L_p CHRISTOFFEL FUNCTIONS, L_p UNIVERSALITY, AND PALEY-WIENER SPACES

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Abstract. Let ω be a regular measure on the unit circle and let $p > 0$. We establish asymptotic behavior, as $n \to \infty$, for the L_p Christoffel function

$$\lambda_{n,p}(\omega, z) = \inf_{\deg(P) \leq n-1} \frac{\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^p d\omega(\theta)}{|P(z)|^p}$$

at Lebesgue points z on the unit circle, where ω' is lower semi-continuous. While bounds for these are classical, asymptotics have never been established for $p \neq 2$. The limit involves an extremal problem in Paley-Wiener space. As a consequence, we deduce universality type limits for the extremal polynomials, which reduce to random-matrix limits involving the sinc kernel in the case $p = 2$. We also present analogous results for L_p Christoffel functions on $[-1, 1]$.

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1. Introduction[1]

Let ω denote a finite positive Borel measure on the unit circle (or equivalently on $[-\pi, \pi]$). We define its L_p Christoffel function

$$\lambda_{n,p}(\omega, z) = \inf_{\deg(P) \leq n-1} \frac{\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^p d\omega(\theta)}{|P(z)|^p}. \tag{1.1}$$

By a compactness argument, there is a polynomial $P^*_{n,p,z}$ of degree $\leq n-1$ with $P^*_{n,p,z}(z) = 1$ and

$$\lambda_{n,p}(\omega, z) = \int_{-\pi}^{\pi} \left|P^*_{n,p,z}\left(e^{i\theta}\right)\right|^p d\omega(\theta). \tag{1.2}$$

When $p \geq 1$, this polynomial is unique. For $p > 1$, this follows from strict convexity of the L_p norm; for $p = 1$, see, for example, [13].

The classical Szegő theory provides asymptotics for $\lambda_{n,p}(\omega, z)$ when $|z| < 1$. For example, if ω is absolutely continuous, then [26, p. 153] for $|z| < 1$,

$$\lim_{n\to\infty} \lambda_{n,p}(\omega, z) = \inf\left\{\int |f|^p d\omega : f \in H^\infty \text{ and } f(z) = 1\right\}.$$

Here H^∞ is the usual Hardy space for the unit disc. Moreover, for general measures, there is an alternative expression involving the Poisson kernel for the unit disc [26, p. 154].

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In this paper, we shall establish asymptotics when z is on the unit circle, for all $p>0$. While estimates such as

$$C_1 \leq n\lambda_{n,p}\left(\omega, z\right) \leq C_2,\ \left|z\right| = 1,$$

are easy to prove under mild conditions on ω, the asymptotics are somewhat deeper. They are new for $p \neq 2$ even for Lebesgue measure on the unit circle.

Of course for $p=2$, $\lambda_{n,p}$ plays a crucial role in analysing orthogonal polynomials and in their applications. In a breakthrough 1991 paper, Maté, Nevai and Totik [20] proved that when ω is *regular*, and satisfies in some subinterval I of $[-\pi,\pi]$,

$$\int_I \log \omega'\left(e^{i\theta}\right) d\theta > -\infty,$$

then for a.e. $\theta \in I$,

$$\lim_{n\to\infty} n\lambda_{n,2}\left(\omega, e^{i\theta}\right) = \omega'\left(\theta\right).$$

Here ω is *regular* if

$$\lim_{n\to\infty}\left(\inf_{\deg(P)\leq n}\frac{\int_{-\pi}^{\pi}\left|P\right|^2 d\omega}{\left\|P\right\|^2_{L_\infty(|z|=1)}}\right)^{1/n} = 1.$$

A sufficient condition for regularity, the so-called Erdős-Turán condition, is that $\omega' > 0$ a.e. in $[-\pi,\pi]$. However, there are pure jump measures, and pure singularly continuous measures that are regular [29].

The asymptotic involves an extremal problem for the Paley-Wiener space L^p_π. This is the set of all entire functions f satisfying

$$\int_{-\infty}^{\infty}\left|f\left(t\right)\right|^p dt < \infty,$$

and for some $C>0$,

$$\left|f\left(z\right)\right| \leq Ce^{\pi|z|},\ z \in \mathbb{C}.$$

We define

$$\mathcal{E}_p = \inf\left\{\int_{-\infty}^{\infty}\left|f\left(t\right)\right|^p dt : f \in L^p_\pi \text{ and } f\left(0\right) = 1\right\}. \tag{1.3}$$

Equivalently,

$$\mathcal{E}_p^{-1/p} = \sup\left\{\left|f\left(0\right)\right| : f \in L^p_\pi \text{ and } \left(\int_{-\infty}^{\infty}\left|f\left(t\right)\right|^p dt\right)^{1/p} \leq 1\right\},$$

the norm of the evaluation functional $f \to f(0)$. Moreover, we let $f_p^* \in L^p_\pi$ be a function attaining the infimum in (1.3), so that $f_p^*(0) = 1$ and

$$\mathcal{E}_p = \int_{-\infty}^{\infty}\left|f_p^*\left(t\right)\right|^p dt.$$

When $p \geq 1$, f_p^* is unique. For $p>1$, this follows from Clarkson's inequalities, see Lemma 3.2 below. For $p=1$, we provide a proof in Section 6. For $p<1$, uniqueness is apparently unresolved.

For $p>1$, we may give an alternate formulation involving the sinc function

$$S\left(t\right) = \frac{\sin \pi t}{\pi t} \tag{1.4}$$

of signal processing and random matrices fame:

$$\mathcal{E}_p = \inf \int_{-\infty}^{\infty} \left| S(t) - \sum_{j=-\infty, j\neq 0}^{\infty} c_j S(t-j) \right|^p dt, \tag{1.5}$$

where the inf is taken over all $\{c_j\} \in \ell_p$, that is over all $\{c_j\}$ satisfying

$$\sum_{j=-\infty, j\neq 0}^{\infty} |c_j|^p < \infty. \tag{1.6}$$

In fact, (cf. [14]) for every $p > 0$, any $f \in L_\pi^p$ has an expansion of the form

$$f(z) = \sum_{j=-\infty}^{\infty} f(j) S(z-j),$$

that converges uniformly in compact subsets of $\mathbb{C}$.

When $p = 2$, the orthonormality of the integer translates $\{S(t-j)\}$ shows that $f_2^* = S$, and

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} S(t)^2 dt = 1.$$

The precise value of $\mathcal{E}_p$ is apparently not known for $p \neq 2$. The estimate

$$\mathcal{E}_p > p^{-1}$$

goes back to 1949, to Korevaar's thesis [2, p.102], [12]. We are grateful to D. Khavinson for this reference. There are some later works in Russian that might be relevant [9], [10], [11] but are inaccessible. The paper [21] contains indirect references to this problem. It seems that while there are estimates, there is not an explicit formula.

We decompose a measure ω on the unit circle as

$$d\omega(z) = \omega'_{ac}(z) d\theta + d\omega_s(z), \; z = e^{i\theta},$$

as a sum of its absolutely continuous and singular parts. Recall that $z_0 = e^{i\theta_0}$ is a Lebesgue point of ω if

$$\lim_{h\to 0+} \frac{1}{2h} \int_{|\theta-\theta_0|\leq h} |\omega'_{ac}(z) - \omega'_{ac}(z_0)| d\theta = 0$$

and

$$\lim_{h\to 0+} \frac{1}{2h} \int_{|\theta-\theta_0|\leq h} d\omega_s(z) = 0.$$

At such a point, we write $\omega'(z_0) = \omega'_{ac}(z_0)$. Recall, too, that ω_{ac} is lower semi-continuous at z_0 if

$$\liminf_{z\to z_0} \omega'_{ac}(z) \geq \omega'_{ac}(z_0).$$

We prove:

Theorem 1.1
Let $p > 0$, and let ω be a finite positive measure supported on the unit circle, and assume that ω is regular. Let $|z_0| = 1$, and assume that z_0 is a Lebesgue point of ω, while ω'_{ac} is lower semi-continuous at z_0.
(a) Then

$$\lim_{n\to\infty} n\lambda_{n,p}(\omega, z_0) = 2\pi \mathcal{E}_p \omega'(z_0). \tag{1.7}$$

(b) If also $\omega'(z_0) > 0$ and $p > 1$, we have

$$\lim_{n\to\infty} P^*_{n,p,z_0}\left(z_0 e^{2\pi i z/n}\right) = e^{i\pi z} f^*_p(z), \tag{1.8}$$

uniformly for z in compact subsets of the plane.

Of course if ω is absolutely continuous in a neighborhood of z_0, and ω' is continuous at z_0, then the local conditions above are satisfied. Even in the case $p = 2$, one needs more than just z_0 being a Lebesue point to prove asymptotics of Christoffel functions. Typical assumptions are continuity of ω' at z_0, or a local Szegő condition [20], [28].

Note that if $p = 2$, (1.8) reduces to a special case of the universality limit of random matrices:

$$\lim_{n\to\infty} \frac{K_n\left(z_0, z_0 e^{2\pi i z/n}\right)}{K_n(z_0, z_0)} = e^{i\pi z} S(z),$$

where K_n is the standard reproducing kernel of orthogonal polynomials, cf. [16].

One consequence of Theorem 1.1. is an asymptotically sharp L_∞, L_p Nikolskii inequality: as $n \to \infty$,

$$\sup_{\deg(P)\le n} \frac{\sup_{\theta\in[-\pi,\pi]} \left|P\left(e^{i\theta}\right)\right|}{\left(n\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p}} \to \left(2\pi\mathcal{E}_p\right)^{-1/p}.$$

We can prove an asymptotic upper bound, along the lines of Maté-Nevai-Totik, without assuming regularity, at each Lebesgue point z_0 of ω:

Theorem 1.2

Let $p > 0$, and let ω be a finite positive measure supported on the unit circle, with infinitely many points in its support. Let z_0 be a point on the unit circle that is a Lebesgue point of ω. Then

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega, z_0) \le 2\pi\mathcal{E}_p \omega'(z_0). \tag{1.9}$$

Our original interest in the L_p Christoffel functions arose from the interval, rather than the unit circle, because of possible applications to $\beta-$ensembles in random matrix theory (cf. [18]). Let μ be a finite positive measure with support $[-1, 1]$. It was probably Paul Nevai, who first systematically studied for measures on $[-1, 1]$, the general L_p Christoffel function

$$\lambda_{n,p}(\mu, x) = \inf_{\deg(P)\le n-1} \frac{\int_{-1}^{1} |P(t)|^p d\mu(t)}{|P(x)|^p}, \tag{1.10}$$

in his 1979 memoir [22]. They were useful in establishing Bernstein and Nikolskii inequalities, in estimating quadrature sums, and in studying convergence of Lagrange interpolation and orthogonal expansions [15], [22], [23].

Nevai and his collaborators established upper and lower bounds on $\lambda_{n,p}(\mu, x)$. For example, if in some open interval $I \subset (-1, 1)$, μ is absolutely continuous, and μ' is bounded above and below by positive constants, then for x in compact subsets of I, and for some $C_1, C_2 > 0$

$$C_1 \le n\lambda_{n,p}(\mu, x) \le C_2, n \ge 1.$$

However, to the best of our knowledge, asymptotics of $\lambda_{n,p}(\mu,x)$ have never been established for $p\neq 2$. In the sequel, we let $P^*_{n,p,\xi}$ denote a polynomial of degree $\leq n-1$ with $P^*_{n,p,\xi}(\xi)=1$, that attains the inf in (1.10).

Let us say that μ is *regular on* $[-1,1]$, or just *regular*, if

$$\lim_{n\to\infty}\left(\inf_{\deg(P)\leq n}\frac{\int_{-1}^{1}P^2d\mu}{\|P\|^2_{L_\infty[-1,1]}}\right)^{1/n}=1.$$

As for the unit circle, a simple sufficient condition for regularity is that $\mu'>0$ a.e. in $[-1,1]$, although it is far from necessary. When μ is regular, and μ' is continuous at a given $x\in(-1,1)$, and absolutely continuous in a neighborhood of x, it is known that

$$\lim_{n\to\infty}n\lambda_{n,2}(\mu,x)=\pi\sqrt{1-x^2}\mu'(x)\,. \tag{1.11}$$

If μ is regular on $[-1,1]$ and satisfies on some subinterval I,

$$\int_I\log\mu'>-\infty,$$

then the 1991 result of Maté, Nevai, and Totik, asserts that (1.11) is true for a.e. $x\in I$. Totik subsequently extended this to measures μ with arbitrary compact support [30].

We prove:

Theorem 1.3
Let $p>0$, and let μ be a finite positive measure supported on $[-1,1]$, and assume that μ is regular. Let $\xi\in(-1,1)$ be a Lebesgue point of μ, and let μ'_{ac} be lower semi-continuous at ξ.
(a) Then

$$\lim_{n\to\infty}n\lambda_{n,p}(\mu,\xi)=\pi\sqrt{1-\xi^2}\mathcal{E}_p\mu'(\xi)\,.$$

(b) If also $\mu'(\xi)>0$ and $p>1$, we have

$$\lim_{n\to\infty}P^*_{n,p,\xi}\left(\xi+\frac{\pi\sqrt{1-\xi^2}z}{n}\right)=f^*_p(z)\,, \tag{1.12}$$

uniformly for z in compact subsets of the plane.

Of course, the definition of a Lebesgue point of μ is entirely analogous to that for ω.

Theorem 1.4
Let $p>0$, and let μ be a finite positive measure supported on $[-1,1]$, with infinitely many points in its support. Let $\xi\in(-1,1)$ be a Lebesgue point of μ. Then

$$\limsup_{n\to\infty}n\lambda_{n,p}(\mu,\xi)\leq\pi\mu'(\xi)\mathcal{E}_p. \tag{1.13}$$

The proofs of the results for the unit circle and the interval follow similar lines. In the former case, we first establish the results for Lebesgue measure on the unit circle, and in the latter case for the Chebyshev weight on $[-1,1]$. In both cases, we then use regularity and "needle polynomials" to extend to general measures. We note that even in the classical $p=2$ case, transferring asymptotics for Christoffel functions from the unit circle to the interval involves more than the substitution $x = \cos\theta$. Indeed, in their landmark 1991 paper, Maté, Nevai and Totik used an identity expressing Christoffel functions on the unit circle in terms of Christoffel functions for two different measures on $[-1, 1]$. We were unable to find an analogue of this identity for L_p Christoffel functions, nor a mechanism to transfer results from the unit circle to the interval. However, the referee was able to find such a mechanism. Since this apparently does not transfer the universality limits for the extremal polynomials, we have kept to our original proofs.

In the sequel, $C, C_1, C_2, \ldots$, denote positive constants independent of n, x, t, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. I^0 denotes the interior of an interval I. This paper is organised as follows: we consider Lebesgue measure on the unit circle in Section 2. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we consider the Chebyshev weight on $[-1, 1]$. We prove Theorems 1.3 and 1.4 in Section 5.

2. Lebesgue Measure on the Unit Circle

In this section, we let

$$d\omega(\theta) = d\theta$$

so that $d\omega$ is Lebesgue measure on the unit circle. We prove:

Theorem 2.1
Let $p > 0$. Then
(a)

$$\lim_{n\to\infty} n\lambda_{n,p}(\omega, 1) = 2\pi\mathcal{E}_p. \tag{2.1}$$

(b) If $p \geq 1$,

$$\lim_{n\to\infty} P^*_{n,p,1}\left(e^{2\pi iz/n}\right) = e^{\pi iz} f^*_p(z), \tag{2.2}$$

uniformly for z in compact subsets of the plane.

If $p < 1$, we cannot prove (2.2), because we do not know the uniqueness of f^*_p. However, our proof shows that every infinite sequence of positive integers, contains a subsequence, say $\mathcal{T}$, such that for some $f^*_p \in L^p_\pi$ satisfying $f^*_p(0) = 1$, and (1.3),

$$\lim_{n\to\infty, n\in\mathcal{T}} P^*_{n,p,1}\left(e^{2\pi iz/n}\right) = e^{\pi iz} f^*_p(z), \tag{2.3}$$

uniformly in compact subsets of the plane.

Our strategy in proving (2.1), will be first to prove an asymptotic upper bound, separately for $p > 1$ and for $0 < p \leq 1$. For the upper bound, we shall use Lagrange interpolation at the roots of unity. We use $[x]$ to denote the greatest integer $\leq x$. Let $n \geq 2$, and for $|j| \leq [n/2]$, we let

$$z_{jn} = e^{2\pi ij/n},$$

and define the corresponding fundamental polynomial

$$\ell_{jn}(z) = \frac{1}{n}\frac{z^n - 1}{z\overline{z_{jn}} - 1}. \tag{2.4}$$

We start with:

Lemma 2.2
Assume that $C > 1$ *and* $k = k(n)$ *is such that*

$$C^{-1} \leq \frac{k}{n} \leq C,\, n \geq 1.$$

Then as $n \to \infty$,

$$\ell_{jk}\left(e^{2\pi it/n}\right) = (-1)^j\, e^{i\pi tk/n} S\left(\frac{tk}{n} - j\right) + o(1)\,, \tag{2.5}$$

uniformly for j *and* t *with*

$$\frac{|j|}{n} = o(1)\,;\ \frac{t}{n} = o(1)\,. \tag{2.6}$$

Proof
We see that

$$\begin{aligned}
\ell_{jk}\left(e^{2\pi it/n}\right) &= \frac{1}{k}\frac{e^{i\pi tk/n}\sin\left(\frac{\pi tk}{n}\right)}{e^{i\pi\left(\frac{t}{n}-\frac{j}{k}\right)}\sin\left(\pi\frac{t}{n}-\pi\frac{j}{k}\right)} \\
&= \frac{1}{k}\frac{e^{i\pi tk/n}(-1)^j\sin\left(\pi\left(\frac{tk}{n}-j\right)\right)}{e^{i\pi\left(\frac{t}{n}-\frac{j}{k}\right)}\sin\left(\frac{\pi}{k}\left(\frac{tk}{n}-j\right)\right)} \\
&= \frac{e^{i\pi tk/n}(-1)^j\, S\left(\frac{tk}{n}-j\right)}{e^{i\pi\left(\frac{t}{n}-\frac{j}{k}\right)}S\left(\frac{1}{k}\left(\frac{tk}{n}-j\right)\right)}.
\end{aligned}$$

Here $e^{i\pi\left(\frac{t}{n}-\frac{j}{k}\right)} = 1 + o(1)$ uniformly for j and t satisfying (2.6). Moreover, by continuity of S at 0, $S\left(\frac{1}{k}\left(\frac{tk}{n}-j\right)\right) = S\left(\frac{t}{n}-\frac{j}{k}\right) = 1 + o(1)$ uniformly for the same range of j and t. ■

Now for each $f \in L^p_\pi$, and any $p > 0$, a result of Plancherel and Polya [4, p. 506], [24] asserts that

$$\sum_{n=-\infty}^{\infty} |f(n)|^p \leq C\int_{-\infty}^{\infty} |f(t)|^p\, dt, \tag{2.7}$$

where C is independent of f. The converse inequality, with appropriate C, holds only for $p > 1$. Thus for $p > 1$, and some C_1, C_2 independent of f, [14, p. 152]

$$C_1\sum_{n=-\infty}^{\infty} |f(n)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p\, dt \leq C_2\sum_{n=-\infty}^{\infty} |f(n)|^p\,. \tag{2.8}$$

As a consequence, any such function f admits an expansion

$$f(z) = \sum_{j=-\infty}^{\infty} f(j)\, S(z-j)\,, \tag{2.9}$$

that converges locally uniformly in the plane. Indeed, for $p > 1$, this follows from the Plancherel-Polya theorem [14, p. 152] that we have just quoted. For $p \leq 1$, (2.7), (2.8) also imply that $f \in L^2_\pi$, so yet again (2.9) holds.

Lemma 2.3
Let $p>1$. Then

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \leq 2\pi\mathcal{E}_p.$$

Proof
Let $f\in L^p_\pi$, with $f(0)=1$. Fix $m\geq 1$ and let

$$S_n(z)=\sum_{|j|\leq m} f(j)(-1)^j \ell_{jn}(z). \tag{2.10}$$

Since $S_n(1)=f(0)=1$, we have

$$\lambda_{n,p}(\omega,1)\leq \int_{-\pi}^{\pi} |S_n(z)|^p\, d\theta.$$

Here, and in the sequel, $z=e^{i\theta}$ in the integral. Now for each $r>0$, Lemma 2.2 gives

$$\lim_{n\to\infty} n\int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^p\, d\theta = 2\pi \lim_{n\to\infty}\int_{-r}^{r}\left|S_n\left(e^{2\pi it/n}\right)\right|^p dt = 2\pi\int_{-r}^{r}\left|\sum_{|j|\leq m} f(j)\, S(t-j)\right|^p dt.$$

(2.11)

Next, we estimate the rest of the integral. Let $z=e^{i\theta},\theta\in[0,\pi]$. If $0\leq j\leq [n/2]$,
(2.12)

$$|\ell_{jn}(z)|\leq \min\left\{1,\frac{2}{n|z-z_{jn}|}\right\}\leq \min\left\{1,\frac{1}{n\left|\sin\left(\frac{\theta-2j\pi/n}{2}\right)\right|}\right\}\leq \min\left\{1,\frac{\pi}{|n\theta-2j\pi|}\right\},$$

by the inequality $|\sin t|\geq \frac{2}{\pi}|t|,\ |t|\leq\frac{\pi}{2}$. For $0>j\geq -[n/2]$, we have instead

$$|\ell_{jn}(z)|\leq |\ell_{-jn}(z)|\leq \min\left\{1,\frac{\pi}{|n\theta-2|j|\pi|}\right\}.$$

Hence if $r\geq 2m$, and $\pi\geq\theta\geq 2\pi r/n$

$$|S_n(z)|\leq\left(\sum_{|j|\leq m}|f(j)|\right)\frac{2\pi}{n|\theta|}.$$

The same estimate holds for $-\pi\leq\theta\leq -2\pi r/n$. Then

$$\begin{aligned} & n\int_{2\pi r/n\leq|\theta|\leq\pi} |S_n(z)|^p\, d\theta \\ \leq\ & \left(2\pi\sum_{|j|\leq m}|f(j)|\right)^p n\int_{2\pi r/n\leq|\theta|\leq\pi}\frac{d\theta}{|n\theta|^p} \\ \leq\ & C\left(2\pi\sum_{|j|\leq m}|f(j)|\right)^p r^{1-p}, \end{aligned} \tag{2.13}$$

where C is independent of n and r. Combined with (2.11), this gives

$$\begin{aligned}\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) &\leq \limsup_{n\to\infty} n\int_{-\pi}^{\pi}|S_n(z)|^p\,d\theta \\ &\leq 2\pi\int_{-r}^{r}\left|\sum_{|j|\leq m} f(j)\,S(t-j)\right|^p dt + C\left(2\pi\sum_{|j|\leq m}|f(j)|\right)^p r^{1-p}.\end{aligned}$$

Recall that m is fixed. Letting $r\to\infty$ gives

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \leq 2\pi\int_{-\infty}^{\infty}\left|\sum_{|j|\leq m} f(j)\,S(t-j)\right|^p dt.$$

Now the triangle inequality and the Polya-Plancherel equivalence (2.8), (2.9) give

$$\begin{aligned}&\left(\int_{-\infty}^{\infty}\left|\sum_{|j|\leq m} f(j)\,S(t-j)\right|^p dt\right)^{1/p} \\ \leq\ & \left(\int_{-\infty}^{\infty}|f(t)|^p\,dt\right)^{1/p} + \left(\int_{-\infty}^{\infty}\left|\sum_{|j|>m} f(j)\,S(t-j)\right|^p dt\right)^{1/p} \\ \leq\ & \left(\int_{-\infty}^{\infty}|f(t)|^p\,dt\right)^{1/p} + C_1\left(\sum_{|j|>m}|f(j)|^p\right)^{1/p}.\end{aligned}$$

Here C_1 is independent of f and m. Thus for any $m\geq 1$,

$$\limsup_{n\to\infty}\left(n\lambda_{n,p}(\omega,1)\right)^{1/p} \leq \left(2\pi\int_{-\infty}^{\infty}|f(t)|^p\,dt\right)^{1/p} + C_2\left(\sum_{|j|>m}|f(j)|^p\right)^{1/p}.$$

Here C_2 is independent of m. Letting $m\to\infty$, gives

$$\limsup_{n\to\infty}\left(n\lambda_{n,p}(\omega,1)\right)^{1/p} \leq \left(2\pi\int_{-\infty}^{\infty}|f(t)|^p\,dt\right)^{1/p}.$$

As we may choose any $f\in L_\pi^p$, with $f(0)=1$, we obtain the result. ■

Next, we handle the more difficult case $p\leq 1$. We let

$$U_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} z^j = \frac{1}{k}\frac{1-z^k}{1-z}.$$

Lemma 2.4
Let $0<p\leq 1$. *Then*

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \leq 2\pi\mathcal{E}_p. \tag{2.14}$$

Proof
Let $f\in L_\pi^p$, with $f(0)=1$. Let $\varepsilon\in\left(0,\frac{1}{2}\right)$. Choose a positive integer k such that

$kp \geq 2$ and let

$$S_n(z) = \left(\sum_{|j| \leq [\log n]} f(j)(-1)^j \ell_{j,n-[\varepsilon n]}(z) \right) U_{\left[\frac{\varepsilon}{k} n\right]}(z)^k,$$

a polynomial of degree $\leq n-1$, with $S_n(1) = 1$. Fix $r > 0$. As $\left| U_{\left[\frac{\varepsilon}{k} n\right]}(z) \right| \leq 1$ for $|z| \leq 1$, we have from Lemma 2.2,

$$\begin{aligned}
& n \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^p \, d\theta \\
\leq \quad & 2\pi \int_{-r}^{r} \left| \sum_{|j| \leq [\log n]} f(j)(-1)^j \ell_{j,n-[\varepsilon n]}\left(e^{2\pi i t/n}\right) \right|^p dt \\
\leq \quad & 2\pi \int_{-r}^{r} \left| \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) + o\left(\sum_{|j| \leq [\log n]} |f(j)| \right) \right|^p dt \\
\leq \quad & 2\pi \int_{-r}^{r} \left| \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) \right|^p dt + o(1).
\end{aligned}$$

Here we are using the fact that

$$D = \sum_{j=-\infty}^{\infty} |f(j)| \leq \|f\|_{L_\infty(\mathbb{R})}^{1-p} \sum_{j=-\infty}^{\infty} |f(j)|^p < \infty,$$

recall (2.7). Next, uniformly for $t \in [-r, r]$,

$$\begin{aligned}
& \left| f(t(1-\varepsilon)) - \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) \right| \\
= \quad & \left| \sum_{|j| > [\log n]} f(j) S(t(1-\varepsilon) - j) \right| \\
\leq \quad & \sum_{|j| > [\log n]} |f(j)| \to 0, \text{ as } n \to \infty.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \limsup_{n \to \infty} n \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^p \, d\theta \\
\leq \quad & 2\pi \int_{-r}^{r} |f(t(1-\varepsilon))|^p \, dt \leq \frac{2\pi}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p \, dt. \qquad (2.15)
\end{aligned}$$

Next, for all $|z| \leq 1$, $|\ell_{jn}(z)| \leq 1$, so with $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$,

$$\begin{aligned}
|S_n(z)| \quad & \leq \quad D \left| U_{\left[\frac{\varepsilon}{k} n\right]}(z) \right|^k \\
& \leq \quad D \left(\frac{2}{\left[\frac{\varepsilon}{k} n\right] |1-z|} \right)^k \leq D \left(\frac{\pi}{\left[\frac{\varepsilon}{k} n\right] |\theta|} \right)^k.
\end{aligned}$$

Hence,

$$\begin{aligned} & n\int_{2\pi r/n\leq|\theta|\leq\pi}|S_n(z)|^p\,d\theta \\ \leq\ & Cn\int_{2\pi r/n\leq|\theta|\leq\pi}\left(\frac{1}{\left[\frac{\varepsilon}{k}n\right]|\theta|}\right)^{kp}d\theta \\ \leq\ & C\int_{|t|\geq 2\pi r}|t|^{-kp}\,dt\leq Cr^{-kp+1}. \end{aligned}$$

Here C is independent of r, n, but depends on ε, k. Combining this with (2.15) gives

$$\begin{aligned} & \limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \\ \leq\ & \limsup_{n\to\infty} n\int_{-\pi}^{\pi}|S_n(z)|^p\,d\theta \\ \leq\ & \frac{2\pi}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p\,dt+Cr^{-kp+1}. \end{aligned}$$

Since the Christoffel function is independent of r, we can let $r\to\infty$ to obtain

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1)\leq\frac{2\pi}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p\,dt.$$

As $\varepsilon>0$ is arbitrary, we obtain

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1)\leq 2\pi\int_{-\infty}^{\infty}|f(t)|^p\,dt,$$

and taking the inf's over all f gives the result. ■

We next turn to the asymptotic lower bound. Recall we defined $P^*_{n,p,1}$ to be a polynomial of degree $\leq n-1$ with $P^*_{n,p,1}(1)=1$ and

$$\int_{-\pi}^{\pi}\left|P^*_{n,p,1}\left(e^{i\theta}\right)\right|^p d\theta=\lambda_{n,p}(\omega,1).$$

We shall simply write

$$P^*_n=P^*_{n,p,1}.$$

The Lagrange interpolation formula asserts that

$$P^*_n(z)=\ell_{0n}(z)+\sum_{|j|\leq[n/2]}P^*_n(z_{jn})\,\ell_{jn}(z).$$

Here if n is even, we omit the term for $j=-[n/2]$, to avoid including the interpolation point -1 twice. We adopt this convention in the sequel, without further mention. We start with estimates for P^*_n :

Lemma 2.5
(a)

$$\sup_{n\geq 1}\sum_{|j|\leq[n/2]}|P^*_n(z_{jn})|^p\leq\Lambda<\infty. \tag{2.16}$$

(b)

$$\sup_{n\geq 1}\|P^*_n\|_{L_\infty(|z|=1)}<\infty. \tag{2.17}$$

Proof
(a) For all $p>0$, there is the Marcinkiewicz-Zygmund inequality

$$\frac{1}{n}\sum_{|j|\leq[n/2]}\left|P_n^*\left(z_{jn}\right)\right|^p\leq C\int_{-\pi}^{\pi}\left|P_n^*\left(z\right)\right|^p d\theta.$$

Here C is independent of n and p. For $p>1$, it is in Zygmund's book [32, Vol. 2, p. 28]; for all $p>0$, see, for example, [19, Thm. 2, p. 533]. Then

$$\frac{1}{n}\sum_{|j|\leq[n/2]}\left|P_n^*\left(z_{jn}\right)\right|^p\leq C\lambda_{n,p}\left(\omega,1\right)\leq\frac{C}{n},$$

by Lemma 2.3 for $p>1$ and Lemma 2.4 for $0<p\leq1$.
(b) Let $M_n=\|P_n^*\|_{L_\infty(|z|=1)}$, and choose z_n such that $\left|P_n^*\left(z_n\right)\right|=M_n$. By Bernstein's inequality,

$$\|P_n^{*\prime}\|_{L_\infty(|z|=1)}\leq nM_n,$$

so if $z=e^{i\theta},z_n=e^{i\theta_n}$,

$$\begin{aligned}&\left|P_n^*\left(z\right)-P_n^*\left(z_n\right)\right|\\=&\left|\int_{\theta_n}^{\theta}P_n^{*\prime}\left(e^{it}\right)ie^{it}dt\right|\leq\frac{1}{2}M_n,\end{aligned}$$

for $|\theta-\theta_n|\leq\frac{1}{2n}$. Hence, for such z, $\left|P_n^*\left(z\right)\right|\geq\frac{1}{2}M_n$. Then using our upper bound on the Christoffel function,

$$\begin{aligned}\frac{C}{n}&\geq\int_{-\pi}^{\pi}\left|P_n^*\left(e^{it}\right)\right|^p dt\\&\geq\left(\frac{M_n}{2}\right)^p\int_{|t-\theta_n|\leq\frac{1}{2n}}dt=\left(\frac{M_n}{2}\right)^p\frac{1}{n},\end{aligned}$$

so

$$M_n\leq2C^{1/p}.$$

■

We shall again separately consider the cases $p>1$ and $p\leq1$.

Lemma 2.6
Let $p>1,r>0$. There exists $M_0=M_0\left(r\right)$ such that for $m\geq M_0$,

$$\int_{|\theta|\leq\frac{2\pi r}{n}}\left|\sum_{|j|>m}P_n^*\left(z_j\right)\ell_{jn}\left(z\right)\right|^p d\theta\leq C\frac{r}{mn}.\tag{2.18}$$

Here M_0 is independent of n and m, and C is independent of r,n and m.
Proof
Now for $|\theta|\leq\frac{r}{n}$, $z=e^{i\theta}$, and $m+1\leq j\leq[n/2]$,

$$\left|\ell_{jn}\left(z\right)\right|\leq\frac{2}{n\left|z-z_{jn}\right|}=\frac{1}{n\left|\sin\left(\frac{\theta-2j\pi/n}{2}\right)\right|}\leq\frac{C}{n\left|r/n-2j\pi/n\right|}\leq\frac{C}{j},\tag{2.19}$$

provided $m \geq M_0(r)$. Hence by Hölder's inequality with $q = p/(p-1)$,

$$\begin{aligned}
& \left| \sum_{|j|>m} P_n^*(z_{jn}) \ell_{jn}(z) \right| \\
\leq & \left(\sum_{|j|>m} |P_n^*(z_{jn})|^p \right)^{1/p} \left(\sum_{|j|>m} \left(\frac{C}{j} \right)^q \right)^{1/q} \\
\leq & C \left(m^{-q+1} \right)^{1/q} = C m^{-1/p},
\end{aligned}$$

by Lemma 2.5. Then

$$\int_{|\theta| \leq \frac{2\pi r}{n}} \left| \sum_{|j|>m} P_n^*(z_{jn}) \ell_{jn}(z) \right|^p d\theta \leq C \frac{r}{mn}.$$

■

Lemma 2.7
Let $p > 1$. Then

$$\liminf_{n \to \infty} n \lambda_{n,p}(\omega, 1) \geq 2\pi \mathcal{E}_p.$$

Proof
Choose a sequence $\mathcal{S}$ of positive integers such that

$$\lim_{n \to \infty, n \in \mathcal{S}} n \lambda_{n,p}(\omega, 1) = \liminf_{n \to \infty} n \lambda_{n,p}(\omega, 1).$$

For each $m \geq 1$, we choose a subsequence $\mathcal{T}_m$ of $\mathcal{S}$ such that for $|j| \leq m$,

$$\lim_{n \to \infty, n \in \mathcal{T}_m} P_n^*(z_{jn}) = d_j, \tag{2.20}$$

with $d_0 = 1$. This is possible in view of Lemma 2.5(a). We can assume that

$$\mathcal{T}_1 \supset \mathcal{T}_2 \supset \mathcal{T}_3 \supset \ldots$$

so that the $\{d_j\}$ are independent of m. By Lemma 2.5(a), for $m \geq 1$,

$$\sum_{|j| \leq m} |d_j|^p \leq \Lambda. \tag{2.21}$$

By Lemma 2.2, for any $r > 0$, and given m,

$$\begin{aligned}
& \lim_{n \to \infty, n \in \mathcal{T}_m} n \int_{|\theta| \leq \frac{2\pi r}{n}} \left| \sum_{|j| \leq m} P_n^*(z_{jn}) \ell_{jn}(z) \right|^p d\theta \\
= & \; 2\pi \int_{-r}^{r} \left| \sum_{|j| \leq m} (-1)^j d_j S(t-j) \right|^p dt.
\end{aligned}$$

Together with Lemma 2.6 and the triangle inequality, this shows that, for $m \geq M_0(r)$,

$$\begin{aligned} & \liminf_{n\to\infty} \left(n\lambda_{n,p}(\omega,1)\right)^{1/p} \\ = & \lim_{n\to\infty, n\in\mathcal{T}_m} \left(n\int_{-\pi}^{\pi} \left|P_n^*\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p} \\ \geq & \lim_{n\to\infty, n\in\mathcal{T}_m} \left(n\int_{-2\pi/r}^{2\pi/r} \left|P_n^*\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p} \\ \geq & \left(2\pi\int_{-r}^{r} \left|\sum_{|j|\leq m} (-1)^j d_j S(t-j)\right|^p dt\right)^{1/p} - \left(C\frac{r}{m}\right)^{1/p}. \end{aligned} \tag{2.22}$$

Moreover, as Λ in (2.21) is independent of m, we can use the Plancherel-Polya inequality (2.8) to show that, for each $r > 0$,

$$\begin{aligned} & \lim_{m\to\infty} \int_{-r}^{r} \left|\sum_{|j|\leq m} (-1)^j d_j S(t-j)\right|^p dt \\ = & \int_{-r}^{r} \left|\sum_{j=-\infty}^{\infty} (-1)^j d_j S(t-j)\right|^p dt. \end{aligned}$$

As $\liminf_{n\to\infty} \left(n\lambda_{n,p}(\omega,1)\right)^{1/p}$ is independent of m, we can let $m \to \infty$ to deduce that

$$\begin{aligned} & \liminf_{n\to\infty} \left(n\lambda_{n,p}(\omega,1)\right)^{1/p} \\ \geq & \left(2\pi\int_{-r}^{r} \left|\sum_{j=-\infty}^{\infty} (-1)^j d_j S(t-j)\right|^p dt\right)^{1/p}. \end{aligned}$$

As the $\{d_j\}$ are independent of r, we can let $r \to \infty$, and use the monotone convergence theorem to deduce

$$\begin{aligned} & \liminf_{n\to\infty} n\lambda_{n,p}(\omega,1) \\ \geq & 2\pi\int_{-\infty}^{\infty} \left|\sum_{j=-\infty}^{\infty} (-1)^j d_j S(t-j)\right|^p dt. \end{aligned}$$

Let

$$f(z) = \sum_{j=-\infty}^{\infty} (-1)^j d_j S(z-j).$$

Since $\{d_j\}$ satisfy (2.21) for all m, we deduce from the Plancherel-Polya inequality that $f \in L_\pi^p$ and $f(0) = 1$, so by definition of $\mathcal{E}_p$,

$$\liminf_{n\to\infty} n\lambda_{n,p}(\omega,1) \geq 2\pi\mathcal{E}_p.$$

■

We turn to the case $p \leq 1$ with an analogue of Lemma 2.6:

Lemma 2.8
Let $0 < p \leq 1, r > 0$*. There exists* $M_0 = M_0(r)$ *such that for* $m \geq M_0$,

$$\int_{|\theta| \leq \frac{2\pi r}{n}} \left| \sum_{|j|>m} P_n^*(z_j)\, \ell_{jn}(z) \right|^p d\theta \leq C \frac{r}{m^p n}. \tag{2.23}$$

Here C *is independent of* r, n *and* m.
Proof
Now for $|\theta| \leq \frac{r}{n}$, and $|j| > m$, we have the estimate (2.19) provided $m \geq M_0(r)$. Hence

$$\left| \sum_{|j|>m} P_n^*(z_j)\, \ell_{jn}(z) \right| \leq C \left(\sum_{|j|>m} |P_n^*(z_j)|^p \Lambda^{1-p} \right) m^{-1} \leq C m^{-1},$$

by Lemma 2.5(a). Then (2.23) follows. ■

Now we can prove:

Lemma 2.9
Let $0 < p \leq 1$*. Then*

$$\liminf_{n\to\infty} n\lambda_{n,p}(\omega, 1) \geq 2\pi \mathcal{E}_p.$$

Proof
This is similar to Lemma 2.7. Choose a sequence $\mathcal{S}$ of positive integers such that

$$\lim_{n\to\infty, n\in\mathcal{S}} n\lambda_{n,p}(\omega, 1) = \liminf_{n\to\infty} n\lambda_{n,p}(\omega, 1).$$

For each $m \geq 1$, we choose a subsequence $\mathcal{T}_m$ of $\mathcal{S}$ such that for $|j| \leq m$, (2.20) holds. This is possible in view of Lemma 2.5(a). We can assume that

$$\mathcal{T}_1 \supset \mathcal{T}_2 \supset \mathcal{T}_3 \supset \ldots$$

so that the $\{d_j\}$ are independent of m. As in Lemma 2.7, (2.21) holds. The inequality $(x+y)^p \leq x^p + y^p, x, y \geq 0$ shows that for $m \geq M_0(r)$,

$$\begin{aligned}
&\liminf_{n\to\infty} n\lambda_{n,p}(\omega, 1) \\
&= \lim_{n\to\infty, n\in\mathcal{T}_m} n \int_{-\pi}^{\pi} \left| P_n^*\left(e^{i\theta}\right) \right|^p d\theta \\
&\geq \lim_{n\to\infty, n\in\mathcal{T}_m} n \int_{-2\pi/r}^{2\pi/r} \left| P_n^*\left(e^{i\theta}\right) \right|^p d\theta \\
&\geq 2\pi \int_{-r}^{r} \left| \sum_{|j|\leq m} (-1)^j d_j S(t-j) \right|^p dt - C\frac{r}{m^p}, \qquad (2.24)
\end{aligned}$$

by Lemma 2.8. Next, as $|S(t)| \leq 1$ for all $t \in \mathbb{R}$, and (2.21) shows that $|d_j| \leq \Lambda^{1/p}$ for all j, we see that

$$\begin{aligned} & \sup_{t\in\mathbb{R}} \left| \sum_{|j|\geq m} (-1)^j d_j S(t-j) \right| \\ \leq & \sum_{|j|\geq m} |d_j| \leq \Lambda^{(1-p)/p} \sum_{|j|\geq m} |d_j|^p \to 0, \end{aligned}$$

as $m \to \infty$. Hence, for each fixed r,

$$\begin{aligned} & \lim_{m\to\infty} \int_{-r}^{r} \left| \sum_{|j|\leq m} (-1)^j d_j S(t-j) \right|^p dt \\ = & \int_{-r}^{r} \left| \sum_{j=-\infty}^{\infty} (-1)^j d_j S(t-j) \right|^p dt. \end{aligned}$$

So we may let $m \to \infty$ in (2.24) to deduce that

$$\begin{aligned} & \liminf_{n\to\infty} n\lambda_{n,p}(\omega,1) \\ \geq & \ 2\pi \int_{-r}^{r} \left| \sum_{j=-\infty}^{\infty} (-1)^j d_j S(t-j) \right|^p dt. \end{aligned} \tag{2.25}$$

We may now let $r \to \infty$ and complete the proof as in Lemma 2.7. ■

Proof of Theorem 2.1(a)
For $p > 1$, we combine Lemmas 2.3 and 2.7. For $0 < p \leq 1$, we combine Lemmas 2.4 and 2.9. ■

Proof of Theorem 2.1(b)
As above, we abbreviate $P^*_{n,p,1}$ as P^*_n. From Lemma 2.5, there exists C_0 such that for all $n \geq 1$,

$$\|P^*_n\|_{L_\infty(|z|=1)} \leq C_0.$$

By Bernstein's growth lemma, for $n \geq 1$, and all $z \in \mathbb{C}$,

$$|P^*_n(z)| \leq C_0 \max\{1, |z|^n\}.$$

Now let

$$f_n(z) = e^{-\pi i z} P^*_n\left(e^{2\pi i z/n}\right).$$

The bound above gives for $n \geq 1$, and all $z \in \mathbb{C}$,

$$|f_n(z)| \leq C_0 e^{\pi|\mathrm{Im}\, z|}.$$

In particular, $\{f_n\}$ is uniformly bounded in compact sets, and hence is a normal family. Let $\mathcal{S}$ be an infinite sequence of positive integers, and $\mathcal{T}$ be a subsequence for which

$$\lim_{n\to\infty, n\in\mathcal{T}} f_n(z) = f(z)$$

uniformly for z in compact sets. As each $f_n(0) = 1$, so $f(0) = 1$. Also

$$|f(z)| \leq C_0 e^{\pi|\mathrm{Im}\, z|},$$

so f is entire of exponential type at most π. Next, given $r>0$, we have

$$\begin{aligned}2\pi\mathcal{E}_p &= \lim_{n\to\infty} n\lambda_{n,p}(\omega,1)\\ &\geq \lim_{n\to\infty,n\in\mathcal{T}} n\int_{-2\pi r/n}^{2\pi r/n}\left|P_n^*\left(e^{i\theta}\right)\right|^p d\theta\\ &= 2\pi\lim_{n\to\infty,n\in\mathcal{T}}\int_{-r}^{r}\left|f_n(t)\right|^p dt\\ &= 2\pi\int_{-r}^{r}\left|f(t)\right|^p dt.\end{aligned}$$

As $r>0$ is arbitrary, we have $f\in L_\pi^p$, and

$$\mathcal{E}_p\geq\int_{-\infty}^{\infty}\left|f(t)\right|^p dt.$$

If $p\geq 1$, uniqueness of the extremal function gives $f=f_p^*$, independent of the subsequence S. Then (2.2) follows. If $p<1$, this argument instead shows that each subsequence of $\{f_n\}$ contains another converging uniformly in compact subsets to *some* extremal function. ■

3. Proof of Theorems 1.1 and 1.2

It suffices to prove both results in the case when $z_0=1$ is a Lebesgue point of ω.

Proof of Theorem 1.2

Let $\varepsilon\in(0,1)$. We can choose $\delta>0$ such that for $0<h\leq\delta$, both the following hold:

$$\begin{aligned}\int_{-h}^{h}\left|\omega'(1)-\omega_{ac}'(z)\right|d\theta &\leq \varepsilon h;\\ \int_{-h}^{h} d\omega_s(z) &\leq \varepsilon h.\end{aligned}\tag{3.1}$$

Recall that as 1 is a Lebesgue point, we use the notation $\omega'(1)=\omega_{ac}'(1)$. Let $\eta\in\left(0,\frac{1}{8}\right)$. We shall use a polynomial U_n of degree $\leq n-1$ of the form

$$U_n=P_{n-[2\eta n]}^*R_nS_n\tag{3.2}$$

to estimate $\lambda_{n,p}(\omega,1)$. Here, for the given p, P_n^* is an abbreviation for $P_{n,p,1}^*$, the extremal polynomial for Lebesgue measure on the unit circle (not our ω here), as in the previous section. In particular, we use Lemma 2.5(b), which shows that for all n,

$$\|P_n^*\|_{L_\infty(|z|=1)}\leq M_\infty<\infty.\tag{3.3}$$

We let

$$R_n(z)=\left(\frac{1+z}{2}\right)^{[\eta n]},\tag{3.4}$$

a polynomial of degree $\leq[\eta n]$, with $R_n(1)=1$, $|R_n(z)|\leq 1$ for $|z|\leq 1$, and

$$\left|R_n\left(e^{i\theta}\right)\right|\leq c^n,\ \delta\leq|\theta|\leq\pi,\tag{3.5}$$

where $c \in (0,1)$ depends on η, δ, but is independent of n. Finally, we let

$$S_n(z) = \ell_{0,[\eta n/k]}(z)^k,$$

where $\ell_{0,[\eta n/k]}$ is defined by (2.4), and where k is a fixed positive integer chosen so that $kp > 1$. Observe that $S_n(1) = 1$, $|S_n(z)| \leq 1$ for $|z| \leq 1$, and (2.12) shows that for $0 < |\theta| \leq \pi$,

$$\left|S_n\left(e^{i\theta}\right)\right| \leq \left(\min\left\{1, \frac{\pi}{[\eta n/k]\,|\theta|}\right\}\right)^k \leq \left(\min\left\{1, \frac{C}{\eta n\,|\theta|}\right\}\right)^k, \tag{3.6}$$

where C depends on k, but is independent of η, n and θ. We have

$$\begin{aligned}
& \lambda_{n,p}(\omega, 1) \\
\leq\ & \int_{-\pi}^{\pi} |U_n|^p\, d\omega \\
\leq\ & \omega'(1) \int_{-\delta}^{\delta} \left|P^*_{n-[2\eta n]}\left(e^{i\theta}\right)\right|^p d\theta \\
& + M_\infty^p \left\{ \int_{-\delta}^{\delta} \left|S_n\left(e^{i\theta}\right)\right|^p \left|\omega'(1) - \omega'_{ac}(\theta)\right| d\theta + \int_{-\delta}^{\delta} \left|S_n\left(e^{i\theta}\right)\right|^p d\omega_s(\theta) \right\} \\
& + M_\infty^p c^{np} \int_{\delta \leq |\theta| \leq \pi} d\theta, \qquad (3.7)
\end{aligned}$$

by (3.3), (3.5). Now by Theorem 2.1,

$$\int_{-\delta}^{\delta} \left|P^*_{n-[2\eta n]}\left(e^{i\theta}\right)\right|^p d\theta \leq \int_{-\pi}^{\pi} \left|P^*_{n-[2\eta n]}\left(e^{i\theta}\right)\right|^p d\theta \leq \frac{2\pi \mathcal{E}_p + o(1)}{n - [2\eta n]}. \tag{3.8}$$

Next, by (3.6),

$$\begin{aligned}
& \int_{-\delta}^{\delta} \left|S_n\left(e^{i\theta}\right)\right|^p \left|\omega'(1) - \omega'_{ac}(\theta)\right| d\theta + \int_{-\delta}^{\delta} \left|S_n\left(e^{i\theta}\right)\right|^p d\omega_s(\theta) \\
\leq\ & \sum_{j=0}^{\infty} \left(\min\left\{1, \frac{C}{|\eta n\delta/2^j|^k}\right\}\right)^p \left\{ \int_{\delta/2^j \geq |\theta| > \delta/2^{j+1}} \left|\omega'(1) - \omega'_{ac}(\theta)\right| d\theta + \int_{\delta/2^j \geq |\theta| > \delta/2^{j+1}} d\omega_s(\theta) \right\} \\
\leq\ & \sum_{j=0}^{\infty} \min\left\{1, \frac{C}{|\eta n\delta/2^j|^{kp}}\right\} \varepsilon\delta 2^{-j},
\end{aligned}$$

by (3.1). We continue this as

$$\begin{aligned}
\leq\ & C\varepsilon\delta\,(\eta n\delta)^{-kp} \sum_{0 \leq j < \log_2(\eta n\delta)} 2^{j(kp-1)} + C\varepsilon\delta \sum_{j \geq \log_2(\eta n\delta)} 2^{-j} \\
\leq\ & C\varepsilon\,(n\eta)^{-1},
\end{aligned}$$

by some simple calculations. Here C is independent of ε, δ and n. Combining this and (3.7), (3.8), gives

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega, 1) \leq \frac{2\pi\mathcal{E}_p\omega'(1)}{1 - 2\eta} + C\frac{\varepsilon}{\eta}. \tag{3.9}$$

Here the left-hand side is independent of ε, η, while C is independent of ε, η. Moreover, ε, η are independent of each other. We can first let ε and then $\eta \to 0+$ to obtain (1.9). ■

For later use, we record more on the polynomials U_n:

Lemma 3.1
Let $\eta \in \left(0, \frac{1}{8}\right)$ and $k \geq 1$ be chosen so that $kp > 1$. Let U_n be the polynomial of degree $\leq n-1$, defined by (3.2). Then $U_n(1) = 1$, and
(a)

$$\limsup_{n\to\infty} n \int_{-\pi}^{\pi} |U_n|^p \, d\omega \leq \frac{2\pi \mathcal{E}_p \omega'(1)}{1-2\eta} + C\eta. \tag{3.10}$$

Here C is independent of η and n.
(b) If $p > 1$, uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty} U_n\left(e^{2\pi i z/n}\right) = e^{i\pi z} f_p^*\left(z\left(1-2\eta\right)\right) S\left(z\eta/k\right)^k. \tag{3.11}$$

Proof
(a) This was proved at (3.9), with the term $C\frac{\varepsilon}{\eta}$ instead of $C\eta$. As ε, η are independent of one another, we can choose $\varepsilon = \eta^2$.
(b) Firstly with $m = m(n) = n - [2\eta n]$,

$$\begin{aligned} P^*_{n-[2\eta n]}\left(e^{2\pi i z/n}\right) &= P_m\left(e^{2\pi i(1-2\eta+o(1))z/m}\right) \\ &= e^{i\pi z(1-2\eta)} f_p^*\left(z\left(1-2\eta\right)\right) + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane, by the uniform convergence in Theorem 2.1(b). Next,

$$\begin{aligned} R_n\left(e^{2\pi i z/n}\right) &= \left(\frac{1+e^{2\pi i z/n}}{2}\right)^{[\eta n]} \\ &= \left(1 + \frac{i\pi z}{n} + o\left(\frac{1}{n}\right)\right)^{[\eta n]} \\ &= e^{i\pi z\eta} + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane. Finally, with $p = p(n) = [\eta n/k]$

$$\begin{aligned} S_n\left(e^{2\pi i z/n}\right) &= \ell_{0p}\left(e^{\left(2\pi i z \frac{p}{n}\right)/p}\right)^k \\ &= e^{\pi i z p k/n} S\left(z\frac{p}{n}\right)^k + o(1) \\ &= e^{i\pi z\eta} S\left(z\eta/k\right)^k + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane, by Lemma 2.2. Combining the three asymptotics gives (3.11). ■

Proof of Theorem 1.1(a)
We may assume $z_0 = 1$. If $\omega'(1) = 0$, the result already follows from Theorem 1.2. So let us assume that $\omega'(1) = \omega'_{ac}(1) > 0$. It suffices to prove that

$$\liminf_{n\to\infty} n\lambda_{n,p}(\omega, 1) \geq 2\pi \mathcal{E}_p \omega'(1). \tag{3.12}$$

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. As ω'_{ac} is lower semi-continuous at 1, we can choose $\delta > 0$ such

$$\omega'_{ac}\left(e^{i\theta}\right) \geq \omega'(1) / (1+\varepsilon), \text{ for } \theta \in [-\delta, \delta]. \tag{3.13}$$

Let $\eta \in \left(0, \frac{1}{8}\right)$ and R_n be the polynomial of degree $\leq [\eta n]$ defined by (3.4). Let $P^{\#}_{n-[\eta n]}$ be the extremal polynomial $P^{*}_{n-[\eta n],p,1}$ for the measure ω, so that $\deg\left(P^{\#}_{n-[\eta n]}\right) \leq n - [\eta n] - 1$; $P^{\#}_{n-[\eta n]}(1) = 1$ and

$$\int_{-\pi}^{\pi} \left|P^{\#}_{n-[\eta n]}\right|^p d\omega = \lambda_{n-[\eta n],p}(\omega, 1).$$

Let ω_L denote Lebesgue measure for the unit circle. We have

$$\begin{aligned}
& \lambda_{n,p}(\omega_L, 1) \\
\leq\ & \int_{-\pi}^{\pi} \left|P^{\#}_{n-[\eta n]}(z) R_n(z)\right|^p d\theta \\
\leq\ & \frac{1+\varepsilon}{\omega'(1)} \int_{-\delta}^{\delta} \left|P^{\#}_{n-[\eta n]}(z)\right|^p \omega'\left(e^{i\theta}\right) d\theta \\
& + \left\|P^{\#}_{n-[\eta n]}\right\|^p_{L_\infty(|z|=1)} c^{np} \int_{\delta \leq |\theta| \leq \pi} d\theta,
\end{aligned} \tag{3.14}$$

by (3.5) and (3.13). Since ω is regular,

$$\left\|P^{\#}_{n-[\eta n]}\right\|^p_{L_\infty(|z|=1)} \leq (1 + o(1))^n \int_{-\pi}^{\pi} \left|P^{\#}_{n-[\eta n]}\right|^p d\omega.$$

Note that although we defined regularity by a relation of this type for L_2 norms, it holds for all L_p norms [29, Thm. 3.4.3, pp. 90-91]. Combining this with (3.14) gives

$$\begin{aligned}
\lambda_{n,p}(\omega_L, 1) &\leq \left(\int_{-\pi}^{\pi} \left|P^{\#}_{n-[\eta n]}\right|^p d\omega\right) \left\{\frac{1+\varepsilon}{\omega'(1)} + 2\pi \left(c(1 + o(1))\right)^n\right\} \\
&\leq \lambda_{n-[\eta n],p}(\omega, 1) \left\{\frac{1+\varepsilon}{\omega'(1)} + o(1)\right\}.
\end{aligned}$$

Using Theorem 2.1 for Lebesgue measure ω_L, we obtain

$$\liminf_{n\to\infty} n\lambda_{n-[\eta n],p}(\omega, 1) \geq 2\pi \mathcal{E}_p \frac{\omega'(1)}{1+\varepsilon}.$$

As η and $\varepsilon > 0$ are independent, we obtain

$$\liminf_{n\to\infty} n\lambda_{n-[\eta n],p}(\omega, 1) \geq 2\pi \mathcal{E}_p \omega'(1). \tag{3.15}$$

Finally the monotonicity of $\lambda_{n,p}$ in n easily yields (3.12). Indeed, given a positive integer m, we choose $n = n(m)$ to be the largest integer with

$$m \geq n - [\eta n].$$

Then

$$n + 1 - [\eta(n+1)] > m,$$

so (3.15) gives

$$\begin{aligned}
m\lambda_{m,p}(\omega, 1) &\geq m\lambda_{n+1-[\eta(n+1)],p}(\omega, 1) \\
&\geq \frac{m}{n+1} \left(2\pi \mathcal{E}_p \omega'(1) + o(1)\right) \\
&\geq \frac{n(1-\eta)}{n+1} \left(2\pi \mathcal{E}_p \omega'(1) + o(1)\right).
\end{aligned}$$

Then

$$\liminf_{m\to\infty} m\lambda_{m,p}(\omega,1) \geq (1-\eta)\, 2\pi\mathcal{E}_p\omega'(1),$$

and we can let $\eta \to 0+$. ■

For the proof of Theorem 1.1(b), we shall use ideas of uniform convexity, and so need Clarkson's inequalities:

Lemma 3.2
Let ν be a finite positive Borel positive measure on the unit circle. Let f, g be measurable complex valued functions on the unit circle.
(a) If $p \geq 2$,

$$\int \left|\frac{f+g}{2}\right|^p d\nu + \int \left|\frac{f-g}{2}\right|^p d\nu \leq \frac{1}{2}\left(\int |f|^p\, d\nu + \int |g|^p\, d\nu\right). \tag{3.16}$$

(b) If $1 < p < 2$, and $q = \frac{p}{p-1}$,

$$\left(\int \left|\frac{f+g}{2}\right|^p d\nu\right)^{\frac{q}{p}} + \left(\int \left|\frac{f-g}{2}\right|^p d\nu\right)^{\frac{q}{p}} \leq \left(\frac{1}{2}\int |f|^p\, d\nu + \frac{1}{2}\int |g|^p\, d\nu\right)^{\frac{q}{p}}. \tag{3.17}$$

Proof
See, for example, [1], [3]. ■

We shall also need a Bernstein-Walsh type estimate:

Lemma 3.3
Let $0 < \tau < \pi$, and $\Gamma = \left\{e^{i\theta} : |\theta| \leq \tau\right\}$. There exist $C_1, C_2 > 0$ depending only on τ, with the following property: given $r > 0$, there exists $n_0 = n_0(r,\tau)$ such that for $n \geq n_0$, polynomials P of degree $\leq n$, and $|z| \leq r$,

$$\left|P\left(e^{2\pi iz/n}\right)\right| \leq C_1 e^{C_2|z|}\, \|P\|_{L_\infty(\Gamma)}. \tag{3.18}$$

Proof
Rather than proving this on Γ, we prove this for the arc $\Delta = \left\{e^{i\theta} : \theta \in [\alpha, 2\pi - \alpha]\right\}$, where $0 < \alpha < \pi$. A reflection $z \to -z$ and substitution $\alpha = \pi - \tau$ give the result above. In [7, p. 213], it was noted that a conformal map Ψ of $\mathbb{C}\backslash\Delta$ onto the exterior of the unit disc, is given by

$$\Psi(z) = \frac{1}{2\cos\frac{\alpha}{2}}\left\{1 + z + \sqrt{z^2 - 2z\cos\alpha + 1}\right\}.$$

Here the branch of the square root is chosen so that $\sqrt{z^2 - 2z\cos\alpha + 1} = z(1 + o(1))$ as $|z| \to \infty$. It follows from the proof of Lemma 3.2 there [7, p. 219-221], that uniformly for v in a subarc Δ_1 of Δ, and $|u| \geq 1$,

$$|\Psi(u) - \Psi(v)| \leq C\,|u - v|$$

and hence

$$|\Psi(u)| \leq 1 + C\,|u - v|.$$

Next, the classical Bernstein-Walsh inequality [25, p.156] asserts that for polynomials of degree $\leq n-1$, and $u \in \mathbb{C}\backslash\Delta$,

$$|P(u)| \leq |\Psi(u)|^n\, \|P\|_{L_\infty(\Delta)},$$

and consequently, for $|z| \le r$, and $n \ge n_0(r)$,

$$\begin{aligned} \left|P\left(e^{2\pi iz/n}\right)\right| &\le \left(1+C\left|e^{2\pi iz/n}-1\right|\right)^n \|P\|_{L_\infty(\Delta)} \\ &\le C_1 e^{C_2|z|}\|P\|_{L_\infty(\Delta)}. \end{aligned}$$

■

It is because of the fact that we don't know $C_2 = \pi$ in (3.18), that we have to use a comparison and localization method for the proof of Theorem 1.1(b). Otherwise, we could have largely followed the proof in Theorem 2.1(b).

Proof of Theorem 1.1(b)

We shall assume that $z_0 = 1$, otherwise we can rotate the variable. Let $P_n^* = P_{n,p,1}^*$. Our strategy will be to construct polynomials $P_n^\#$ of degree $\le n-1$, with $P_n^\#(1) = 1$, satisfying

$$\lim_{n\to\infty} n\int_{-\pi}^{\pi}\left|P_n^\#\left(e^{i\theta}\right)\right|^p d\omega(\theta) = 2\pi\mathcal{E}_p\omega'(1) \tag{3.19}$$

and, uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty} P_n^\#\left(e^{2\pi iz/n}\right) = e^{i\pi z} f_p^*(z), \tag{3.20}$$

where f_p^* is the extremal function defined in Section 1. We shall also show that given any infinite sequence of positive integers, it contains a subsequence $\mathcal{T}$ such that uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty, n\in\mathcal{T}} P_n^*\left(e^{2\pi iz/n}\right) = e^{i\pi z} g(z) \tag{3.21}$$

for some entire function g. Once we have these, we can use Clarkson's inequalities to finish the proof. Indeed, if $p \ge 2$, the Clarkson inequalities (3.16) give

$$\begin{aligned} & n\int_{-\pi}^{\pi}\left|\frac{P_n^*-P_n^\#}{2}\right|^p d\omega(\theta) \\ \le\ & \frac{1}{2}\left(n\int_{-\pi}^{\pi}\left|P_n^*\left(e^{i\theta}\right)\right|^p d\omega(\theta) + n\int_{-\pi}^{\pi}\left|P_n^\#\left(e^{i\theta}\right)\right|^p d\omega(\theta)\right) - n\int_{-\pi}^{\pi}\left|\frac{P_n^*+P_n^\#}{2}\right|^p d\omega(\theta) \\ \le\ & \frac{1}{2}\left(n\int_{-\pi}^{\pi}\left|P_n^*\left(e^{i\theta}\right)\right|^p d\omega(\theta) + n\int_{-\pi}^{\pi}\left|P_n^\#\left(e^{i\theta}\right)\right|^p d\omega(\theta)\right) - n\lambda_{n,p}(\omega,1) \\ \to\ & \pi\mathcal{E}_p\omega'(1) + \pi\mathcal{E}_p\omega'(1) - 2\pi\mathcal{E}_p\omega'(1) = 0 \end{aligned}$$

as $n \to \infty$, by Theorem 1.1(a) and (3.19). The case $1 < p < 2$ is similar. Then, given $r > 0$, (3.20) and (3.21) give

$$\begin{aligned} \int_{-r}^{r}\left|\frac{g-f_p^*}{2}\right|^p(t)\,dt &= \lim_{n\to\infty, n\in\mathcal{T}}\int_{-r}^{r}\left|\frac{P_n^*-P_n^\#}{2}\right|^p\left(e^{2\pi it/n}\right)dt \\ &= \frac{1}{2\pi}\lim_{n\to\infty, n\in\mathcal{T}} n\int_{-2\pi r/n}^{2\pi r/n}\left|\frac{P_n^*-P_n^\#}{2}\right|^p\left(e^{i\theta}\right)d\theta \\ &\le C\lim_{n\to\infty, n\in\mathcal{T}} n\int_{-2\pi r/n}^{2\pi r/n}\left|\frac{P_n^*-P_n^\#}{2}\right|^p d\omega(\theta) = 0. \end{aligned}$$

Here we have used the lower semicontinuity of ω'_{ac} at 1 and that $\omega'(1) > 0$. Thus $g = f_p^*$ a.e in $[-r, r]$, and as both are entire, $g = f_p^*$ in the plane. As this is true of every subsequence, we have shown that uniformly for z in compact subsets of the plane, as desired,

$$\lim_{n\to\infty} P_n^*\left(e^{2\pi iz/n}\right) = e^{i\pi z} f_p^*(z). \tag{3.22}$$

We now turn to the proof of (3.19-3.21).

Proof of (3.21)

Now by hypothesis, there exists $\rho \in (0,\pi)$, such that $\omega'_{ac} \geq C$ on the arc $\Gamma_1 = \left\{e^{i\theta} : |\theta| \leq \rho\right\}$. Hence

$$C\int_{-\rho}^{\rho} \left|P_n^*\left(e^{i\theta}\right)\right|^p d\theta \leq \int_{-\pi}^{\pi} \left|P_n^*\left(e^{i\theta}\right)\right|^p d\omega(\theta) = \lambda_{n,p}(\omega, 1) \leq C_5/n.$$

Next, we use bounds on Christoffel functions for an arc of the unit circle, which follow from much deeper bounds on orthogonal polynomials, due to Golinskii [6, p. 256, Proposition 11]. These have the immediate consequence that if $0 < \tau < \rho$,

$$\sup\left\{\left|P_n^*\left(e^{i\theta}\right)\right|^p : |\theta| \leq \tau\right\} \leq Cn\int_{-\rho}^{\rho} \left|P_n^*\left(e^{i\theta}\right)\right|^p d\theta \leq C.$$

Alternatively, one can use a special case of the quadrature sum in [7, Theorem 1.1, p. 208]. Then Lemma 3.3 shows that, given $r > 0$, there exists $n_0(r)$ such that for $n \geq n_0(r)$, and all $|z| \leq r$,

$$\left|P_n^*\left(e^{2\pi iz/n}\right)\right| \leq C_1 e^{C_2|z|}.$$

It follows that the functions $\left\{P_n^*\left(e^{2\pi iz/n}\right)\right\}_n$ are a normal family in $\mathbb{C}$, and hence given any infinite sequence of positive integers, we can extract a subsequence $\mathcal{T}$, with (3.21) holding for some entire function g.

Proof of (3.19-3.20)

For each $\eta > 0$, we constructed a sequence of polynomials $\{U_n\}$ satisfying the conclusions of Lemma 3.1. Now choose a decreasing sequence of positive numbers $\{\eta_j\}$ with limit 0, and an increasing sequence of positive numbers $\{r_j\}$ with limit ∞. For each j, we can construct a sequence of polynomials $\{U_n\}$ satisfying the conclusions of Lemma 3.1, in the following form: for some positive integer n_j and $n \geq n_j$,

$$n\int_{-\pi}^{\pi} |U_n|^p d\omega \leq \frac{2\pi\mathcal{E}_p\omega'(1)}{1-2\eta_j} + C\eta_j. \tag{3.23}$$

$$\sup_{|z|\leq r_j} \left|U_n\left(e^{2\pi iz/n}\right) - e^{i\pi z} f_p^*\left(z\left(1-2\eta_j\right)\right) S\left(z\eta_j/k\right)^k\right| \leq \eta_j.$$

Here C is independent of n and η_j, while k was a fixed positive integer such that $kp > 1$ - and in particular is independent of $n, \{\eta_j\}$. We may assume that $\{n_j\}$ is strictly increasing. Of course, there is a slight abuse of notation since $\{U_n\}$ depend on the particular η_j. We now take $P_n^{\#} = U_n$ for η_j for $n_j \leq n < n_{j+1}$. It is easily seen that these satisfy (3.19) and (3.20). Note that (3.23) involves only an upper bound, but the corresponding lower bound follows from the minimum property of Christoffel functions and Theorem 1.1. ∎

4. The Chebyshev Weight

In this section, we let

$$v(x)=\frac{1}{\sqrt{1-x^2}},x\in(-1,1),$$

and establish asymptotics for $\lambda_{n,p}(v,x)$, using methods analogous to those we used for Lebesgue measure on the unit circle in Section 2. We prove

Theorem 4.1
Let $p>0$ and $\xi\in(-1,1)$. Then
(a)

$$\lim_{n\to\infty}n\lambda_{n,p}(v,\xi)=\pi\mathcal{E}_p. \tag{4.1}$$

(b) If $p\geq 1$, the extremal polynomials $\left\{P^*_{n,p,\xi}\right\}$ satisfy

$$\lim_{n\to\infty}P^*_{n,p,\xi}\left(\xi+\frac{\pi\sqrt{1-\xi^2}z}{n}\right)=f^*_p(z), \tag{4.2}$$

uniformly for z in compact subsets of the plane.

In this section, we let

$$K_n(x,t)=\frac{1}{\pi}\frac{T_n(x)T_{n-1}(t)-T_{n-1}(x)T_n(t)}{x-t},n\geq 1,$$

denote the reproducing kernel for the Chebyshev weight. Of course, T_n denotes the classical Chebyshev polynomial of degree n. We fix $\xi\in(-1,1)$, and let $\{t_{jn}\}_{j\neq 0}$ denote the at most $n-1$ zeros of $K_n(\xi,t)$ [5, p. 19], ordered as follows:

$$...<t_{-1,n}<t_{0n}=\xi<t_{1n}<t_{2n}<... \tag{4.3}$$

Note that there will $n-1$ zeros unless $T_{n-1}(\xi)=0$. Of course, the sequence terminates for both positive and negative subscripts. We let $\{\ell_{jn}\}$ denote the fundamental polynomials of Lagrange interpolation at the $\{t_{jn}\}$, so that

$$\ell_{jn}(t)=\frac{K_n(t_{jn},t)}{K_n(t_{jn},t_{jn})}.$$

In particular,

$$\ell_{0n}(t)=\frac{K_n(\xi,t)}{K_n(\xi,\xi)}.$$

Lemma 4.2
Let $\rho\in(0,1)$.
(a) Uniformly for $\xi\in(-\rho,\rho)$, and for integers j,

$$\lim_{n\to\infty}n(t_{jn}-\xi)=j\pi\sqrt{1-\xi^2}. \tag{4.4}$$

(b) Uniformly for $\xi\in(-\rho,\rho)$, for t in a compact set, and for fixed integers j,

$$\lim_{n\to\infty}\ell_{jn}\left(\xi+\frac{t\pi\sqrt{1-\xi^2}}{n}\right)=S(t-j). \tag{4.5}$$

(c) Uniformly for $\xi\in(-\rho,\rho)$, $t\in\mathbb{R}$, and for all all $|j|\leq[\log n]$

$$|\ell_{jn}(t)|\leq C_1\min\left\{1,\frac{1}{n|t-t_{jn}|}\right\}. \tag{4.6}$$

(d) There exists an increasing sequence of positive integers $\{L_n\}$ with limit ∞, such that for each fixed $r > 0$, (4.5) holds uniformly for $t \in (-r, r)$ and $|j| \leq L_n$.

Proof

(a) This follows immediately from Theorem V.8.1 in [5, p. 266].

(b) We use the well known universality limit

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n}\right)}{K_n(\xi,\xi)} = S(a-b),$$

uniformly for a, b in compact sets, as well as the Christoffel function limit

$$\lim_{n\to\infty} \frac{\pi}{n} K_n(y,y) = 1,$$

which holds uniformly for y in compact subsets of $(-1,1)$. See, for example [17], [27], [31]. Using the uniform convergence, and (a), gives

$$\begin{aligned}
&\ell_{jn}\left(\xi + \frac{t\pi\sqrt{1-\xi^2}}{n}\right) \\
= &\frac{K_n\left(\xi + \frac{j\pi\sqrt{1-\xi^2}}{n}(1+o(1)), \xi + \frac{t\pi\sqrt{1-\xi^2}}{n}\right)}{K_n(\xi,\xi)} \frac{K_n(\xi,\xi)}{K_n(t_{jn},t_{jn})} \\
= &S(t-j)(1+o(1)).
\end{aligned}$$

(c) From the Christoffel-Darboux formula, for all $x, t \in [-1,1]$,

$$|K_n(x,t)| \leq \frac{2}{\pi|x-t|}, \tag{4.7}$$

and also

$$|K_n(x,t)| = \left|\frac{1}{\pi} + \frac{2}{\pi}\sum_{j=1}^{n-1} T_j(x)T_j(t)\right| \leq \frac{2}{\pi}n. \tag{4.8}$$

This and the lower bound $K_n(x,x) \geq Cn$, which holds uniformly for x in a compact subset of $(-1,1)$, gives the result.

(d) This follows easily from (b). ■

Next, we give an analogue of Lemma 2.3:

Lemma 4.3

Let $\xi \in (-1,1)$. Let $p > 1$. Then

$$\limsup_{n\to\infty} n\lambda_{n,p}(v,\xi) \leq \pi\mathcal{E}_p.$$

Proof

Let $f \in L^p_\pi$, with $f(0) = 1$. Fix $m \geq 1$ and let

$$S_n(z) = \sum_{|j|\leq m} f(j)\,\ell_{jn}(z).$$

Since $S_n(\xi) = f(0)\,\ell_{0n}(\xi) = 1$, we have

$$\lambda_{n,p}(\omega,\xi) \leq \int_{-1}^{1} |S_n(x)|^p \frac{dx}{\sqrt{1-x^2}}.$$

Now for each $r>0$, Lemma 4.2(b) gives

$$\begin{aligned}
&\lim_{n\to\infty} n\int_{\xi-\pi\sqrt{1-\xi^2}r/n}^{\xi+\pi\sqrt{1-\xi^2}r/n} \left|S_n\left(x\right)\right|^p \frac{dx}{\sqrt{1-x^2}} \\
&= \pi\lim_{n\to\infty}\int_{-r}^{r}\left|S_n\left(\xi+\frac{t\pi\sqrt{1-\xi^2}}{n}\right)\right|^p dt = \pi\int_{-r}^{r}\left|\sum_{|j|\le m} f\left(j\right)S\left(t-j\right)\right|^p dt.
\end{aligned}$$

(4.9)

Next, we estimate the rest of the integral. Choose r so large that for all large enough n, $|\xi - t_{\pm mn}| \le \frac{1}{2}r\pi\sqrt{1-\xi^2}/n$. This is possible because of (4.4). Then for $|x-\xi| \ge r\pi\sqrt{1-\xi^2}/n$, Lemma 4.2(c) gives

$$\begin{aligned}
\left|S_n\left(x\right)\right| &\le \frac{C}{n}\sum_{|j|\le m}\left|f\left(j\right)\right|/\left|x-t_{jn}\right| \\
&\le \frac{C}{n\left|x-\xi\right|}\left(\sum_{|j|\le m}\left|f\left(j\right)\right|\right),
\end{aligned}$$

so if $J_n = [-1,1]\setminus\left[\xi-\pi\sqrt{1-\xi^2}r/n, \xi+\pi\sqrt{1-\xi^2}r/n\right]$,

$$\begin{aligned}
& n\int_{J_n}\left|S_n\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}} \\
\le\ & C\left(\sum_{|j|\le m}\left|f\left(j\right)\right|\right)^p n^{1-p}\int_{J_n}\left|x-\xi\right|^{-p}\frac{dx}{\sqrt{1-x^2}} \\
\le\ & C_1\left(\sum_{|j|\le m}\left|f\left(j\right)\right|\right)^p r^{1-p},
\end{aligned}$$

where C_1 is independent of n, r, but depends on ξ. In estimating the integral in the second last line, one splits the integral into a range over $\left\{x : \pi\sqrt{1-\xi^2}r/n \le |x-\xi| \le \frac{1}{8}\sqrt{1-\xi^2}\right\}$, and the rest of J_n. Combined with (4.9), this gives

$$\begin{aligned}
\limsup_{n\to\infty} n\lambda_{n,p}\left(v,\xi\right) &\le \limsup_{n\to\infty} n\int_{-1}^{1}\left|S_n\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}}. \\
&\le \pi\int_{-r}^{r}\left|\sum_{|j|\le m} f\left(j\right)S\left(t-j\right)\right|^p dt + C_2\left(\sum_{|j|\le m}\left|f\left(j\right)\right|\right)^p r^{1-p}.
\end{aligned}$$

First letting $r\to\infty$, and then using the triangle inequality and (2.8), (2.9) as in the proof of Lemma 2.3, with $m\to\infty$, gives the result. ■

Next, we handle the more difficult case $p \le 1$. We need careful estimates on needle polynomials:

Lemma 4.4

Let $\xi \in (-1,1)$*. For* $k \geq 1$*, let*

$$U_k(x) = \frac{K_k(x,\xi)}{K_k(\xi,\xi)}.$$

(a) Then $U_k(\xi) = 1$ *and for some* C_ξ *depending on* ξ*, but independent of* x, k*,*

$$|U_k(x)| \leq 1 + C_\xi |x-\xi|, \; x \in [-1,1]. \tag{4.10}$$

(b) In particular, for some C'_ξ *independent of* x, k*,*

$$|U_k(x)| \leq C'_\xi, \; x \in [-1,1]. \tag{4.11}$$

(c)

$$|U_k(x)| \leq \frac{C}{1 + k|x-\xi|}. \tag{4.12}$$

Proof
(a) The Cauchy-Schwarz inequality gives

$$U_k(x) \leq \sqrt{\frac{K_k(x,x)}{K_k(\xi,\xi)}}.$$

Here, uniformly for x in compact subsets of $(-1,1)$, classical limits for Christoffel functions give

$$\lim_{k\to\infty} \frac{K_k(x,x)}{K_k(\xi,\xi)} = \sqrt{\frac{1-\xi^2}{1-x^2}}.$$

It follows from the differentiability of the right-hand side that if $r = \frac{1}{2}(1-|\xi|)$, there exist k_0 and C depending only on ξ such that

$$\frac{K_k(x,x)}{K_k(\xi,\xi)} \leq 1 + C|x-\xi| \text{ for } |x-\xi| \leq r \text{ and } k \geq k_0.$$

In view of (4.8), we also have for $k \geq k_0$, and $x \in [-1,1]$ with $|x-\xi| \geq r$,

$$\frac{K_k(x,x)}{K_k(\xi,\xi)} \leq C_1 \leq 1 + C_2|x-\xi|.$$

Thus we obtain (4.10) for $k \geq k_0$ and all $x \in [-1,1]$. By increasing the constant in (4.10), we obtain it for all $k \geq 1$ and $x \in [-1,1]$, using just the differentiability of $K_n(x,\xi)/K_n(\xi,\xi)$.
(b) follows directly from (a).
(c) follows from (4.7), (4.8), and our lower bound for $K_n(\xi,\xi)$. ■

Lemma 4.5
Let $0 < p \leq 1$*. Then*

$$\limsup_{n\to\infty} n\lambda_{n,p}(v,\xi) \leq \pi\mathcal{E}_p. \tag{4.13}$$

Proof
Let $f \in L^p_\pi$, with $f(0) = 1$. Let $\varepsilon \in \left(0, \frac{1}{2}\right)$. Choose a positive integer k such that $kp \geq 2$ and let

$$S_n(x) = \left(\sum_{|j| \leq L_{n-[\varepsilon n]}} f(j)\,\ell_{j,n-[\varepsilon n]}(x)\right) U_{\left[\frac{\varepsilon}{k}n\right]}(x)^k = V_n(x)\, U_{\left[\frac{\varepsilon}{k}n\right]}(x)^k,$$

say, a polynomial of degree $\leq n-1$, with $S_n(\xi)=1$. Here $L_{n-[\varepsilon n]}$ is as in Lemma 4.2(d). Fix $r>0$. By (4.10), $\left|U_{\left[\frac{\varepsilon}{k}n\right]}(z)\right| \leq 1+o(1)$ for $|x-\xi| \leq \pi\sqrt{1-\xi^2}r/n$, so Lemma 4.2 gives,

$$\begin{aligned}
& n\int_{\xi-\pi\sqrt{1-\xi^2}r/n}^{\xi+\pi\sqrt{1-\xi^2}r/n} |S_n(x)|^p \frac{dx}{\sqrt{1-x^2}} \\
\leq\ & (1+o(1))\,\pi\int_{-r}^{r}\left|V_n\left(\xi+\frac{t\pi\sqrt{1-\xi^2}}{n-[\varepsilon n]}\frac{n-[\varepsilon n]}{n}\right)\right|^p dt \\
\leq\ & (1+o(1))\,\pi\int_{-r}^{r}\left|\sum_{|j|\leq L_{n-[\varepsilon n]}} f(j)\,S(t(1-\varepsilon)-j)+o(1)\sum_{|j|\leq L_{n-[\varepsilon n]}}|f(j)|\right|^p dt.
\end{aligned}$$

Here we are using the fact that

$$D=\sum_{j=-\infty}^{\infty}|f(j)| \leq \|f\|_{L_\infty(\mathbb{R})}^{1-p}\sum_{j=-\infty}^{\infty}|f(j)|^p<\infty,$$

recall (2.7). Next, uniformly for $t\in[-r,r]$,

$$\begin{aligned}
& \left|f(t(1-\varepsilon))-\sum_{|j|\leq L_{n-[\varepsilon n]}} f(j)\,S(t(1-\varepsilon)-j)\right| \\
=\ & \left|\sum_{|j|>L_{n-[\varepsilon n]}} f(j)\,S(t(1-\varepsilon)-j)\right| \leq \sum_{|j|>L_{n-[\varepsilon n]}}|f(j)| \to 0,
\end{aligned}$$

as $n\to\infty$. It follows that

$$\begin{aligned}
& \limsup_{n\to\infty} n\int_{\xi-\pi\sqrt{1-\xi^2}r/n}^{\xi+\pi\sqrt{1-\xi^2}r/n}|S_n(x)|^p\frac{dx}{\sqrt{1-x^2}} \\
\leq\ & \limsup_{n\to\infty}\pi\int_{-r}^{r}|f(t(1-\varepsilon))|^p dt \leq \frac{\pi}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p dt. \qquad (4.14)
\end{aligned}$$

Next, Lemma 4.2(c) followed by Lemma 4.4(c), gives

$$\begin{aligned}
|S_n(x)| &\leq C_1 D\left|U_{\left[\frac{\varepsilon}{k}n\right]}(x)\right|^k \\
&\leq C_1 D\left(\frac{C}{1+\frac{\varepsilon}{k}n|x-\xi|}\right)^k.
\end{aligned}$$

Hence, if $J_n=[-1,1]\setminus\left[\xi-\pi\sqrt{1-\xi^2}r/n,\xi+\pi\sqrt{1-\xi^2}r/n\right]$,

$$\begin{aligned}
& n\int_{J_n}|S_n(x)|^p\frac{dx}{\sqrt{1-x^2}} \\
\leq\ & C\varepsilon^{-kp}n^{1-kp}\int_{J_n}|x-\xi|^{-kp}\frac{dx}{\sqrt{1-x^2}} \\
\leq\ & C_2\varepsilon^{-kp}r^{1-kp}.
\end{aligned}$$

Here C_2 is independent of r, n, but depends on ξ, k. Combining this with (4.14) gives

$$\begin{aligned}
& \limsup_{n\to\infty} n\lambda_{n,p}(v,\xi) \\
\leq\ & \limsup_{n\to\infty} n \int_{-1}^{1} |S_n(x)|^p \frac{dx}{\sqrt{1-x^2}} \\
\leq\ & \frac{\pi}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p \, dt + C\varepsilon^{-kp} r^{-kp+1}.
\end{aligned}$$

We can now complete the proof as in Lemma 2.4: let $r \to \infty$, then $\varepsilon \to 0+$, and finally take inf's over f. ■

We next turn to the asymptotic lower bound. We defined P_n^* to be a polynomial of degree $\leq n-1$ with $P_n^*(\xi) = 1$ and

$$\int_{-1}^{1} |P_n^*(x)|^p \frac{dx}{\sqrt{1-x^2}} = \lambda_{n,p}(v,\xi).$$

The Lagrange interpolation formula asserts that

$$P_n^*(x) = \ell_{0n}(x) + \sum_{j\neq 0} P_n^*(t_{jn}) \, \ell_{jn}(x),$$

provided there are n distinct $\{t_{jn}\}$, including $t_{0n} = \xi$. This occurs unless $T_{n-1}(\xi) = 0$. We let $\mathcal{Z}$ denote the possibly empty sequence of positive integers n for which $T_{n-1}(\xi) = 0$. Note that no two successive integers can both belong to $\mathcal{Z}$. We start with estimates for P_n^* :

Lemma 4.6
Let $p > 0$.
(a)

$$\sup_{n\geq 1, n\notin\mathcal{Z}} \sum_j |P_n^*(t_{jn})|^p \leq \Lambda < \infty. \tag{4.15}$$

(b)

$$\sup_{n\geq 1, n\notin\mathcal{Z}} \|P_n^*\|_{L_\infty[-1,1]} < \infty. \tag{4.16}$$

Proof
(a) For all $p > 0$, there is the Marcinkiewicz-Zygmund inequality [19, Thm. 2, p. 533]

$$\frac{1}{n} \sum_j |P_n^*(t_{jn})|^p \leq C \int_{-1}^{1} |P_n^*(x)|^p \frac{dx}{\sqrt{1-x^2}} = C\lambda_{n,p}(v,\xi).$$

Here C is independent of n and p. Then by Lemma 4.3 for $p > 1$, and Lemma 4.5 for $0 < p \leq 1$,

$$\frac{1}{n} \sum_j |P_n^*(t_{jn})|^p \leq \frac{C}{n}.$$

(b) We use Nevai's estimates for $\lambda_{n,p}(v,\xi)$ [22, p. 120]

$$\begin{aligned}\|P_n^*\|^p_{L_\infty[-1,1]} &\leq \left\|\lambda_{n,p}^{-1}(v,\cdot)\right\|_{L_\infty[-r,r]}\int_{-1}^{1}|P_n^*(x)|^p\frac{dx}{\sqrt{1-x^2}}\\ &\leq Cn\cdot\frac{1}{n}.\end{aligned}$$

■

We shall again separately consider the case $p>1$ and $p\leq 1$.

Lemma 4.7

Let $p>1, r>0$. There exists $M_0=M_0(r)$ such that for $m\geq M_0$, and all $n\notin\mathcal{Z}$,

$$\int_{\xi-\pi\sqrt{1-\xi^2}r/n}^{\xi+\pi\sqrt{1-\xi^2}r/n}\left|\sum_{|j|>m}P_n^*(t_{jn})\ell_{jn}(x)\right|^p\frac{dx}{\sqrt{1-x^2}}\leq C\frac{r}{mn}. \tag{4.17}$$

Here C and M_0 are independent of r,n and m.

Proof

Now the $\{t_{jn}\}$ interlace the zeros of T_n [5, p.19], so the the well known spacing of the latter gives

$$|t_{jn}-\xi|\geq C_1\frac{j}{n}$$

for $|j|\geq 2$ and all n. Then for $|x-\xi|\leq\frac{r\pi\sqrt{1-\xi^2}}{n}$, and $|j|\geq m+1$,

$$|\ell_{jn}(x)|\leq\frac{C}{n|(x-\xi)-(t_{jn}-\xi)|}\leq\frac{C}{n\left|\frac{r\pi\sqrt{1-\xi^2}}{n}-C_1\frac{j}{n}\right|}\leq\frac{C}{j}, \tag{4.18}$$

provided $m\geq M_0(r)$. Hence by Hölder's inequality with $q=p/(p-1)$,

$$\begin{aligned}&\left|\sum_{|j|>m}P_n^*(t_{jn})\ell_{jn}(x)\right|\\ \leq\ &\left(\sum_{|j|>m}|P_n^*(t_{jn})|^p\right)^{1/p}\left(\sum_{|j|>m}\left(\frac{C}{j}\right)^q\right)^{1/q}\\ \leq\ &C\left(m^{-q+1}\right)^{1/q}=Cm^{-1/p},\end{aligned}$$

by Lemma 4.6(a). Then

$$\int_{|x-\xi|\leq\frac{r\pi\sqrt{1-\xi^2}}{n}}\left|\sum_{|j|>m}P_n^*(t_{jn})\ell_{jn}(x)\right|^p\frac{dx}{\sqrt{1-x^2}}\leq C\frac{r}{mn}.$$

■

Lemma 4.8

Let $p>1$. Then

$$\liminf_{n\to\infty}n\lambda_{n,p}(v,\xi)\geq\pi\mathcal{E}_p.$$

Proof

Choose a sequence $\mathcal{S}$ of positive integers such that

$$\lim_{n\to\infty,n\in\mathcal{S}} n\lambda_{n,p}\left(v,\xi\right) = \liminf_{n\to\infty} n\lambda_{n,p}\left(v,\xi\right).$$

Because of the monotonicity of $\lambda_{n,p}$, and because no two successive integers belong to $\mathcal{Z}$, we may assume that $\mathcal{S}\cap\mathcal{Z}=\varnothing$. For each $m\geq 1$, we choose a subsequence $\mathcal{T}_m$ of $\mathcal{S}$ such that for $|j|\leq m$,

$$\lim_{n\to\infty,n\in\mathcal{T}_m} P_n^*\left(t_{jn}\right) = d_j, \tag{4.19}$$

with $d_0=1$. This is possible in view of Lemma 4.6(a). We can assume that

$$\mathcal{T}_1\supset\mathcal{T}_2\supset\mathcal{T}_3\supset ...$$

so that the $\{d_j\}$ are independent of m. Lemma 4.6 also gives, for all $m\geq 1$,

$$\sum_{|j|\leq m}|d_j|^p\leq\Lambda. \tag{4.20}$$

By Lemma 4.2(b), for any $r>0$, and given m,

$$\begin{aligned} &\lim_{n\to\infty,n\in\mathcal{T}}\int_{|x-\xi|\leq\frac{r\pi\sqrt{1-\xi^2}}{n}}\left|\sum_{|j|\leq m}P_n^*\left(t_{jn}\right)\ell_{jn}\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}} \\ &= \pi\int_{-r}^{r}\left|\sum_{|j|\leq m}d_jS\left(t-j\right)\right|^p dt. \end{aligned} \tag{4.21}$$

Together with Lemma 4.7 and the triangle inequality, this shows that, for $m\geq M_0\left(r\right)$,

$$\begin{aligned} &\liminf_{n\to\infty}\left(n\lambda_{n,p}\left(v,\xi\right)\right)^{1/p} \\ =\;& \lim_{n\to\infty,n\in\mathcal{T}}\left(n\int_{-1}^{1}\left|P_n^*\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}}\right)^{1/p} \\ \geq\;& \lim_{n\to\infty,n\in\mathcal{T}}\left(n\int_{|x-\xi|\leq\frac{\pi r\sqrt{1-\xi^2}}{n}}\left|P_n^*\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}}\right)^{1/p} \\ \geq\;& \left(\pi\int_{-r}^{r}\left|\sum_{|j|\leq m}d_jS\left(t-j\right)\right|^p dt\right)^{1/p}-\left(C\frac{r}{m}\right)^{1/p}. \end{aligned}$$

The proof may now be completed as in the proof of Lemma 2.7: we first let $m\to\infty$, and then $r\to\infty$. ■

We turn to the case $p\leq 1$ with an analogue of Lemma 2.8:

Lemma 4.9

Let $0<p\leq 1, r>0$. There exists $M_0=M_0\left(r\right)$ such that for $m\geq M_0$,

$$\int_{\xi-\frac{\pi r\sqrt{1-\xi^2}}{n}}^{\xi+\frac{\pi r\sqrt{1-\xi^2}}{n}}\left|\sum_{|j|>m}P_n^*\left(t_{jn}\right)\ell_{jn}\left(x\right)\right|^p\frac{dx}{\sqrt{1-x^2}}\leq C\frac{r}{m^pn}. \tag{4.22}$$

Here C is independent of r, n and m.

Proof

Now for $|x-\xi| \leq \frac{r\pi\sqrt{1-\xi^2}}{n}$, and $|j| > m$, we have the estimate (4.18) provided $m \geq M_0(r)$. Hence

$$\left|\sum_{|j|>m} P_n^*(t_{jn})\,\ell_{jn}(x)\right| \leq C\left(\sum_{|j|>m} |P_n^*(t_{jn})|^p \Lambda^{1-p}\right) m^{-1} \leq Cm^{-1}.$$

Then (4.22) follows. ■

Now we can prove:

Lemma 4.10

Let $0 < p \leq 1$. Then

$$\liminf_{n\to\infty} n\lambda_{n,p}(v,\xi) \geq \pi\mathcal{E}_p.$$

Proof

This is similar to Lemmas 2.9 and 4.8. Choose a sequence $\mathcal{S}$ of positive integers such that

$$\lim_{n\to\infty, n\in\mathcal{S}} n\lambda_{n,p}(v,\xi) = \liminf_{n\to\infty} n\lambda_{n,p}(v,\xi).$$

As in Lemma 4.8, we may assume that $\mathcal{S} \cap \mathcal{Z} = \varnothing$. For each $m \geq 1$, we choose a subsequence $\mathcal{T}_m$ of $\mathcal{S}$ such that for $|j| \leq m$, (4.19) holds. As usual, we can assume that

$$\mathcal{T}_1 \supset \mathcal{T}_2 \supset \mathcal{T}_3 \supset ...$$

so that the $\{d_j\}$ are independent of m. Again, (4.20) and (4.21) hold. The inequality $(x+y)^p \leq x^p + y^p, x, y \geq 0$ shows that for $m \geq M_0(r)$,

$$\begin{aligned}
&\liminf_{n\to\infty} n\lambda_{n,p}(v,\xi) \\
&= \lim_{n\to\infty, n\in\mathcal{T}} n\int_{-1}^{1} |P_n^*(x)|^p \frac{dx}{\sqrt{1-x^2}} \\
&\geq \lim_{n\to\infty, n\in\mathcal{T}} n\int_{|x-\xi|\leq\frac{r\pi\sqrt{1-\xi^2}}{n}} |P_n^*(x)|^p \frac{dx}{\sqrt{1-x^2}} \\
&\geq \pi\int_{-r}^{r}\left|\sum_{|j|\leq m}(-1)^j d_j S(t-j)\right|^p dt - C\frac{r}{m^p},
\end{aligned} \tag{4.23}$$

by Lemmas 4.2 and 4.9. The proof may now be completed as in the proof of Lemma 2.9: we let $m \to \infty$, and then $r \to \infty$. ■

Proof of Theorem 4.1(a)

For $p > 1$, we combine Lemmas 4.3 and 4.8. For $0 < p \leq 1$, we combine Lemmas 4.5 and 4.10. ■

Proof of Theorem 4.1(b)

From Lemma 4.6(b), there exists C_0 such that for all $n \geq 1$,

$$\|P_n^*\|_{L_\infty[-1,1]} \leq C_0.$$

Let

$$f_n(t) = P_n^*\left(\xi + \frac{\pi\sqrt{1-\xi^2}t}{n}\right).$$

We extend the bound on P_n^* to one on f_n using the equilibrium potential,

$$V(z) = \int_{-1}^{1} \log|z-s| \frac{ds}{\pi\sqrt{1-s^2}} = \log\left|\frac{z+\sqrt{z^2-1}}{2}\right|.$$

One form of Bernstein's growth lemma (cf. [25, p. 156]) gives for all $z \in \mathbb{C}$,

$$|P_n^*(z)| \le C_0 e^{n[V(z)+\log 2]}.$$

In particular, with $z = \xi + \frac{\pi t\sqrt{1-\xi^2}}{n}$, where $t \in \mathbb{C}$, some straightforward estimation gives

$$\begin{aligned}
& V\left(\xi + \frac{\pi t\sqrt{1-\xi^2}}{n}\right) + \log 2 \\
= \; & V\left(\xi + \frac{\pi t\sqrt{1-\xi^2}}{n}\right) - V\left(\xi + \frac{\pi(\operatorname{Re} t)\sqrt{1-\xi^2}}{n}\right) \\
= \; & \frac{1}{2}\int_{-1}^{1} \log\left(1 + \left(\frac{\frac{\pi \operatorname{Im} t\sqrt{1-\xi^2}}{n}}{\xi + \frac{\pi(\operatorname{Re} t)\sqrt{1-\xi^2}}{n} - s}\right)^2\right) \frac{ds}{\pi\sqrt{1-s^2}} \\
= \; & \frac{1}{2}\frac{1+o(1)}{\pi\sqrt{1-\xi^2}} \int_{\left|\xi + \frac{\pi(\operatorname{Re} t)\sqrt{1-\xi^2}}{n} - s\right| \le n^{-1/4}} \log\left(1 + \left(\frac{\frac{\pi \operatorname{Im} t\sqrt{1-\xi^2}}{n}}{\xi + \frac{\pi(\operatorname{Re} t)\sqrt{1-\xi^2}}{n} - s}\right)^2\right) ds + O\left(n^{-3/2}\right) \\
= \; & \frac{1}{2}\frac{1+o(1)}{\pi\sqrt{1-\xi^2}} \int_{|u| \le n^{-1/4}} \log\left(1 + \left(\frac{\pi \operatorname{Im} t\sqrt{1-\xi^2}}{nu}\right)^2\right) du + O\left(n^{-3/2}\right) \\
= \; & \frac{1}{2}\frac{|\operatorname{Im} t|}{n} \int_{-\infty}^{\infty} \log\left(1 + \frac{1}{x^2}\right) dx + o\left(\frac{1}{n}\right) \\
= \; & \pi\frac{|\operatorname{Im} t|}{n} + o\left(\frac{1}{n}\right),
\end{aligned}$$

cf. [8, p. 525, no. 4.222.1]. This holds uniformly for t in compact sets. Thus, uniformly for t in compact subsets of $\mathbb{C}$, and all n,

$$|f_n(t)| \le C_0 e^{\pi|\operatorname{Im} t| + o(1)}.$$

In particular, $\{f_n\}$ is uniformly bounded in compact sets, and hence is a normal family. Let $\mathcal{S}$ be an infinite sequence of positive integers, and $\mathcal{T}$ be a subsequence for which

$$\lim_{n\to\infty, n\in\mathcal{T}} f_n(z) = f(z)$$

uniformly for z in compact sets. As each $f_n(0) = 1$, so $f(0) = 1$. Also

$$|f(z)| \le C_0 e^{\pi|\operatorname{Im} z|},$$

so f is entire of exponential type at most π. Next, given $r>0$, we have

$$\begin{aligned}\pi\mathcal{E}_p &= \lim_{n\to\infty} n\lambda_{n,p}(v,\xi)\\ &\geq \lim_{n\to\infty,n\in\mathcal{T}} n\int_{\xi-\pi r\sqrt{1-\xi^2}/n}^{\xi+\pi r\sqrt{1-\xi^2}/n} |P_n^*(x)|^p \frac{dx}{\sqrt{1-x^2}}\\ &= \pi\lim_{n\to\infty,n\in\mathcal{T}}\int_{-r}^{r}|f_n(t)|^p\,dt\,(1+o(1))\\ &= \pi\int_{-r}^{r}|f(t)|^p\,dt.\end{aligned}$$

As $r>0$ is arbitrary, we have $f\in L_\pi^p$, and

$$\mathcal{E}_p \geq \int_{-\infty}^{\infty}|f(t)|^p\,dt.$$

If $p\geq 1$, uniqueness of the extremal function gives $f=f_p^*$, independent of the subsequence S. Then (4.2) follows. ■

5. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.4

Let $\varepsilon\in(0,1)$. We can choose $\delta\in\left(0,\frac{1}{2}\left(1-|\xi|^2\right)\right)$ such that for $0<h\leq\delta$, both

$$\begin{aligned}\int_{\xi-h}^{\xi+h}|\mu'(\xi)-\mu'_{ac}(t)|\,dt &\leq \varepsilon h;\\ \int_{\xi-h}^{\xi+h}d\mu_s(t) &\leq \varepsilon h.\end{aligned}\tag{5.1}$$

Recall here, our notation $\mu'(\xi)=\mu'_{ac}(\xi)$ if ξ is a Lebesgue point. Let $\eta\in\left(0,\frac{1}{8}\right)$. We shall use a polynomial U_n of degree $\leq n-1$ of the form

$$U_n = P^*_{n-[2\eta n]}R_nS_n \tag{5.2}$$

to estimate $\lambda_{n,p}(\mu,\xi)$. Here, for the given p, P_n^* is an abbreviation for $P^*_{n,p,\xi}$, the extremal polynomial for the Chebyshev weight on $[-1,1]$. In particular, we use Lemma 4.6(b), which shows that for all n,

$$\|P_n^*\|_{L_\infty(|z|=1)}\leq M_\infty<\infty. \tag{5.3}$$

We let

$$R_n(x)=\left(1-\left(\frac{x-\xi}{2}\right)^2\right)^{[\eta n/2]}, \tag{5.4}$$

a polynomial of degree $\leq[\eta n]$, with $R_n(\xi)=1$, $|R_n(x)|\leq 1$ for $x\in[-1,1]$, and

$$|R_n(x)|\leq c^n,\text{ for } x\in[-1,1]\text{ with } |x-\xi|\geq\delta. \tag{5.5}$$

Here $c\in(0,1)$ depends on η,δ, but is independent of n. Finally, we let

$$S_n(x)=\ell_{0,[\eta n/k]}(x)^k,$$

where k is a fixed positive integer chosen so that $kp > 1$, and $\ell_{0,[\eta n/k]}$ is the fundamental polynomial of Lagrange interpolation at the points $\{t_{jn}\}$ for the Chebyshev weight, taking value 1 at ξ, as in Section 4. Observe that $S_n(\xi) = 1$, $|S_n(x)| \leq 1 + C|x - \xi|$ for $x \in [-1,1]$, and (4.6) shows that for $x \in [-1,1]$,

$$|S_n(x)| \leq C \min\left\{1, \frac{1}{|\eta n (x-\xi)|^k}\right\}, \tag{5.6}$$

where C depends on k, but is independent of η, n and x. We have

$$\begin{aligned}
&\lambda_{n,p}(\mu,\xi) \\
\leq\ & \int_{-1}^{1} |U_n(x)|^p \, d\mu(x) \\
\leq\ & \left[\mu'(\xi)\sqrt{1-(|\xi|-\delta)^2}\right] \int_{\xi-\delta}^{\xi+\delta} \left|P^*_{n-[2\eta n]}(x)\right|^p \frac{dx}{\sqrt{1-x^2}}(1+o(1)) \\
& + M_\infty^p \left\{ \int_{\xi-\delta}^{\xi+\delta} |S_n(x)|^p \left|\mu'(\xi) - \mu'_{ac}(x)\right| dx + \int_{\xi-\delta}^{\xi+\delta} |S_n(x)|^p \, d\mu_s(x) \right\} \\
& + M_\infty^p C c^{np} \int_{[-1,1]\setminus[\xi-\delta,\xi+\delta]} d\mu(x),
\end{aligned} \tag{5.7}$$

by (5.3), (5.4) and (5.6). Now by Theorem 4.1,

$$\int_{\xi-\delta}^{\xi+\delta} \left|P^*_{n-[2\eta n]}(x)\right|^p \frac{dx}{\sqrt{1-x^2}} \leq \int_{-1}^{1} \left|P^*_{n-[2\eta n]}(x)\right|^p \frac{dx}{\sqrt{1-x^2}} \leq \frac{\pi \mathcal{E}_p + o(1)}{n - [2\eta n]}. \tag{5.8}$$

Next, by (5.6) and then (5.1),

$$\begin{aligned}
& \int_{\xi-\delta}^{\xi+\delta} |S_n(x)|^p \left|\mu'(\xi) - \mu'_{ac}(x)\right| dx + \int_{\xi-\delta}^{\xi+\delta} |S_n(x)|^p \, d\mu_s(x) \\
\leq\ & \sum_{j=0}^{\infty} \left(C \min\left\{1, \frac{1}{|\eta n \delta / 2^j|^k}\right\}\right)^p \left\{ \int_{\delta/2^j \geq |x-\xi| \geq \delta/2^{j+1}} \left|\mu'(\xi) - \mu'_{ac}(x)\right| dx + \int_{\delta/2^j \geq |x-\xi| \geq \delta/2^{j+1}} d\mu_s(x) \right\} \\
\leq\ & C \sum_{j=0}^{\infty} \min\left\{1, \frac{1}{|\eta n \delta/2^j|^{kp}}\right\} \varepsilon \delta 2^{-j}.
\end{aligned}$$

We continue this as

$$\begin{aligned}
\leq\ & C\varepsilon\delta (\eta n \delta)^{-kp} \sum_{0 \leq j < \log_2(\eta n \delta)} 2^{j(kp-1)} + C\varepsilon\delta \sum_{j \geq \log_2(\eta n\delta)} 2^{-j} \\
\leq\ & C \frac{\varepsilon}{\eta n},
\end{aligned}$$

by some simple calculations. Here C is independent of $\varepsilon, \delta, \eta$ and n. Combining this and (5.7), (5.8), gives

$$\limsup_{n\to\infty} n \lambda_{n,p}(\mu,\xi) \leq \frac{\pi \mathcal{E}_p \mu'(\xi) \left[\sqrt{1-\xi^2} + C\delta\right]}{1 - 2\eta} + C\frac{\varepsilon}{\eta}. \tag{5.9}$$

Here the left-hand side is independent of $\varepsilon, \eta, \delta$, while C is independent of $\varepsilon, \eta, \delta$. Moreover, ε and η are independent of each other. We can let first ε, δ and then $\eta \to 0+$ to obtain the result. ■

For later use, we record more on the polynomials U_n:

Lemma 5.1
Let $\eta \in \left(0, \frac{1}{8}\right)$ and $k \geq 1$ be chosen so that $kp > 1$. Let U_n be the polynomial of degree $\leq n-1$, defined by (5.2). Then
(a)

$$\limsup_{n\to\infty} n \int_{-1}^{1} |U_n|^p \, d\mu \leq \frac{\pi \mathcal{E}_p \mu'(\xi) \sqrt{1-\xi^2}}{1-2\eta} + C\eta, \tag{5.10}$$

where C is independent of η and n.
(b) Uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty} U_n \left(\xi + \frac{\pi\sqrt{1-\xi^2} z}{n} \right) = f_p^* \left(z \left(1 - 2\eta \right) \right) S \left(z\eta/k \right)^k. \tag{5.11}$$

Proof
(a) This was proved at (5.9), if we choose $\varepsilon = \eta^2$ there. Note that there $\{U_n\}$ and C are independent of ε.
(b) Firstly with $m = m(n) = n - [2\eta n]$,

$$\begin{aligned} P_{n-[2\eta n]}^* \left(\xi + \frac{\pi\sqrt{1-\xi^2} z}{n} \right) &= P_m^* \left(\xi + \frac{\pi\sqrt{1-\xi^2} z}{m} \frac{m}{n} \right) \\ &= f_p^* \left(z \left(1 - 2\eta \right) \right) + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane, by the uniform convergence in Theorem 4.1(b). Next, R_n of (5.4) satisfies

$$\begin{aligned} R_n \left(\xi + \frac{\pi\sqrt{1-\xi^2} z}{n} \right) &= \left(1 - \left(\frac{\pi\sqrt{1-\xi^2} z}{2n} \right)^2 \right)^{[\eta n/2]} \\ &= 1 + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane. Finally, with $p = p(n) = [\eta n/k]$

$$\begin{aligned} S_n \left(e^{2\pi i z/n} \right) &= \ell_{0p} \left(\xi + \frac{\pi\sqrt{1-\xi^2} z}{n} \right)^k \\ &= S \left(z \frac{p}{n} \right)^k + o(1) \\ &= S \left(z\eta/k \right)^k + o(1), \end{aligned}$$

uniformly for z in compact subsets of the plane, by Lemma 4.2. Combining the three asymptotics gives (5.11). ■

Proof of Theorem 1.3(a)
If $\mu'(\xi) = 0$, the result already follows from Theorem 1.4. So let us assume that $\mu'_{ac}(\xi) = \mu'(\xi) > 0$. It suffices to prove that

$$\liminf_{n\to\infty} n \lambda_{n,p} \left(\mu, \xi \right) \geq \pi \mathcal{E}_p \mu'(\xi). \tag{5.12}$$

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$, and choose $\delta > 0$ such that

$$\mu'_{ac}(x)\sqrt{1-x^2} \geq \mu'_{ac}(\xi)\sqrt{1-\xi^2}/(1+\varepsilon), \text{ for } |x-\xi| \leq \delta. \tag{5.13}$$

This is possible as μ'_{ac} is lower semi-continuous at ξ. Let $\eta \in \left(0, \frac{1}{8}\right)$ and R_n be the polynomial of degree of $\leq [\eta n]$, defined by (5.4). Let $P^{\#}_{n-[\eta n]}$ be the extremal polynomial $P^{*}_{n-[\eta n],p,\xi}$ for the measure μ, so that $P^{\#}_{n-[\eta n]}(\xi) = 1$ and

$$\int_{-1}^{1} \left|P^{\#}_{n-[\eta n]}(x)\right|^p d\mu(x) = \lambda_{n-[\eta n],p}(\mu, \xi). \tag{5.14}$$

We have

$$\begin{aligned}
&\lambda_{n,p}(v,\xi) \\
\leq\ & \int_{-1}^{1} \left|P^{\#}_{n-[\eta n]}(x) R_n(x)\right|^p \frac{dx}{\sqrt{1-x^2}} \\
\leq\ & \frac{1+\varepsilon}{\mu'(\xi)\sqrt{1-\xi^2}} \int_{\xi-\delta}^{\xi+\delta} \left|P^{\#}_{n-[\eta n]}(x)\right|^p \mu'_{ac}(x)\,dx + \left\|P^{\#}_{n-[\eta n]}\right\|^p_{L_\infty[-1,1]} c^{np} \int_{|x-\xi|\geq\delta} dx,
\end{aligned}$$

(5.15)

by (5.5) and (5.13). Since μ is regular,

$$\left\|P^{\#}_{n-[\eta n]}\right\|^p_{L_\infty[-1,1]} \leq (1+o(1))^n \int_{-1}^{1} \left|P^{\#}_{n-[\eta n]}(x)\right|^p d\mu(x).$$

Combining this with (5.15), gives

$$\begin{aligned}
\lambda_{n,p}(v,\xi) &\leq \left(\int_{-1}^{1} \left|P^{\#}_{n-[\eta n]}(x)\right|^p d\mu(x)\right) \left\{\frac{1+\varepsilon}{\mu'(\xi)\sqrt{1-\xi^2}} + 2\left(c^p(1+o(1))\right)^n\right\} \\
&\leq \lambda_{n-[\eta n],p}(\mu,\xi) \left\{\frac{1+\varepsilon}{\mu'(\xi)\sqrt{1-\xi^2}} + o(1)\right\}.
\end{aligned}$$

Using Theorem 4.1, we obtain

$$\liminf_{n\to\infty} n\lambda_{n-[\eta n],p}(\mu,\xi) \geq \pi\mathcal{E}_p \frac{\mu'(\xi)\sqrt{1-\xi^2}}{1+\varepsilon}.$$

We can now complete the proof as in that of Theorem 1.1(a). ■

Proof of Theorem 1.3(b)

Let $P^*_n = P^*_{n,p,\xi}$ for the measure μ. Our strategy will be to construct polynomials $P^{\#}_n$ of degree $\leq n-1$, with $P^{\#}_n(\xi) = 1$, satisfying

$$\lim_{n\to\infty} n \int_{-1}^{1} \left|P^{\#}_n(x)\right|^p d\mu(x) = \pi\mathcal{E}_p\mu'(\xi) \tag{5.16}$$

and, uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty} P^{\#}_n\left(\xi + \frac{\pi\sqrt{1-\xi^2}z}{n}\right) = f^*_p(z), \tag{5.17}$$

where f_p^* is the extremal function defined in Section 1. We shall also show that given any infinite sequence of positive integers, it contains a subsequence $\mathcal{T}$ such that uniformly for z in compact subsets of the plane,

$$\lim_{n\to\infty, n\in\mathcal{T}} P_n^*\left(\xi+\frac{\pi\sqrt{1-\xi^2}z}{n}\right)=g(z) \tag{5.18}$$

for some entire function g. Once we have these, we can use Clarkson's inequalities to complete the proof as in the proof of Theorem 1.1(b).

Proof of (5.18)

Choose $\rho, C>0$, such that $\mu_{ac}'\geq C$ in $[\xi-\rho,\xi+\rho]$. Let $0<\tau<\rho$. By standard estimates for Christoffel functions,

$$\|P_n^*\|_{L_\infty[\xi-\tau,\xi+\tau]}^p\leq Cn\int_{\xi-\rho}^{\xi+\rho}|P_n^*(x)|^p\,dx\leq Cn\int_{\xi-\rho}^{\xi+\rho}|P_n^*(x)|^p\,d\mu(x)\leq C.$$

A standard application of the Bernstein growth lemma for polynomials (cf. the proof of Theorem 4.1(b)), gives for large enough n, and all $|z|\leq r$,

$$\left|P_n^*\left(\xi+\frac{\pi\sqrt{1-\xi^2}z}{n}\right)\right|\leq C_1e^{C_2|z|}.$$

Thus $\left\{P_n^*\left(\xi+\frac{\pi\sqrt{1-\xi^2}z}{n}\right)\right\}$ is a normal family in z, and (5.18) follows.

Proof of (5.16) and (5.17)

These follow from Lemma 5.1 exactly as in the proof of Theorem 1.1(b). ■

6. Uniqueness of the Extremal Function for $p=1$

We prove:

Lemma 6.1

The function f_1^* is unique.

Proof

(I) We first show that all zeros of f_1^* are real. Now, because of its minimal norm, f_1^* must have real coefficients, so if they exist, complex zeros will occur in conjugate pairs. If

$$f_1^*(x)=\left\{(x-a)^2+b^2\right\}g(x),$$

where g is entire, and $b>0, a\in\mathbb{R}$, we could form

$$h(x)=\left\{(x-a)^2+b^2-\varepsilon x^2\right\}g(x),$$

which also has $h(0)=1, h\in L_\pi^1$, and has smaller L_1 norm for small enough positive ε. Indeed, a straightforward calculation shows that $(x-a)^2+b^2>\varepsilon x^2$ for all real x iff ε is so small that $b^2>\frac{a^2\varepsilon}{1-\varepsilon}$.

(II) Next, we show that if g^* is another extremal function, then $f_1^*g^*\geq 0$ throughout $\mathbb{R}$. Indeed, if $f_1^*g^*<0$ on some interval, we have $|f_1^*+g^*|<|f_1^*|+|g^*|$ there, and then it easily follows that $\frac{1}{2}(f_1^*+g^*)\in L_\pi^1$, takes value 1 at 0, and has smaller L_1 norm.

(III) In view of (I), (II), all zeros of $f_1^*g^*$ are of even order, so we may define a branch of $h = \sqrt{f_1^* g^*}$, such that $h(0) = 1$, and h is entire of type $\leq \pi$. By the arithmetic-geometric inequality,

$$|h| \leq \frac{1}{2}\left(|f_1^*| + |g^*|\right),$$

with strict inequality unless $|f_1^*| = |g^*|$. Since by (II), f_1^*, g^* have the same sign, h will have smaller L_1 norm unless $|f_1^*| = |g^*|$ identically, that is, unless $f_1^* = g^*$ identically. ■

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