relation is Theorem 1.18 in [8, p. 22]. The upper bound follows because of our bound on $T(a_{\pm n})$ in the previous lemma.
(ii) See [8, Theorem 1.19(f), p. 23].

(iii) This is part of Theorem 13.6 in [8, p. 362].
(iv) The asymptotic (18) with the estimate (20) on the error term ε_n is Theorem 1.24 in [8, p. 26]. The uniform bound (19) follows from the estimate [8, Theorem 1.17, p. 22]

$$\sup_{x\in I} |p_n W| (x) \left|(x - a_{-n}) (x - a_n)\right|^{1/4} \leq C.$$

Then the substitution $x = L_n (\cos\theta)$ gives uniformly for $n \geq 1$ and $\theta \in [0, \pi]$,

$$\delta_n^{1/2} |p_n W| \left(L_n^{[-1]} (\cos\theta)\right) \sqrt{\sin\theta} \leq C.$$

Thus the left-hand side of (18) is bounded uniformly in $n \geq 1$ and $\theta \in [0, \pi]$, and the first term on the right-hand side of (18) is also uniformly bounded. Then the same is true of ε_n.
(v) From (20) and (10),

$$\begin{aligned} &\int_{n^{-\kappa}}^{\pi - n^{-\kappa}} |\varepsilon_n (\theta)| \, Q\left(L_n^{[-1]} (\cos\theta)\right) d\theta \\ \leq \quad & C n^{-\kappa} \max\{Q(a_n), Q(a_{-n})\} \leq C n^{1-\kappa}. \end{aligned}$$

Next, the bound (19) on ε_n and the bound (10) on $Q(a_{\pm n})$ give

$$\begin{aligned} &\left(\int_0^{n^{-\kappa}} + \int_{\pi - n^{-\kappa}}^{\pi}\right) |\varepsilon_n (\theta)| \, Q\left(L_n^{[-1]} (\cos\theta)\right) d\theta \\ \leq \quad & C n \cdot n^{-\kappa}. \end{aligned}$$

Now add these last two estimates. ■

4 Estimation of the Integrals

In this section, we finish the estimation of the "J" integrals defined in the proof of Theorem 2(I). We use the same notation as there. We first bound some "tail" integrals.

Lemma 4.1
There exists $\kappa > 0$ such that the following hold:
(a)

$$1 - \int_{a_{-n}}^{a_n} (p_n W)^2 = O\left(n^{-\kappa}\right).$$

(b)

$$J_{22} = \int_{I \setminus [a_{-n}, a_n]} (p_n W)^2 \, Q = O\left(n^{1-\kappa}\right).$$

Proof

(a) This is really contained in Theorem 8.4(b) of [8, p. 238]. Now

$$\begin{aligned}
1 &= \int_I (p_n W)^2 \\
&\geq \int_{a_{-n}}^{a_n} (p_n W)^2 \\
&\geq \gamma_n^2 \inf_{\deg(S)<n} \int_{a_{-n}}^{a_n} (x^n - S(x))^2 W(x)^2 \, dx \\
&= \inf_{\deg(S)<n} \int_{a_{-n}}^{a_n} (x^n - S(x))^2 W(x)^2 \, dx / \inf_{\deg(S)<n} \int_I (x^n - S(x))^2 W(x)^2 \, dx \\
&= 1 + O\left(n^{-\kappa}\right),
\end{aligned}$$

by (8.30) in Theorem 8.4(b) [8, p. 238]. Here we are using the extremal properties of leading coefficients of orthogonal polynomials, and the fact that our class of weights is contained in the class $\mathcal{F}\left(s^{1/2}\right)$ considered there.
(b) Now

$$\begin{aligned}
&\int_{a_n}^{a_{3n}} (p_n W)^2 Q \\
\leq\ & Q(a_{3n}) \int_{a_n}^{a_{3n}} (p_n W)^2 \\
\leq\ & C n^{1-\kappa}, \qquad (21)
\end{aligned}$$

by (10) and (a) of this lemma. To handle the integral over $[a_{3n}, d)$, we use a restricted range inequality from [8, (4.12), Lemma 4.4, p. 99]. Choosing $p = \infty$ and $\Omega = t = n$ there, gives

$$|p_n W|(x) \leq \exp(U_n(x)) \parallel p_n W \parallel_{L_\infty(I)}$$

where U_n is an explicitly given function. We shall not need this explicit form; instead we need the estimate (4.18) from Lemma 4.5 there. It yields

$$U_n(a_r) \leq -C_1 r^C, r \geq 3n, \qquad (22)$$

where $C, C_1 > 0$ are independent of n and r. Then

$$\begin{aligned}
\int_{a_{3n}}^{d} (p_n W)^2 Q &= \int_{3n}^{\infty} (p_n W)^2 (a_r) Q(a_{3r}) a_r' \, dr \\
&\leq C \parallel p_n W \parallel_{L_\infty(I)}^2 \int_{3n}^{\infty} \exp\left(-C_1 r^C\right) r \, a_r' \, dr.
\end{aligned}$$

Here we have used our bound (10) on $Q(a_r)$. Next, we note the relation [8, (3.47), Theorem 3.10(b), p. 79]

$$a_r' \sim \frac{a_r}{rT(a_r)} \leq C_2, r > 0. \qquad (23)$$

This and our bound from Lemma 3.2(i) on $\| p_n W \|_{L_\infty(I)}$ give

$$\int_{a_{3n}}^{d} (p_n W)^2 Q \leq C_3 \exp(-C_4 n^C).$$

In summary, this inequality, and (21) give

$$\int_{a_n}^{d} (p_n W)^2 Q \leq C n^{1-\kappa},$$

and a similar relation holds over (c, a_{-n}). ■

In the estimation of $J_{21,2}$, we shall need smoothness properties of the function

$$\Phi_n(\theta) = 2\pi \int_{\cos\theta}^{1} \sigma_n^*(t)\, dt, \theta \in [0, \pi].$$

Lemma 4.2
(a) For $n \geq 1$ and $\theta \in [0, \pi]$,

$$C_1 \geq \Phi_n'(\theta) \geq C_2 (\sin\theta)^2. \tag{24}$$

(b) For $n \geq 1$ and $\theta, \phi \in (0, \pi)$,

$$|\Phi_n'(\theta) - \Phi_n'(\phi)| \leq C \left(\frac{|\theta - \phi|}{|\sin\theta|^2} \right)^{1/4}. \tag{25}$$

(c) For $n \geq 1$ and $u, v \in [0, 2\pi]$,

$$C_1 |u - v| \leq \left| \Phi_n^{[-1]}(u) - \Phi_n^{[-1]}(v) \right| \leq C_2 |u - v|^{1/3}. \tag{26}$$

(d) Let $\rho > 0$. For $n \geq 1$ and $u, v \in [n^{-\rho}, 2\pi - n^{-\rho}]$,

$$\left| \frac{1}{\Phi_n'\left(\Phi_n^{[-1]}(u)\right)} - \frac{1}{\Phi_n'\left(\Phi_n^{[-1]}(v)\right)} \right| \leq C |u - v|^{1/12} n^{5\rho}. \tag{27}$$

Proof
(a) Theorem 6.1(b) in [8, p. 146] asserts that for $n \geq 1$ and $u \in [-1, 1]$,

$$C_1 \frac{\sqrt{1 - u^2}}{h_n^*(u)} \leq \sigma_n^*(u) \leq \frac{C_2}{\sqrt{h_n^*(u)}}$$

where

$$h_n^*(u) = (1 - u + \chi_n)\left(1 + u + \chi_{-n}\right)$$

and

$$\chi_{\pm n} = \frac{|a_{\pm n}|}{\delta_n T(a_{\pm n})}.$$

Then we deduce that

$$C_1\sqrt{1-u^2} \le \sigma_n^*(u) \le \frac{C_2}{\sqrt{1-u^2}}.$$

Then

$$\Phi_n'(\theta) = 2\pi\sigma_n^*(\cos\theta)\sin\theta$$

satisfies (24).

(b) By Theorem 6.3(a) in [8, p. 148], with $\psi(s) = s^{1/2}$, and by this last identity,

$$\begin{aligned} |\Phi_n'(\theta) - \Phi_n'(\phi)| &\le C\left(\frac{|\cos\theta - \cos\phi|}{h_n^*(\cos\theta)}\right)^{1/4} \\ &\le C\left(\frac{|\theta-\phi|}{\sin^2\theta}\right)^{1/4}. \end{aligned}$$

(c) It follows from (a) that for $n \ge 1$ and $\theta \in [0,\pi]$,

$$C_1 \ge \Phi_n'(\theta) \ge C_2 \min\{\theta, \pi-\theta\}^2.$$

On integrating this inequality, we see that for $\theta, \phi \in [0,\pi]$,

$$C_1|\theta-\phi| \ge |\Phi_n(\theta) - \Phi_n(\phi)| \ge C_2|\theta-\phi|^3.$$

Setting $\theta = \Phi_n^{[-1]}(u)$ and $\phi = \Phi_n^{[-1]}(v)$ gives for all n and $u, v \in [0, 2\pi]$,

$$C_1\left|\Phi_n^{[-1]}(u) - \Phi_n^{[-1]}(v)\right| \ge |u-v| \ge C_2\left|\Phi_n^{[-1]}(u) - \Phi_n^{[-1]}(v)\right|^3.$$

(d) Now the left-hand side of (27) equals

$$\begin{aligned} &\left|\frac{\Phi_n'\left(\Phi_n^{[-1]}(v)\right) - \Phi_n'\left(\Phi_n^{[-1]}(u)\right)}{\Phi_n'\left(\Phi_n^{[-1]}(u)\right)\Phi_n'\left(\Phi_n^{[-1]}(v)\right)}\right| \\ \le\ & C\left[\frac{\left|\Phi_n^{[-1]}(v) - \Phi_n^{[-1]}(u)\right|}{\left(\sin\Phi_n^{[-1]}(v)\right)^2}\right]^{1/4} \frac{1}{\left(\sin\Phi_n^{[-1]}(u)\right)^2\left(\sin\Phi_n^{[-1]}(v)\right)^2}, \end{aligned}$$

by (24) and (25). We continue this using (26) as

$$\le |v-u|^{1/12}\min\left\{\sin\Phi_n^{[-1]}(u), \sin\Phi_n^{[-1]}(v)\right\}^{-5}.$$

Now assume that $u, v \in [n^{-\rho}, 2\pi - n^{-\rho}]$. Then as $\Phi_n^{[-1]}(0) = 0$ and $\Phi_n^{[-1]}(2\pi) = \pi$, we see from (26) that

$$\Phi_n^{[-1]}(u), \Phi_n^{[-1]}(v) \in \left[C_3 n^{-\rho}, \pi - C_4 n^{-\rho}\right]$$

and so

$$\min\left\{\sin\Phi_n^{[-1]}\left(u\right),\sin\Phi_n^{[-1]}\left(v\right)\right\}\geq C_5 n^{-\rho}.$$

Then (27) follows. ■

Now we turn to estimation of $J_{21,2}$:

Lemma 4.3
For some $\kappa>0$,

$$J_{21,2}=\frac{1}{\pi}\int_0^{\pi}Q\left(L_n^{[-1]}\left(\cos\theta\right)\right)\sin\theta\cos n\Phi_n\left(\theta\right)d\theta=O\left(n^{1-\kappa}\right).$$

Proof
As in the proof of Theorem 2, Jackson type theorems yield the estimate

$$\left|J_{21,2}\right|\leq C\sup_{|u|\leq 1/n}\int_0^{2\pi}\left|g_n\left(\phi+u\right)-g_n\left(\phi\right)\right|d\phi,$$

where

$$g_n\left(\phi\right)=Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}\left(\phi\right)\right)\right)\sin\Phi_n^{[-1]}\left(\phi\right)/\Phi_n'\left(\Phi_n^{[-1]}\left(\phi\right)\right).$$

(We take the difference $g_n\left(\phi+u\right)-g_n\left(\phi\right)$ as 0 if $\phi+u$ lies outside $[0,2\pi]$). Let $\rho=\frac{1}{100}$. From (10), for $|u|\leq 1/n$,

$$\begin{aligned}
&\int_{2\pi-n^{-\rho}}^{2\pi}\left|g_n\left(\phi+u\right)-g_n\left(\phi\right)\right|d\phi\\
\leq\ & Cn\int_{2\pi-2n^{-\rho}}^{2\pi}d\phi/\Phi_n'\left(\Phi_n^{[-1]}\left(\phi\right)\right)\\
=\ & Cn\left[\Phi_n^{[-1]}\left(2\pi\right)-\Phi_n^{[-1]}\left(2\pi-2n^{-\rho}\right)\right]\\
\leq\ & Cn^{1-\rho/3},
\end{aligned}\tag{28}$$

by (26). A similar estimate holds for the integral over $[0,n^{-\rho}]$. Now consider $\phi\in[n^{-\rho},2\pi-n^{-\rho}]$ and $|u|\leq 1/n$. For some ξ between $\phi+u$ and ϕ,

$$\begin{aligned}
&\left|Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}\left(\phi+u\right)\right)\right)-Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}\left(\phi\right)\right)\right)\right|\\
=\ &\left|Q'\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}\left(\xi\right)\right)\right)\delta_n\sin\Phi_n^{[-1]}\left(\xi\right)/\Phi_n'\left(\Phi_n^{[-1]}\left(\xi\right)\right)\right|\left|u\right|\\
\leq\ & C/\left(\sin\Phi_n^{[-1]}\left(\xi\right)\right)^2,
\end{aligned}$$

by (24) and Lemma 3.1(iii). Here the lower bound in (26) and the fact that $\Phi_n\left(0\right)=0,\Phi_n\left(\pi\right)=2\pi$ show that

$$\Phi_n^{[-1]}\left(\xi\right)\in\left[C_1n^{-\rho},\pi-C_2n^{-\rho}\right]$$

and hence,

$$\left|Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}(\phi+u)\right)\right)-Q\left(L_n^{[-1]}\left(\cos\Phi_n^{[-1]}(\phi)\right)\right)\right|\leq Cn^{2\rho}.$$

Also, from (27),

$$\left|\frac{1}{\Phi_n'\left(\Phi_n^{[-1]}(\phi+u)\right)}-\frac{1}{\Phi_n'\left(\Phi_n^{[-1]}(\phi)\right)}\right|\leq Cn^{-1/12+5\rho},$$

and from the upper bound in (26),

$$\left|\sin\Phi_n^{[-1]}(\phi+u)-\sin\Phi_n^{[-1]}(\phi)\right|\leq Cn^{-1/3}.$$

Combining these estimates in the obvious way and using our bound (10) for Q gives

$$\begin{aligned}&|g_n(\phi+u)-g_n(\phi)|\\ \leq\ & Cn^{2\rho}/\Phi_n'\left(\Phi_n^{[-1]}(\phi+u)\right)+Cn\cdot n^{-1/3}/\Phi_n'\left(\Phi_n^{[-1]}(\phi+u)\right)+Cn\cdot n^{-1/12+5\rho}.\end{aligned}$$

Integrating and making the obvious substitution gives

$$\int_{n^{-\rho}}^{2\pi-n^{-\rho}}|g_n(\phi+u)-g_n(\phi)|\,d\phi\leq Cn^{\max\{2\rho,11/12+5\rho\}}=Cn^{11/12+5\rho},$$

recall that $\rho=1/100$. Together with (28), this shows that for some $\kappa>0$,

$$|J_{21,2}|\leq Cn^{1-\kappa}.$$

■

Finally, we note that a similar estimate holds for $J_{21,3}$.

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