ORTHOGONAL POLYNOMIALS FOR WEIGHTS CLOSE TO INDETERMINACY

E. LEVIN [1], D. S. LUBINSKY[2]

Abstract. We obtain estimates for Christoffel functions and orthogonal polynomials for even weights $W : \mathbb{R} \to [0, \infty)$ that are 'close' to indeterminate weights. Our main example is $\exp\left(- |x| (\log |x|)^{\beta}\right)$, with β real, possibly modified near 0, but our results also apply to $\exp\left(- |x|^{\alpha} (\log |x|)^{\beta}\right), \alpha < 1$. These types of weights exhibit interesting properties largely because they are either indeterminate, or are close to the border between determinacy and indeterminacy in the classical moment problem.

1. [1]Introduction and Results

Let $Q : \mathbb{R} \to [0, \infty)$ be even, and $W = \exp(-Q)$, with all power moments

$$\int_{\mathbb{R}} x^j W^2 (x)\, dx,$$

$j = 0, 1, 2, ...$ finite. Then we may define orthonormal polynomials

$$p_n (x) = p_n \left(W^2, x\right) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int_{\mathbb{R}} p_n p_m W^2 = \delta_{mn}.$$

The study of orthonormal polynomials for such weights, and related applications, has been a major theme in analysis in the twentieth century.

Typical examples are the Freud type weights

$$W_{\alpha} (x) = \exp\left(- |x|^{\alpha}\right), \alpha > 0. \tag{1.1}$$

For $\alpha \geq 1$, these weights are determinate, that is they are the only non-negative function W solving the moment problem

Date: August 5, 2005.

[1]Research of second author supported by NSF grant DMS 0400446

$$\int_{\mathbb{R}} x^j W^2(x)\,dx = \int_{\mathbb{R}} x^j W_\alpha^2(x)\,dx, j \geq 0.$$

For $\alpha < 1$, there are other solutions to the moment problem, that is the corresponding moment problem is indeterminate [5], [20]. So the weight $\exp(-|x|)$ sits on the boundary between determinacy and indeterminacy. This boundary extends to issues such as density of weighted polynomials (the so-called Bernstein approximation problem), Jackson type theorems, and other issues [1], [5], [13], [15], [17]. From the point of view of this article, however, it is the difficulty in analyzing their orthogonal polynomials, that forms our focus.

Orthogonal polynomials for weights $\exp(-2Q)$, where Q grows at least as fast as $|x|^\alpha$, some $\alpha > 1$, have been analyzed in many works [6], [10], [15], [17]. Weights like $\exp(-|x|^\alpha), \alpha \leq 1$, have been analyzed in [1], [2], [4], [7], [9], [6], [18]. In particular, it is known that for each $\alpha > 0$, the orthonormal polynomials $p_n(W_\alpha^2, x)$ admit the bound

$$\left|p_n(W_\alpha^2, x)\right| W_\alpha(x) \leq C_1 n^{-1/2\alpha}, |x| \leq C_2 n^{1/\alpha}, \tag{1.2}$$

for some C_1 and C_2 independent of n. Such bounds are useful in studying weighted approximation, numerical quadrature, Lagrange interpolation... . The case $\alpha \leq 1$ is much more difficult to analyze than the case $\alpha > 1$, partly because $Q(x) = |x|^\alpha$ is strictly convex only for $\alpha > 1$. Convexity of Q is an essential part of one of the traditional approaches to Freud weights. The authors [9] established a bound like (1.2) for part of the range $|x| \leq C_2 n^{1/\alpha}$ when $\alpha \leq 1$, but the full bound was proved only recently [6], as part of sharper asymptotics derived using Riemann-Hilbert methods.

In this paper, we study orthonormal polynomials and Christoffel functions for weights that behave roughly like $\exp(-|x|^\alpha)$, some $\alpha \leq 1$. Some of our motivation comes from weighted approximation - in the special case of $\exp(-|x|)$, bounds on orthonormal polynomials are useful in establishing Jackson theorems [14]. One of our key examples is the case

$$Q(x) = |x| (\log |x|)^\beta, |x| \geq 2, \tag{1.3}$$

with any real β. (We omit a neighborhood of 0, because of the singularity of $\log |x|$ at 0, redefining it suitably in that neighborhood).

In analysis of Freud weights $W = e^{-Q}$, an important descriptive quantity is the Mhaskar-Rakhmanov-Saff number a_n, the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}}, n > 0. \tag{1.4}$$

One of its features is the Mhaskar-Saff identity [15], [16], [19]

$$\| PW \|_{L_\infty(\mathbb{R})} = \| PW \|_{L_\infty[-a_n,a_n]},$$

valid for polynomials P of degree $\leq n$. In the case $Q(x) = |x|^\alpha$,

$$a_n = C_\alpha n^{1/\alpha},$$

with C_α a constant admitting a representation in terms of gamma functions.

Following is our class of weights:

Definition 1.1
Let $Q : \mathbb{R} \to \mathbb{R}$ be continuous with
(a) Q'' existing and $xQ'(x)$ positive and increasing in $(0, \infty)$.
(b)

$$\liminf_{x\to\infty} \frac{(xQ'(x))'}{Q'(x)} > 0. \tag{1.5}$$

(c)

$$\limsup_{x\to\infty} \frac{(xQ'(x))'}{Q'(x)} \leq 1. \tag{1.6}$$

Then we write $W = \exp(-Q) \in \mathcal{SF}$.

We write $W \in \mathcal{SF}^+$ if in addition for some $0 < A \leq 1 \leq B$,

$$A \leq \frac{(xQ'(x))'}{Q'(x)} \leq B, x \in (0, \infty). \tag{1.7}$$

Remarks
(a) Consider

$$Q(x) = |x|^\alpha (\log(|x|))^\beta, |x| \geq L;$$

where $0 < \alpha \leq 1, \beta \in \mathbb{R}$, some large enough L. This Q satisifes both (1.5) and (1.6), but clearly there is a problem for $|x| \leq 1$. We could define it to be constant in $[-L, L]$ but this violates the first condition. In such a case, we shall find it convenient to modify Q near 0, see below. For large enough L, and $\beta > -1$,

$$Q(x) = |x|^\alpha \left(\log\left(L^2 + x^2\right)\right)^\beta$$

does satisfy (1.5) through (1.7). For $\beta = -1$, the lower bound in (1.7) fails for x close to L, irrespective of how large is L.
(b) We use $\mathcal{SF}$ or $\mathcal{SF}^+$ as an abbreviation for slow Freud, indicating that the exponent Q grows slowly to ∞. The bound in (1.5) ensures that Q grows as $x \to \infty$ ate least as fast as some positive power of x,

while that in (1.6) ensures that it grows not much faster than x.
(c) The assumption that $xQ'(x)$ is increasing in $(0,\infty)$ guarantees that a_n exists for all n. For many purposes, however, we only need it and (1.7), or some analogue, for large x. In particular, this is true for estimates on Christoffel functions. When (1.7) fails for small $|x|$, one simply replaces Q for small $|x|$ by a quartic polynomial S as follows: choose L such that for $x \geq L$, and some $A \leq 1$,

$$0 < A \leq \frac{(xQ'(x))'}{Q'(x)} \leq 2$$

and determine

$$S(x) = ax^4 + bx^2 + c$$

by the relations

$$S^{(k)}(L) = Q^{(k)}(L), k = 0,1,2.$$

A little calculation shows that

$$a = \frac{LQ''(L) - Q'(L)}{8L^3}; b = \frac{3Q'(L) - LQ''(L)}{4L}.$$

The condition (1.5) for $x = L$ shows that $a < 0$, $b > 0$, while for $x \in [0,L]$,

$$\frac{1}{x}S'(x) = 4ax^2 + 2b \geq 4aL^2 + b = \frac{1}{4L}(xQ'(x))'_{|x=L} > 0,$$

so $S'(x) > 0$ for $x \in [0,L]$. Next,

$$\frac{(xS'(x))'}{S'(x)} = 2\frac{4ax^2+b}{2ax^2+b}$$

is decreasing in $(0,L]$. For $x = L$, the left-hand side coincides with the value of $\frac{(xQ'(x))'}{Q'(x)}_{|x=L}$, which is $\geq A$. An upper bound for $\frac{(xS'(x))'}{S'(x)}$ is 2, the value at 0. Defining

$$\widetilde{Q}(x) := \begin{cases} S(x), & |x| \leq L \\ Q(x), & |x| > L \end{cases},$$

we then obtain a new weight $\widetilde{W} = \exp\left(-\widetilde{Q}\right)$ such that

$$0 < A \leq \frac{\left(x\widetilde{Q}'(x)\right)'}{\widetilde{Q}'(x)} \leq 2, x \in (0,\infty)$$

so $\widetilde{W} \in \mathcal{SF}^+$. Moreover, $W/\widetilde{W}$ is bounded above and below by positive constants and

$$\int_0^1 \frac{\widetilde{Q}'(x)}{x}dx < \infty.$$

In analyzing orthogonal polynomials, and in other contexts, one needs the Christoffel functions

$$\lambda_n\left(W^2,x\right)=\inf_{\deg(P)<n}\frac{\int_{-\infty}^{\infty}(PW)^2}{P^2(x)}.$$

It is well known that

$$\lambda_n\left(W^2,x\right)=1/\sum_{j=0}^{n-1}p_j^2\left(W^2,x\right).$$

Lower bounds for $\lambda_n(W^2,x)$ for weights including those we consider in this paper were established in [8], building on many previous works. There, however, the main focus was Freud weights whose exponent Q grows at least as fast as $|x|^\alpha$, some $\alpha>1$. For $W_\alpha,\alpha\leq 1$, corresponding upper bounds were established in [9]. For W_1, upper and lower bounds had been established earlier by Freud, Giroux and Rahman [4]. Here we shall find upper bounds for all the weights in $\mathcal{SF}$ to match the already established lower bounds. The description of these involves the functions

$$\rho_n(x)=\int_{\max\{1,|x|\}}^{a_n}\frac{Q'(s)}{s}ds, x\in[-a_n,a_n] \tag{1.8}$$

and

$$\varphi_n(x)=\frac{a_n}{n}\left(\max\left\{n^{-2/3},1-\frac{|x|}{a_n}\right\}\right)^{-1/2}, x\in\mathbb{R}. \tag{1.9}$$

We combine them as

$$\Lambda_n(x)=\begin{cases}1/\rho_n(x), & |x|\leq\frac{1}{2}a_n;\\ \varphi_n(x), & |x|>\frac{1}{2}a_n\end{cases}. \tag{1.10}$$

For sequences $(x_n),(y_n)$ of non-zero real numbers, we write

$$x_n\sim y_n$$

if for some $C_1,C_2>0$,

$$C_1\leq x_n/y_n\leq C_2, n\geq 1.$$

Similar notation is used for sequences and sequences of functions. Throughout, $C,C_1,C_2,...$ denote positive constants independent of n,x and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences.

Theorem 1.2

Let $W \in \mathcal{SF}$, and $\varepsilon \in (0,1)\,, L > 0$.
(a) Uniformly for $n \geq 1$ and $|x| \leq a_n(1+Ln^{-2/3})$,

$$\lambda_n\left(W^2,x\right)W^{-2}(x) \sim \Lambda_n(x)\,. \tag{1.11}$$

(b) Moreover, for some $C > 0$ and all $|x| \geq \varepsilon a_n$,

$$\lambda_n\left(W^2,x\right)W^{-2}(x) \geq C\varphi_n(x)\,. \tag{1.12}$$

Remarks
(a) It follows easily from the technical estimates of Section 3 that

$$\rho_n(x) \sim \int_{\max\{1,|x|\}}^{Q^{[-1]}(Cn)} \frac{Q'(s)}{s}ds = \int_{Q(\max\{1,|x|\})}^{Cn} \frac{dt}{Q^{[-1]}(t)},$$

where $Q^{[-1]}$ denotes the inverse function of Q. It is then easy to recognize the lower bounds implicit in (1.11) as following from Theorem 1.7 in [8, pp. 468-9]. So all we have to obtain is an upper bound for $\lambda_n(W^2,x)$, and it is in the proof of those that the main novelty of this paper lies. In [9], we treated the weights $\exp(-|x|^{\alpha})\,, \alpha \leq 1$ and used canonical products; here we avoid this by directly using polynomials that arise from discretising a potential, in the explicit formula for Christoffel functions for Bernstein-Szegö weights.
(b) In the overlap region $[\varepsilon a_n, \eta a_n]$, any $0 < \varepsilon < \eta < 1$, (see Lemma 3.2)

$$\frac{1}{\rho_n(x)} \sim \varphi_n(x) \sim \frac{a_n}{n}$$

so the two functions defining Λ_n agree there.

Corollary 1.3
Let $\varepsilon \in (0,1)$, $\beta \in \mathbb{R}$ and

$$Q(x) = |x|(\log|x|)^{\beta}\,,$$

for large enough $|x|$, with extension to $[-L,L]$ as described above. Then

$$a_n \sim \frac{n}{(\log n)^{\beta}}.$$

Moreover,
(a) If $\beta > -1$,

$$\lambda_n\left(W^2,x\right)W^{-2}(x) \sim \frac{1}{\log^{\beta} n}\frac{1}{\log\frac{a_n}{1+|x|}}, \quad |x| \leq \varepsilon a_n\,. \tag{1.13}$$

(b) If $\beta = -1$,

$$\lambda_n\left(W^2,x\right)W^{-2}(x) \sim \frac{1}{\log\left(\frac{\log a_n}{\log(1+|x|)}\right)}, \quad |x| \leq \varepsilon a_n\,. \tag{1.14}$$

(c) If $\beta < -1$,

$$\lambda_n \left(W^2, x\right) W^{-2}(x) \sim \frac{\log n}{\log^{\beta+1}(1+|x|)} \frac{1}{\log \frac{a_n}{1+|x|}}, \quad |x| \leq \varepsilon a_n \ . \tag{1.15}$$

For all three cases, and for $n \geq 1$ *and* $\varepsilon a_n \leq |x| \leq a_n$,

$$\lambda_n \left(W^2, x\right) W^{-2}(x) \sim \frac{1}{(\log n)^\beta} \left(\max \left\{ n^{-2/3}, 1 - \frac{|x|}{a_n} \right\} \right)^{1/2}. \tag{1.16}$$

The bounds on $\lambda_n(W^2, x)$ in Theorem 1.2 allow us to estimate spacing between successive zeros of $p_n(W^2, x)$: let us denote the zeros of $p_n(W^2, x)$ by

$$-\infty < x_{nn} < x_{n-1,n} < x_{n-2,n} < \ldots < x_{2n} < x_{1n} < \infty.$$

Corollary 1.4

Let $W \in \mathcal{SF}$, *and* $\varepsilon \in (0,1)$. *Then for some* n_0 *and* $n \geq n_0$,

$$|1 - x_{1n}/a_n| \leq C n^{-2/3} \tag{1.17}$$

and for $2 \leq j \leq n-1$,

$$x_{j-1,n} - x_{j+1,n} \sim \Lambda_n(x_{jn}). \tag{1.18}$$

Finally we state some bounds on orthogonal polynomials:

Theorem 1.5

Let $W \in \mathcal{SF}$.

(a) Let $\varepsilon \in (0,1), L > 0$. *Then for* $\varepsilon a_n \leq |x| \leq a_n \left(1 + L n^{-2/3}\right)$,

$$\left|p_n\left(W^2, x\right)\right| W(x) \leq C a_n^{-1/2} \left(\max \left\{ n^{-2/3}, 1 - \frac{|x|}{a_n} \right\} \right)^{-1/2}. \tag{1.19}$$

(b) If in addition, $W \in \mathcal{SF}^+$ *and* $Q'(x)$ *and* $xQ''(x)$ *are bounded in* $(0, C]$ *for each* $C > 0$, *while*

$$\lim_{x \to \infty} \frac{\left(xQ'(x)\right)'}{Q'(x)} = 1, \tag{1.20}$$

and

$$\int_1^\infty \frac{Q'(x)}{x} dx = \infty \tag{1.21}$$

then

$$\|p_n W\|_{L_\infty(\mathbb{R})} \sim a_n^{-1/2} n^{1/6}. \tag{1.22}$$

Remarks

(a) We expect the bound (1.19) to hold for all $|x| \leq a_n$. For the special case $Q(x) = |x|^{\alpha}, \alpha \leq 1$, this follows from the deep asymptotics of Kriecherbauer and McLaughlin [6].
(b) Note that the conditions in (b) are are satisfied if

$$Q(x) = |x| (\log (L + |x|))^{\beta}, \beta > -1,$$

with L large enough (depending on β). If $\beta \leq -1$, then (1.21) fails.

This paper is organised as follows: in Section 2, we give most of the proof of Theorem 1.2, deferring some technical details till later. In Section 3, we present technical estimates related to Q, equilibrium measures and the like. In Section 4, we construct polynomials that approximate W^{-1}, and in Section 5, we prove Corollary 1.3. in Section 6 Corollary 1.4 and in Section 7, Theorem 1.5.

2. Proof of Theorem 1.2

As after Definition 1.1, we can assume that $W \in \mathcal{SF}^+$, since the modified weight $\widetilde{W}$ there has $\lambda_n(W^2, x) \sim \lambda_n\left(\widetilde{W}^2, x\right)$, uniformly in n and x. Moreover, it is easily seen that if a_n and $\widetilde{a}_n$ denote the Mhaskar-Rakhmanov-Saff numbers for W and $\widetilde{W}$ respectively, then $\widetilde{a}_n = a_{n+O(1)}$. Recall from the remark after Theorem 1.2 that we only need the upper bounds for λ_n. We establish these in this section, based on auxiliary results to be established in Sections 3 and 4. It is shown there (see Lemma 4.2) that for $n \geq n_0$, there exist polynomials R_{2n} of degree $2n$, such that uniformly for $n \geq n_0$, and $t \in [-1, 1]$,

$$R_{2n}(t) W^2(a_n t) \sim 1, t \in [-1, 1]. \tag{2.1}$$

This and the restricted range inequality (Lemma 3.4 below) yield for $x \in [-a_n, a_n]$,

$$\begin{aligned}
\lambda_{n+1}\left(W^2, x\right) W^{-2}(x) &= \inf_{P \in \mathcal{P}_n} \frac{\int_{\mathbb{R}} (PW)^2(s)\, ds}{(PW)^2(x)} \\
&\leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{-a_n}^{a_n} (PW)^2(s)\, ds}{(PW)^2(x)} \\
&\leq C \inf_{P \in \mathcal{P}_n} \frac{\int_{-a_n}^{a_n} P^2(s) R_{2n}^{-1}\left(\frac{s}{a_n}\right) ds}{P^2(x) R_{2n}^{-1}\left(\frac{x}{a_n}\right)} \\
&= C a_n \inf_{P \in \mathcal{P}_n} \frac{\int_{-1}^{1} P^2(t) R_{2n}^{-1}(t)\, dt}{P^2\left(\frac{x}{a_n}\right)} R_{2n}\left(\frac{x}{a_n}\right).
\end{aligned}$$

If we now define a weight w_n on $[-1,1]$ by

$$w_n(t) = \left(1-t^2\right)^{-1/2} R_{2n}^{-1}(t), t \in (-1,1),$$

then we deduce from the above that

$$\lambda_{n+1}\left(W^2, x\right) W^{-2}(x) \leq C a_n \lambda_{n+1}\left(w_n, \frac{x}{a_n}\right) R_{2n}\left(\frac{x}{a_n}\right). \tag{2.2}$$

Since $R_{2n} > 0$ in $[-1,1]$, we may write for $z \in \mathbb{C} \setminus \{0\}$,

$$R_{2n}\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right) = h_{2n}(z) \overline{h_{2n}\left(\frac{1}{\bar{z}}\right)},$$

where h_{2n} is a polynomial of degree $2n$, having all its zeros in $|z| > 1$. It is known [21, (13.4.10), p. 320] that if

$$t = \cos\theta, z = e^{i\theta}, \theta \in (0,\pi), \tag{2.3}$$

then

$$\begin{aligned} & \pi\lambda_{n+1}^{-1}(w_n, t)\left(1-t^2\right)^{1/2} w_n(t) \\ = \; & n + \frac{1}{2} - \operatorname{Re}\left\{\frac{z h_{2n}'(z)}{h_{2n}(z)}\right\} + (2\sin\theta)^{-1} \operatorname{Im}\left\{z^{2n+1}\frac{\overline{h_{2n}(z)}}{h_{2n}(z)}\right\}. \end{aligned}$$

$$= n - \operatorname{Re}\left\{\frac{z h_{2n}'(z)}{h_{2n}(z)}\right\} + O(1), \tag{2.4}$$

provided $|t| \leq \frac{1}{2}$, say. We show in Lemma 4.4 that for some $C_1, C_2 > 0, \varepsilon \in \left(0, \frac{1}{2}\right)$, all $|t| \leq \varepsilon$, and all $n \geq 1$,

$$-\operatorname{Re}\left\{\frac{z h_{2n}'(z)}{h_{2n}(z)}\right\} \geq C_1 a_n \rho_n(a_n t) - C_2 n. \tag{2.5}$$

Here it is crucial that C_2 does not depend on ε. Moreover, we show in Lemma 3.3 that if ε is small enough, then for $|t| \leq \varepsilon$,

$$a_n \rho_n(a_n t)/n \geq 2C_2/C_1.$$

Setting $t = x/a_n$, we deduce from (2.2) to (2.5) that for some $\varepsilon > 0$, and $|x| \leq \varepsilon a_n$,

$$\lambda_{n+1}\left(W^2, x\right) W^{-2}(x) \leq C\rho_n(x) = C\Lambda_n(x).$$

So we have the required upper bound implicit in (1.9) for some $\varepsilon < 1$. Since for any $0 < \varepsilon < \eta < 1$,

$$\rho_n(x) \sim \frac{1}{\varphi_n(x)} \sim \frac{a_n}{n}, \varepsilon a_n \leq |x| \leq \eta a_n,$$

(see Lemma 3.2) it remains to establish the upper bound implicit in (1.11). This was done in [8, pp. 515-517], under the additional assumption that the constant in A in (1.5) is larger than 1. This assumption was however used for only one purpose - to show that

$$\lambda_{m,\infty}(W,x) = \inf_{P\in\mathcal{P}_{m-1}} \frac{\|PW\|_{L_\infty(\mathbb{R})}}{|P(x)|} \le CW(x), |x| \le a_n\left(1+Ln^{-2/3}\right),$$

with the appropriate choice of m there. This relation in our case follows from Lemma 4.3. We may repeat word for word the proof in [8, pp. 515-517] and this completes the proof. ■

3. Auxiliary Results

Throughout this section, unless otherwise specified, we assume that $W \in \mathcal{SF}^+$.

Lemma 3.1
(a)

$$t^A \le \frac{tQ'(tx)}{Q'(x)} \le t^B, x>0, t\ge 1. \tag{3.1}$$

(b) If $0<a<b<\infty$, *then uniformly for* $x\in[a,b]$ *and* $n\ge 1$,

$$a_n x Q'(a_n x) \sim Q(a_n x) \sim n. \tag{3.2}$$

(c)

$$a_1 n^{1/B} \le a_n \le a_1 n^{1/A}. \tag{3.3}$$

(d) For $\frac{1}{2} \le \frac{m}{n} \le 2$,

$$\left|1-\frac{a_m}{a_n}\right| \sim \left|1-\frac{m}{n}\right|. \tag{3.4}$$

(e) Let $L>1$. *There exists* $C_L>0$ *such that for* $y\ge x\ge C_L$,

$$\frac{Q'(y)}{Q'(x)} \le \left(\frac{y}{x}\right)^{1/L}. \tag{3.5}$$

Proof
(a) - (d) See Lemma 3.1 in [7, p. 1071] and Lemma 5.2(b), (c) in [8, p. 478].
(e) By (1.6) in Definition 1.1, there exists C_L such that

$$\frac{(sQ'(s))'}{Q'(s)} \le 1+\frac{1}{L}, s\ge C_L.$$

Then

$$\begin{aligned} \frac{yQ'(y)}{xQ'(x)} &= \exp\left(\int_x^y \frac{(sQ'(s))'}{sQ'(s)}ds\right) \\ &\leq \exp\left(\int_x^y \left(1+\frac{1}{L}\right)\frac{1}{s}ds\right) \\ &= \left(\frac{y}{x}\right)^{1+\frac{1}{L}}. \end{aligned}$$

■

In the sequel, we need the equilibrium measures $\{\mu_n\}$ associated with the external field Q. Our condition that $xQ'(x)$ is increasing implies that the support of μ_n is the interval $[-a_n, a_n]$. Moreover, $d\mu_n(x) = \sigma_n(x)\,dx$, where the density σ_n is even and continuous in $(0, a_n]$ [10, Chapter 2], [19]. After our modification, it is continuous at 0 as well (??). We shall also use the contracted density σ_n^*, defined by

$$\sigma_n^*(t) = \frac{a_n}{n}\sigma_n(a_n t), t \in [-1, 1]. \tag{3.6}$$

It satisfies

$$\int_{-1}^{1} \sigma_n^* = 1$$

and it is given by [7, (2.10), p. 1070], [19, (3.21), p. 226]

$$\sigma_n^*(t) = \frac{2}{\pi^2}\int_0^1 \frac{\sqrt{1-t^2}}{\sqrt{1-s^2}}\frac{a_n sQ'(a_n s) - a_n tQ'(a_n t)}{n(s^2-t^2)}ds. \tag{3.7}$$

Lemma 3.2
Let $0 < \varepsilon < \eta < 1$. Then uniformly for $n \geq n_0$,
(a)

$$\sigma_n^*(t) \sim \frac{a_n}{n}\int_t^1 \frac{Q'(a_n s)}{s}ds = \rho_n(a_n t), t \in [0, \eta], \tag{3.8}$$

(b)

$$\sigma_n^*(t) \sim \sigma_n^*\left(\frac{L}{a_n}\right) \geq C, t \in \left[0, \frac{L}{a_n}\right]. \tag{3.9}$$

(c)

$$\sigma_n^*(t) \sim \sqrt{1-t^2}, t \in [\eta, 1). \tag{3.10}$$

(d)

$$\sigma_n^*(t) \sim 1, t \in [\varepsilon, \eta]. \tag{3.11}$$

(e)

$$\sigma_{n+1}(x) \sim \sigma_n(x) \sim \rho_n(x), |x| \leq \eta a_n. \tag{3.12}$$

(f)

$$\sigma_n(x) \sim 1/\varphi_n(x), \varepsilon a_n \leq |x| \leq a_n\left(1-\varepsilon n^{-2/3}\right). \tag{3.13}$$

(g)

$$\sigma_n(x) \sim \frac{1}{\varphi_n(x)} \sim \frac{a_n}{n}, \varepsilon a_n \leq |x| \leq \eta a_n. \tag{3.14}$$

Proof

(a), (d) The upper bound implicit in (3.8) was proved in [7, Lemma 4.1, p. 1074]. There the upper limit in the integral was chosen to be 2, but this is inessential, since for any fixed $0 < a < b$, we have by (3.2),

$$\frac{a_n}{n}\int_a^b \frac{Q'(a_n s)}{s} ds \sim \int_a^b \frac{1}{s^2} ds \sim 1. \tag{3.15}$$

Note that (3.15) also gives (3.11). Hence, in proving the lower bound implicit in (3.8), we may assume that $t < \eta < \frac{1}{4}$. Then we obtain from the formula (3.7) for σ_n^* :

$$\sigma_n^*(t) \geq C\frac{a_n}{n}\int_t^1 \Delta \frac{ds}{s},$$

where

$$\Delta = \frac{a_n s Q'(a_n s) - a_n t Q'(a_n t)}{a_n s - a_n t}.$$

It remains to show that

$$\Delta \geq CQ'(a_n s).$$

Indeed if $s \in [2t, 1]$, then (recall that $uQ'(u)$ is increasing),

$$\begin{aligned}
\Delta &\geq \frac{sQ'(a_n s) - \frac{s}{2}Q'\left(a_n \frac{s}{2}\right)}{s} \\
&= Q'(a_n s) - \frac{1}{2}Q'\left(a_n \frac{s}{2}\right) \\
&\geq \left(2^{A-1} - 2^{-1}\right) Q'\left(a_n \frac{s}{2}\right) \\
&\geq \left(2^{A-1} - 2^{-1}\right) 2^{1-B} Q'(a_n s)
\end{aligned}$$

where we used (3.1). For $s \in [t, 2t]$, we observe that

$$\Delta = (uQ'(u))'$$

for some u in $[a_n t, 2a_n t]$. Hence $u \sim a_n s$, and (1.5), (3.1) yield

$$\Delta \geq AQ'(u) \geq CQ'(a_n s).$$

So we have proved (3.8) and (3.11).
(b) From (a), for $t \in \left[0, \frac{L}{a_n}\right]$,

$$C_1 \frac{a_n}{n} \int_0^1 \frac{Q'(a_n s)}{s} ds \geq \sigma_n^*(t) \geq C_2 \frac{a_n}{n} \int_{L/a_n}^1 \frac{Q'(a_n s)}{s} ds.$$

We must show that the integral on the left $\sim$ that on the right. This follows easily from the fact that for any $D > 0$,

$$\frac{a_n}{n} \int_0^{D/a_n} \frac{Q'(a_n s)}{s} ds = \frac{a_n}{n} \int_0^D \frac{Q'(u)}{u} du \sim \frac{a_n}{n}.$$

Finally the lower bound

$$\sigma_n^* \left(\frac{L}{a_n} \right) \geq C$$

follows from (3.8) and (3.11).
(c) The relation (3.10) was established in [8, Lemma 7.2, pp. 486-487].
(e) Next, the second $\sim$ relation in (3.12) follows immediately from (3.8) and the relation (3.6) between σ_n^* and σ_n. The first $\sim$ relation is equivalent to $\sigma_{n+1}^*(t) \sim \sigma_n^*(t), t \in [0, \eta]$, which follows from (3.8) (substitute $s = \frac{a_{n+1}}{a_n} u$ and use (3.1)).
(f), (g) Finally (3.13) is a consequence of (3.10) and the definition of φ_n, and then (3.14) is trivial. ■

Lemma 3.3
(a) Let $K > 0$. Then there exists $\varepsilon \in (0, 1)$ and $n_0 = n_0(\varepsilon)$ such that for $n \geq n_0$,

$$\frac{a_n}{n} \rho_n(a_n \varepsilon) = \frac{a_n}{n} \int_\varepsilon^1 \frac{Q'(a_n t)}{t} dt \geq K. \tag{3.16}$$

(b) Uniformly for $n \geq n_0$ and $t \in \left[0, \frac{1}{2} a_n\right]$,

$$\rho_n \left(\frac{t}{2} \right) \sim \rho_n(t). \tag{3.17}$$

(c) Uniformly for $n \geq n_0, x \in \mathbb{R}$ and $m \leq 4n^{1/3}$,

$$\Lambda_n(x) \sim \Lambda_{n-m}(x). \tag{3.18}$$

(d)

$$\rho_n(0) \leq \left\{ \begin{array}{rl} C n a_n^{-A}, & A < 1 \\ C n a_n^{-1} \log n, & A = 1 \end{array} \right. . \tag{3.19}$$

Proof

(a) Suppose $L \geq 1$ to be chosen as later, and C_L is as in Lemma 3.1(d). Let $\varepsilon \in (0,1)$ with $a_n\varepsilon \geq C_L$. For $t \in (0,1)$

$$\frac{Q'(a_n)}{Q'(a_nt)} \leq \left(\frac{1}{t}\right)^{\frac{1}{L}}.$$

Then

$$\begin{aligned} & \frac{a_n}{n}\int_\varepsilon^1 \frac{Q'(a_nt)}{t}dt \\ \geq\ & \frac{a_nQ'(a_n)}{n}\int_\varepsilon^1 t^{-1+\frac{1}{L}}\frac{dt}{t} \\ \geq\ & C^*L\left(1-\varepsilon^{\frac{1}{L}}\right), \end{aligned} \tag{3.20}$$

by (3.2). Here it is crucial that C^* is independent of ε,L and n. We now choose ε so small that for the given K,

$$\frac{3}{4}C^*\log\frac{1}{\varepsilon} \geq K$$

and then choose L so large that

$$\frac{|\log\varepsilon|}{L} \leq \frac{1}{2}.$$

Finally we choose n_0 such that for $n \geq n_0$, $a_n\varepsilon \geq C_L$. Then using the inequality

$$1-e^{-u} \geq \frac{3}{4}u, u \in \left[0,\frac{1}{2}\right],$$

we see that

$$1-\varepsilon^{\frac{1}{L}} = 1-\exp\left(-\frac{|\log\varepsilon|}{L}\right) \geq \frac{3}{4}\frac{|\log\varepsilon|}{L}.$$

We can then continue (3.20) for $n \geq n_0$, as

$$\frac{a_n}{n}\int_\varepsilon^1 \frac{Q'(a_nt)}{t}dt \geq C^*\frac{3}{4}|\log\varepsilon| \geq K.$$

(b)

$$\begin{aligned}\rho_n\left(\frac{t}{2}\right)-\rho_n(t) &= \int_{\max\left\{1,\frac{t}{2}\right\}}^{\max\{1,t\}}\frac{Q'(s)}{s}ds\\ &= \int_{\max\{2,t\}}^{\max\{2,2t\}}\frac{Q'\left(\frac{u}{2}\right)}{u}du\\ &\leq 2^{1-A}\int_{\max\{2,t\}}^{\max\{2,2t\}}\frac{Q'(u)}{u}du\leq 2^{1-A}\rho_n(t),\end{aligned}$$

by (3.1) of Lemma 3.1 and as $2t\leq a_n$. Then as ρ_n is decreasing,

$$\rho_n\left(\frac{t}{2}\right)\leq\rho_n(t)\leq\left(1+2^{1-A}\right)\rho_n\left(\frac{t}{2}\right).$$

(c) If $|x|\leq\frac{1}{2}a_{n-m}$,

$$\begin{aligned}\Lambda_{n-m}^{-1}(x)-\Lambda_n^{-1}(x) &= \rho_{n-m}(x)-\rho_n(x)\\ &= \int_{a_{n-m}}^{a_n}\frac{Q'(s)}{s}ds\\ &\leq CQ'(a_n)\log\left(\frac{a_n}{a_{n-m}}\right)\\ &\leq C\frac{n}{a_n}\frac{m}{n}=o\left(\frac{n}{a_n}\right).\end{aligned}$$

In the last line, we used (3.4). Since $\Lambda_n^{-1}(x)=\rho_n(x)\geq C\frac{n}{a_n}$, we obtain for $n\geq n_0$,

$$\Lambda_{n-m}^{-1}(x)-\Lambda_n^{-1}(x)\leq C\Lambda_n^{-1}(x).$$

Thus

$$\Lambda_n^{-1}(x)\leq\Lambda_{n-m}^{-1}(x)\leq(1+C)\Lambda_n^{-1}(x).$$

If $\frac{1}{2}a_{n-m}\leq|x|\leq\frac{1}{2}a_n$, $\Lambda_{n-m}(x)\sim\Lambda_n(x)\sim\frac{a_n}{n}$. If $|x|\geq\frac{1}{2}a_n$, then we need to show

$$\varphi_{n-m}(x)\sim\varphi_n(x)$$

or equivalently,

$$\max\left\{1-\frac{|x|}{a_{n-m}},n^{-2/3}\right\}\sim\max\left\{1-\frac{|x|}{a_n},n^{-2/3}\right\}. \tag{3.21}$$

We see that if $|x| \leq a_{n-m}\left(1-n^{-2/3}\right)$,

$$\begin{aligned} 0 &\leq \frac{1-\frac{|x|}{a_n}}{1-\frac{|x|}{a_{n-m}}}-1=\frac{\frac{|x|}{a_{n-m}}\left(1-\frac{a_{n-m}}{a_n}\right)}{1-\frac{|x|}{a_{n-m}}} \\ &\leq C\frac{m}{n\left(1-\frac{|x|}{a_{n-m}}\right)} \leq C, \end{aligned}$$

recall that $m/n = O\left(n^{2/3}\right)$. Then (3.21) follows for this range of x. The remaining ranges are easily handled with the aid of (3.4).

(d) This is an easy consequence of (3.1), and (3.2): for example if $A < 1$,

$$\rho_n(0) = \int_1^{a_n} \frac{Q'(s)}{s} ds \leq Q'(a_n)\, a_n^{1-A} \int_1^{a_n} s^{A-2} ds.$$

■

Next we state two lemmas that apply to the larger class of weights $\mathcal{SF}$. First, a lemma relating Mhaskar-Rakhmanov-Saff numbers for W and its modified weight W :

Lemma 3.4

Let $W \in \mathcal{SF}$ and $\widetilde{W}$ be the modified weight as after Definition 1.1 Let a_n and $\widetilde{a}_n$ denote the Mhaskar-Rakhmanov-Saff numbers for W and $\widetilde{W}$ respectively. Then

$$a_n = \widetilde{a}_{n+O(1/a_n)} = \widetilde{a}_n + O\left(\frac{1}{n}\right). \tag{3.22}$$

Proof

Since $tQ'(t)$ and $t\widetilde{Q}'(t)$ are increasing, we see that

$$\int_0^{1/a_n} \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt, \int_0^{1/a_n} \frac{a_n t \widetilde{Q}'(a_n t)}{\sqrt{1-t^2}} dt = O\left(\frac{1}{a_n}\right).$$

Then as $Q'(a_n t) = \widetilde{Q}'(a_n t)$ for $|t| \geq C/a_n$,

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt = \frac{2}{\pi}\int_0^1 \frac{a_n t \widetilde{Q}'(a_n t)}{\sqrt{1-t^2}} dt + O(1/a_n).$$

Uniqueness of the Mhaskar-Rakhmanov-Saff number $\widetilde{a}_n$ for $\widetilde{Q}$ then gives the first relation in (3.22), and (3.4) applied to $\widetilde{a}_{n+O(1/a_n)}$ and $\widetilde{a}_n$ then gives the second. ■

We note that the two sets of Mhaskar-Rakhmanov-Saff numbers are so close that they can be interchanged for all purposes, at least for large enough n. This has the consequence that estimates like (3.2) to (3.5) and (3.16) to (3.19) can be applied to $W \in \mathcal{SF}$ for large enough x or n. Finally, a restricted range inequality that we use in estimating the largest zero of p_n :

Lemma 3.5
Let $W \in \mathcal{SF}$, $\varepsilon > 0$ and $0 < p \leq \infty$.
(a) There exist $K > 0$ and n_0 such that for $n \geq n_0$ and polynomials P of degree $\leq n$,

$$\|PW\|_{L_p\left(|x|\geq a_n\left(1+Kn^{-2/3}\right)\right)} \leq \varepsilon \|PW\|_{L_p\left(|x|\leq a_n\left(1+Kn^{-2/3}\right)\right)}. \tag{3.23}$$

(b) Let $K > 0$. There exist $C, n_0 > 0$ such that for $n \geq n_0$ and polynomials P of degree $\leq n$,

$$\|PW\|_{L_p(\mathbb{R})} \leq C \|PW\|_{L_p\left(|x|\leq a_n\left(1-Kn^{-2/3}\right)\right)}. \tag{3.24}$$

Proof
(a) Let $\widetilde{W}$ be the usual modified weight. Let P be a polynomial of degree $\leq n$. In [10, Lemma 4.4, p. 99] we showed (with $\Omega = n, t = n+\frac{2}{p}$ there) that

$$\left\|P\widetilde{W}e^{-U_{n+2/p}}\right\|_{L_p(\mathbb{R}\setminus[-\widetilde{a}_{n+2/p},\widetilde{a}_{n+2/p}]} \leq \left\|P\widetilde{W}\right\|_{L_p[-\widetilde{a}_{n+2/p},\widetilde{a}_{n+2/p}]}, \tag{3.25}$$

where

$$U_t(x) = -\left[V^{\mu_t}(x) + \widetilde{Q}(x) - c_t\right]$$

and $V^{\mu_t}(x)$ is an equilibrium potential, while c_t is an equilibrium constant. While Q was assumed convex there, the proof goes through without any changes for $\widetilde{W}$. In fact, for a class of weights containing $\widetilde{W}$, Mhaskar proved a very similar inequality in [15, p. 142, Theorem 6.2.4]. In [10, p. 101, Lemma 4.5], it is shown that

$$U_{n+2/p}(x) \leq -C\left(\frac{\frac{x}{\widetilde{a}_{n+2/p}} - 1}{n^{2/3}}\right)^{3/2}, x \in \left[\widetilde{a}_{n+2/p}, \widetilde{a}_{2n}\right],$$

with C independent of n, x. Again it was assume there that Q is convex, but the proof goes through. In fact with different notation, this estimate was proved in [8, p. 485, (7.14)] and in [15, p. 148, Corollary 6.2.7] for a class of weights containing $\widetilde{W}$. Then we see that for some C independent of K,

$$-U_{n+2/p}(x) \geq CK^{3/2}, |x| \geq \widetilde{a}_n\left(1 + Kn^{-2/3}\right)$$

Now we substitute this in (3.25) and use $W = \widetilde{W}$ outside a finite interval, while $W/\widetilde{W} \leq C_1$ on the real line. We obtain

$$\|PW\|_{L_p\left(|x|\geq\widetilde{a}_n\left(1+Kn^{-2/3}\right)\right)} \leq C_1 \exp\left(-CK^{3/2}\right) \|PW\|_{L_p[-\widetilde{a}_{n+2/p},\widetilde{a}_{n+2/p}]}.$$

As C_1 and C are independent of K, we can ensure that by choosing K large enough, $C_1 \exp\left(-CK^{3/2}\right)$ is as small as we please. Applying Lemma 3.5, and (3.4) on $\widetilde{a}_{n+2/p}, \widetilde{a}_n$ then gives the result.
(b) This is a special case of Theorem 1.8 in [10, p. 469], at least when $W \in \mathcal{SF}^+$. When $W \in \mathcal{SF}$, we modify W as per usual, and this only increases the size of the constant in (3.24). ■

4. Weighted Polynomials

Our next task is to construct polynomials that in some sense approximate W^{-1}. Throughout we assume that $W \in \mathcal{SF}$. The method we used is standard, based on the discretisation of the potential

$$V^{\sigma_n^*}(z) = \int_{-1}^{1} \log|z-t|^{-1} \sigma_n^*(t)\, dt. \tag{4.1}$$

For a given n, we choose

$$-1 = t_0 < t_1 < t_2 < ... < t_n = 1 \tag{4.2}$$

by the conditions

$$\int_{t_{k-1}}^{t_k} \sigma_n^* = \frac{1}{n}, 0 \leq k \leq n-1, \tag{4.3}$$

and let

$$I_k = [t_{k-1}, t_k] \text{ and } |I_k| = t_k - t_{k-1}.$$

Lemma 4.1
Uniformly for $n \geq 1, 2 \leq k \leq n-1,$ *and* $t \in I_k$,

$$n\sigma_n^*(t)\,|I_k| \sim 1. \tag{4.4}$$

For $k = 1$ *and* n*, this relation persists if we omit an interval of length* $\varepsilon\,|I_k|$ *(with* $\varepsilon \in (0,1)$ *fixed) at the endpoint* ± 1.
Proof
We first consider $I_k = [t_{k-1}, t_k] \subset [-1,1]$ with $|t_{k-1}| \leq \frac{1}{2}$. We split this into two cases:
Case I: $t_k \leq 2t_{k-1}$ **and** $t_{k-1} \leq \frac{1}{2}$
As ρ_n is decreasing, (3.17) gives for $t \in I_k$,

$$\rho_n(a_n t_k) \leq \rho_n(a_n t_{k-1}) \leq \rho_n\left(a_n \frac{t_k}{2}\right) \sim \rho_n(a_n t_k).$$

Then

$$\rho_n(a_n t) \sim \rho_n(a_n t_k), t \in I_k$$

and hence from (3.8),

$$\sigma_n^*(t) \sim \sigma_n(t_k), t \in I_k,$$

giving (4.4).

Case II: $t_k > 2t_{k-1}$ **and** $t_{k-1} \leq \frac{1}{2}$

Then

$$\begin{aligned} \frac{1}{n} &= \int_{t_{k-1}}^{t_k} \sigma_n^* \geq \int_{t_k/2}^{t_k} \sigma_n^* \\ &\sim \frac{a_n}{n} t_k \rho_n(a_n t_k), \end{aligned} \tag{4.5}$$

in view of (3.8). But

$$\rho_n(a_n t_k) \geq \int_{t_k}^{2t_k} \frac{Q'(a_n s)}{s} ds \geq C Q'(a_n t_k) \log 2,$$

by (3.1). Then we can continue (4.5) as

$$C \geq a_n t_k Q'(a_n t_k).$$

Since $xQ'(x) \sim Q(x)$ increases to ∞ as $x \to \infty$, this forces $a_n t_k \leq C_1$. Then $t_{k-1}, t_k \in \left[0, \frac{C_1}{a_n}\right]$, so (3.9) gives

$$\sigma_n(t) \sim \sigma_n\left(\frac{C_1}{a_n}\right), t \in I_k,$$

and again (4.4) follows.

Finally, we consider $t_{k-1} > \frac{1}{2}$. In this case, we use that from (3.10), uniformly in n,

$$\sigma_n^*(t) \sim \sqrt{1-t}, t \in \left[\frac{1}{2}, 1\right]$$

to deduce that

$$\begin{aligned} \frac{1}{n} &= \int_{t_{k-1}}^{t_k} \sigma_n^* \\ &\sim (1-t_{k-1})^{3/2} - (1-t_k)^{3/2} \end{aligned}$$

so

$$\left(\frac{1-t_{k-1}}{1-t_k}\right)^{3/2} \leq 1 + \frac{C}{n(1-t_k)^{3/2}} \leq C,$$

since for $k = n$, we obtain,

$$\frac{1}{n} \sim (1 - t_n)^{3/2}.$$

Then

$$1 - t_{k-1} \sim 1 - t_k$$

and hence

$$\sigma_n^* (t_{k-1}) \sim \sigma_n^* (t_k) \sim \sigma_n^* (t), t \in I_k.$$

■

Lemma 4.2
There exists n_0 and for $n \geq n_0$ polynomials R_{2n} of degree $2n$ such that uniformly for $t \in [-1, 1]$ and $n \geq n_0$,

$$R_{2n} (t) W^2 (a_n t) \sim 1, t \in [-1, 1]. \tag{4.6}$$

Proof
Since $t_k \in I_k \cap I_{k-1}$, we see from (4.4) that uniformly in k, n,

$$|I_k| \sim |I_{k-1}|. \tag{4.7}$$

Choose 'weight points' $\xi_k \in I_k$ by

$$\int_{I_k} (t - \xi_k) \sigma_n^* (t) dt = 0,$$

$1 \leq k \leq n$. We shall see that for some real constant κ_n, the complex polynomials

$$S_n (t) = \kappa_n \prod_{k=1}^{n} (t - \xi_k + i\eta_k)$$

satisfy

$$|S_n (t)| W (a_n t) \geq 1, t \in [-1, 1], n \geq 1, \tag{4.8}$$

and

$$|S_n (t)| W (a_n t) \leq C, t \in \mathbb{R}, n \geq 1. \tag{4.9}$$

Once these properties are verified, it remains to set

$$R_{2n} (t) = |S_n (t)|^2 = \kappa_n^2 \prod_{k=1}^{n} \left((t - \xi_k)^2 + \eta_k^2 \right)^2. \tag{4.10}$$

To establish these, we proceed exactly as in [10, Chapter 7]. The method of discretisation that we use has a long history. In its most powerful variant, it is due to Totik [22]. The basic idea is that if we define the potential

$$V^{\sigma_n}(z) = \int_{-a_n}^{a_n} \log \frac{1}{|z-t|} \sigma_n(t)\, dt,$$

then

$$V^{\sigma_n}(x) + Q(x) = c_n, x \in [-a_n, a_n],$$

where c_n is a constant. After a transformation $t = a_n s$, $x = a_n u$, we obtain

$$nV^{\sigma_n^*}(u) + W(a_n u) = c_n^*, u \in [-1, 1],$$

where

$$V^{\sigma_n^*}(z) = \int_{-1}^{1} \log \frac{1}{|z-s|} \sigma_n^*(s)\, ds.$$

We choose $\kappa_n = e^{-c_n}$ in S_n and see that

$$\begin{aligned} & \log |S_n(u) W(a_n u)| \\ = & \sum_{k=1}^{n} \log |u - (\xi_k + i\eta_k)| - n \int_{-1}^{1} \log |u-s| \sigma_n^*(s)\, ds \\ = & n \sum_{k=1}^{n} \Gamma_{n,k}(u), \end{aligned}$$

where

$$\Gamma_{n,k}(u) := n \int_{I_k} \log \left| \frac{u - (\xi_k + i\eta_k)}{u-s} \right| \sigma_n^*(s)\, ds$$

and we have used (4.3). Exactly as in Lemma 7.6 in [10, p. 175] with $d_n = 2$ there, we see that

$$\Gamma_{n,j}(u) \geq 0, u \in \mathbb{R}.$$

Next, recall the properties (4.4), (4.7) and (as shown in Lemma 4.1),

$$1 - t_n^2 \sim 1 - t_1^2 \sim n^{-2/3}.$$

These coincide with those of Lemma 7.16 in [10, pp. 194-195]. Suppose that $u \in [-1, 1]$ and we choose k_0 such that $u \in I_{k_0}$. Proceeding as in Lemma 7.20 there, with $d_n = 2$, we see that for $|k - k_0| < 4$,

$$\Gamma_{n,k}(u) \leq C.$$

With the aid of the same Lemma 7.16, we can proceed as in [10, Section 7.6] to show that if $u \in I_{k_0}$, then

$$\sum_{k:|k-k_k|\geq 4} \Gamma_{n,k}(u) \leq C.$$

Altogether, we obtain that

$$0 \leq \Gamma_n(u) = \sum_{k=0}^{n} \Gamma_{n,k}(u) \leq C.$$

This means that (4.8), (4.9) are satisfied, as required. ■

Lemma 4.3

There exists n_0 and for $n \geq n_0$ polynomials P_n of degree$\leq n$ such that uniformly in n, x

$$P_n(x) W(x) \sim 1, x \in [-a_n, a_n]. \tag{4.11}$$

Proof

Assume that n is even and construct R_{2m} as in Lemma 4.2, with $m = n/2$ and with the weight $W^{1/2}$ instread of W. Then

$$P_n(x) = R_{2(n/2)}(x/a_n)$$

will do the job. See [10, pp. 177-178.]. ■

Lemma 4.4

Let R_{2n} be as in Lemma 4.2, and let h_{2n} be the polynomial of degree $2n$, with all zeros in $|z| > 1$, and such that

$$R_{2n}\left(\frac{1}{2}\left(z + \frac{1}{z}\right)\right) = h_{2n}(z)\overline{h_{2n}\left(\frac{1}{\overline{z}}\right)}. \tag{4.12}$$

Let

$$t = \cos\theta, z = e^{i\theta}, \theta \in (0, \pi). \tag{4.13}$$

There exist n_0 and $\varepsilon > 0$ such that for $n \geq n_0$ and $\left|\theta - \frac{\pi}{2}\right| \leq \varepsilon$,

$$\begin{aligned} -\operatorname{Re}\left(z\frac{h_{2n}'(z)}{h_{2n}(z)}\right) &\geq C_1 n \sigma_n^*(t) - C_2 n \\ &\geq C_3 a_n \rho_n(a_n t) - C_2 n. \end{aligned} \tag{4.14}$$

Proof

By (4.10), R_{2n} has zeros at $\xi_k \pm i\eta_k$, $1 \leq k \leq n$. Hence h_{2n} can be written in the form

$$h(z) = h_{2n}(z) = c_n \prod_{k=1}^{n} (z - z_k)(z - \overline{z_k})$$

where $z_k = x_k + iy_k, 1 \leq k \leq n$ are uniquely determined by the requirements

$$\frac{1}{2}\left(z_k + \frac{1}{z_k}\right) = \xi_k + i\eta_k \text{ or } \xi_k - i\eta_k; \tag{4.15}$$

$$|z_k| > 1, \operatorname{Im}(z_k) > 0. \tag{4.16}$$

Note that this implies

$$|\xi_k| = \frac{1}{2}|x_k|\left(1+\frac{1}{|z_k|^2}\right) < |x_k|.$$

Now

$$-\operatorname{Re}\frac{zh'(z)}{h(z)} = \sum_{k=1}^{n}\operatorname{Re}\frac{-z}{z-z_k} + \sum_{k=1}^{n}\operatorname{Re}\frac{-z}{z-\overline{z_k}}.$$

Assuming that $\left|\theta - \frac{\pi}{2}\right| < \varepsilon$, some small ε, we see that

$$\operatorname{Im}(z-\overline{z_k}) = \sin\theta + y_k \geq \sin\theta \geq \frac{1}{2}$$

while

$$|\operatorname{Re}(z-z_k)| = |\cos(\theta-\theta_k)| \geq |x_k| - |\cos\theta| > |\xi_k| - \varepsilon.$$

Therefore

$$-\operatorname{Re}\frac{zh'(z)}{h(z)} \geq -O(n) + \sum\nolimits_k' \operatorname{Re}\frac{-z}{z-z_k},$$

where the summation in $\sum_k'$ is over those k for which $|\xi_k| < 2\varepsilon$. For such k, we may write

$$\xi_k = \cos\theta_k, |\theta-\theta_k| < c\varepsilon.$$

Now recall that $\xi_k \in I_k$ and $\eta_k = 2|I_k|$. Since σ_n^* is bounded below, uniformly in n, in any compact subinterval of $(-1,1)$, we deduce from Lemma 4.1 that

$$|I_k| = O\left(n^{-1}\right)$$

uniformly for $I_k \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. Therefore $\eta_k = O(n^{-1})$ uniformly for all k in $\sum_k'$. Next, we claim that for all such k and for n large enough, $z_k = x_k + iy_k$ is given by

$$x_k = \cos\theta_k + \eta_k\cot\theta_k + O\left(\eta_k^3\right); \tag{4.17}$$

$$y_k = \sin\theta_k + \eta_k + \frac{1}{2\sin^3\theta_k}\eta_k^2 + O\left(\eta_k^3\right), \tag{4.18}$$

with the order terms uniform in k. Assuming these are true, we continue as follows: Write

$$\operatorname{Re}\frac{-z}{z-z_k} = \operatorname{Re}\frac{z\overline{z_k}-1}{|z-z_k|^2} = \frac{x_k\cos\theta + y_k\sin\theta - 1}{(x_k-\cos\theta)^2 + (y_k-\sin\theta)^2}.$$

By (4.17) and (4.18), we obtain for n large enough,

$$\begin{aligned} & x_k \cos\theta + y_k \sin\theta - 1 \\ = \; & \cos(\theta-\theta_k) - 1 + \eta_k \frac{\cos(\theta-\theta_k)}{\sin\theta_k} + O\left(\eta_k^2\right) \\ \geq \; & \frac{1}{2}\eta_k - \frac{1}{2}(\theta-\theta_k)^2 . \end{aligned}$$

(Recall that θ and θ_k are both close to $\frac{\pi}{2}$). Similarly we obtain, after simple manipulations,

$$\begin{aligned} & (x_k-\cos\theta)^2 + (y_k-\sin\theta)^2 \\ = \; & 2(1-\cos(\theta-\theta_k)) + 2\eta_k \frac{1-\cos(\theta-\theta_k)}{\sin\theta} + 2\frac{\eta_k^2}{\sin^2\theta_k} \\ & + \text{smaller terms} \\ \sim \; & (\theta-\theta_k)^2 + \eta_k^2, \end{aligned}$$

provided θ, θ_k are close enough to $\frac{\pi}{2}$ and n is large enough. Therefore

$$\begin{aligned} \sum\nolimits_k' \; \geq \; & C\sum\nolimits_k' \frac{\eta_k}{(\theta-\theta_k)^2+\eta_k^2} - C_1 \sum\nolimits_k' \frac{(\theta-\theta_k)^2}{(\theta-\theta_k)^2+\eta_k^2} \\ = \; & C\sum\nolimits_k' \frac{\eta_k}{(\theta-\theta_k)^2+\eta_k^2} - O(n). \end{aligned}$$

Now let $|t|$ be small enough, so that $t = \cos\theta \in I_k$, for some index k that appears in $\sum_k'$. Since

$$|\theta-\theta_k| \sim |\cos(\theta-\theta_k)| = |t-\xi_k| < |I_k|,$$

we see that the corresponding term of $\sum_k'$ contributes at least $C/|I_k|$ which is $\sim n\sigma_n^*(t)$, by Lemma 4.1. Other terms in $\sum_k'$ are positive, so we obtain

$$-\operatorname{Re}\frac{zh'(z)}{h(z)} \geq C_1 n\sigma_n^*(t) - O(n),$$

as required. The second relation in (4.14) follows from (3.8).

It remains to establish (4.17) and (4.18). Let us consider the conditions (4.15), (4.16) with the index k omitted, for simplicity. Then we have from (4.15),

$$z = \cos\theta \pm i\eta + \sqrt{(\cos\theta \pm i\eta)^2 - 1}.$$

On choosing the + sign, we continue this as

$$z = \cos\theta + i\eta + i\sin\theta\sqrt{1 - 2i\eta\frac{\cos\theta}{\sin^2\theta} + \frac{\eta^2}{\sin^2\theta}}.$$

Since θ is close to $\frac{\pi}{2}$ and η is small, we may continue this as

$$\begin{aligned} z &= \cos\theta + i\eta + i\sin\theta\left(1 - i\eta\frac{\cos\theta}{\sin^2\theta} + \frac{\eta^2}{2\sin^2\theta} + \frac{\eta^2\cos^2\theta}{\sin^4\theta} + O\left(\eta^3\right)\right) \\ &= \left(\cos\theta + \eta\cot\theta\right) + O\left(\eta^3\right) + i\left(\sin\theta + \eta + \frac{\eta^2}{2\sin^3\theta} + O\left(\eta^3\right)\right), \end{aligned}$$

giving (4.17) and (4.18). For $\eta > 0$ small enough, this also gives (4.18). ■

5. Proof of Corollary 1.3

Proof of Corollary 1.3

It is easy to check that $Q(x) = |x|(\log|x|)^\beta$ satisfies the conditions of Definition 1.1 for $|x| \geq L$ and some L. Since it does not affect $\lambda_n(W^2, x)$ up to $\sim$, we modify W as after Definition 1.1. We must estimate the function appearing in the estimate (1.11) of the Christoffel functions, namely

$$\Lambda_n(x)^{-1} = \rho_n(x) = \int_{\max\{1,|x|\}}^{a_n} \frac{Q'(s)}{s}ds. \tag{5.1}$$

Since given $L > 1$, we have

$$Q'(s) \sim (\log s)^\beta, s \geq L,$$

and in particular (recall (3.2))

$$n \sim a_n Q'(a_n) \sim a_n (\log a_n)^\beta,$$

whence

$$a_n \sim \frac{n}{(\log n)^\beta}. \tag{5.2}$$

We deduce that for $\frac{1}{2}a_n \geq |x| \geq L$,

$$\begin{aligned} \rho_n(x) &\sim \int_{|x|}^{a_n} \frac{(\log s)^\beta}{s}ds \\ &\sim \begin{cases} \left|(\log a_n)^{\beta+1} - (\log|x|)^{\beta+1}\right|, & \beta \neq -1 \\ \log\log a_n - \log\log|x|, & \beta = -1 \end{cases}. \end{aligned}$$

If $\beta > -1$, we use

$$1 - u^{\beta+1} \sim 1 - u, u \in (0, 1),$$

so that

$$\begin{aligned}
&\left|(\log a_n)^{\beta+1} - (\log|x|)^{\beta+1}\right| \\
= &\ (\log a_n)^{\beta+1}\left|1-\left(\frac{\log|x|}{\log a_n}\right)^{\beta+1}\right| \\
\sim &\ (\log n)^{\beta+1}\left|1-\frac{\log|x|}{\log a_n}\right| \sim (\log n)^{\beta}\log\frac{a_n}{|x|}.
\end{aligned}$$

Together with (1.9) and (5.1), this gives the result for $L \leq |x| \leq \varepsilon a_n$. For $|x| \leq L$, we redefine Q as an even quartic polynomial, as after Definition 1.1. The redefined Q has $Q'(0) = 0$ and $Q'(x) = O(x), x \to 0+$, so

$$\int_0^L \frac{Q'(s)}{s} ds < \infty.$$

Then for $|x| \leq L$, $\rho_n(x)$ admits the same estimate as for $|x| \geq L$.

If $\beta = -1$, then we already have the result. If $\beta < -1$, we use instead

$$\begin{aligned}
&\left|(\log a_n)^{\beta+1} - (\log|x|)^{\beta+1}\right| \\
= &\ (\log|x|)^{\beta+1}\left|1-\left(\frac{\log a_n}{\log|x|}\right)^{-(\beta+1)}\right| \\
\sim &\ (\log|x|)^{\beta+1}\left|1-\frac{\log|x|}{\log a_n}\right| \\
\sim &\ (\log|x|)^{\beta+1}\frac{\log\frac{a_n}{|x|}}{\log n}.
\end{aligned}$$

Again, together with (1.9) and (5.1), this gives the result. ■.

6. Zeros of Orthogonal Polynomials

The proofs of this section are similar to those in [9, Section 5], but we provide the details. We begin with the largest zero:

Proof of (1.17) of Corollary 1.4

We use the well known extremal property

$$x_{1n} = \sup \int_{-\infty}^{\infty} xP(x)W^2(x)\,dx \Big/ \int_{-\infty}^{\infty} P(x)W^2(x)\,dx,$$

where the sup is taken over all polynomials P of degree $\leq 2n-2$ that are non-negative in $\mathbb{R}$. (Each such P is the square of a real polynomial of degree$\leq n-1$). This is a consequence of the Gauss quadrature formula. Then

$$a_n - x_{1n} = \inf \int_{-\infty}^{\infty} (a_n - x)\, P(x)\, W^2(x)\, dx \Big/ \int_{-\infty}^{\infty} P(x)\, W^2(x)\, dx,$$

where the inf is over the same set of polynomials. Since a_{2n} for W^2 is a_n for W^2, we can use Lemma 3.4(b) (with $p=1$ there and W^2 rather than W) to deduce that

$$a_n - x_{1n} \leq C \inf \int_{-a_n}^{a_n} (a_n - x)\, P(x)\, W^2(x)\, dx \Big/ \int_{-\infty}^{\infty} P(x)\, W^2(x)\, dx.$$

Now we choose P. Choose a positive even integer $k \geq 4$ so large that for n large enough,

$$n^{\frac{5-2k}{3}} a_n^{1-A} \log n \leq 1$$

Next, let

$$m = \left[n^{1/3}/k\right]$$

where $[x]$ denotes the greatest integer $\leq x$. This choice of m and k ensures that (by (3.19)),

$$m^{-2k} \rho_n(0) \leq C \frac{n}{a_n} m^{-5}. \tag{6.1}$$

Next, let

$$P(x) = \lambda_{n-km}^{-1}\left(W^2, x\right) \ell(a_n x)^k$$

where ℓ is the fundamental polynomial of Lagrange interpolation at the zeros $\left\{x_{jm}^*\right\}_{j=1}^{m}$ of the Chebyshev polynomial T_m of degree m, associated with the largest zero $x_{1m}^* = \cos\left(\frac{\pi}{2m}\right)$ of T_m. Thus for $1 \leq j \leq m$,

$$\ell\left(x_{jm}^*\right) = \delta_{1m}.$$

It follows from our Theorem 1.2 and (3.18) that

$$\lambda_{n-2m}^{-1}\left(W^2, x\right) W^2(x) \sim \Lambda_n^{-1}(x),\ |x| \leq a_n,$$

as $a_n = a_{n-2m}\left(1 + O\left(n^{-2/3}\right)\right)$. Using a substitution, we see that

$$a_n - x_{1n} \leq C a_n \int_{-1}^{1} (1-s)\, \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \Big/ \int_{-1}^{1} \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds. \tag{6.2}$$

Now it is known that for some $C_1, C_2 > 0$, [8, p. 531]

$$|\ell(s)| \leq C \min\left\{\frac{1}{m^2\left|s - x_{1m}^*\right|}, 1\right\},\ s \in [-1, 1] \tag{6.3}$$

and

$$|\ell(s)| \geq \frac{1}{2},\ \left|s - x_{1m}^*\right| \leq C_2 m^{-2}. \tag{6.4}$$

We split

$$\begin{aligned} & \int_{-1}^{1} (1-s)\, \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \\ = & \left[\int_{-1}^{1/2} + \int_{1/2}^{x_{1m}^* - C_2 m^{-2}} + \int_{x_{1m}^* - C_2 m^{-2}}^{x_{1m}^* + C_2 m^{-2}} + \int_{x_{1m}^* + C_2 m^{-2}}^{1} \right] (1-s)\, \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \\ = & : I_1 + I_2 + I_3 + I_4. \end{aligned}$$

In I_1, $\Lambda_n^{-1}(a_n s) \le C\rho_n(0)$ and hence, from (6.1),

$$I_1 \le C m^{-2k} \rho_n(0) \le C \frac{n}{a_n} m^{-5}.$$

Next in I_2, $\Lambda_n^{-1}(a_n s) \le C \frac{n}{a_n} \left(1 - s + n^{-2/3}\right)^{1/2}$, so

$$\begin{aligned} I_2 & \le C m^{-2k} \frac{n}{a_n} \int_{1/2}^{x_{1m}^* - C m^{-2}} \left| s - x_{1m}^* \right|^{-k} \left(1 - s + n^{-2/3}\right)^{3/2} ds \\ & \le C m^{-2k} \frac{n}{a_n} \int_{1/2}^{x_{1m}^* - C m^{-2}} \left[\left| s - x_{1m}^* \right|^{3/2 - k} + \left| s - x_{1m}^* \right|^{-k} n^{-3} \right] ds \\ & \le \frac{n}{a_n} m^{-5}. \end{aligned}$$

(Recall that $1 - x_{1m}^* \sim m^{-2} \sim n^{-2/3}$). Also,

$$\begin{aligned} I_3 & \sim \frac{n}{a_n} \int_{x_{1m}^* - C_2 m^{-2}}^{x_{1m}^* + C_2 m^{-2}} (1-s)^{3/2} ds \\ & \sim \frac{n}{a_n} m^{-5}. \end{aligned}$$

Finally, we can estimate I_4 much as I_2,

$$I_4 \le C \frac{n}{a_n} m^{-5}.$$

Thus

$$\int_{-1}^{1} (1-s)\, \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \sim \frac{n}{a_n} m^{-5}.$$

Similarly,

$$\int_{-1}^{1} \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \ge \int_{x_{1m}^* - C_2 m^{-2}}^{x_{1m}^* + C_2 m^{-2}} \ell(s)^k\, \Lambda_n^{-1}(a_n s)\, ds \sim \frac{n}{a_n} m^{-3}.$$

Hence

$$a_n - x_{1n} \le C a_n m^{-2} \sim a_n n^{-2/3}.$$

The corresponding lower bound is easier. By Lemma 3.4(a), (with $\varepsilon = p = 1$ and W replacing W^2 there, and using a_{2n} for W^2 is a_n for W), if L is sufficiently large, then for all polynomials S of degree $\leq 2n$,

$$\int_{|x| \geq a_n\left(1+Ln^{-2/3}\right)} \left|SW^2\right|(x)\,dx \leq \int_{|x| \leq a_n\left(1+Ln^{-2/3}\right)} \left|SW^2\right|(x)\,dx.$$

In particuler, if $S(x) = \left(a_n\left(1+Ln^{-2/3}\right) - x\right) P_{n-1}^2(x)$ where P_{n-1} has degree $\leq n-1$, it follows that

$$\begin{aligned} & \int_{|x| \geq a_n\left(1+Ln^{-2/3}\right)} \left|a_n\left(1+Ln^{-2/3}\right) - x\right| \left(P_{n-1}W\right)^2(x)\,dx \\ \leq & \int_{|x| \leq a_n\left(1+Ln^{-2/3}\right)} \left(a_n\left(1+Ln^{-2/3}\right) - x\right) \left(P_{n-1}W\right)^2(x)\,dx \end{aligned}$$

(the integrand is non-negative in the right-hand integral) and hence

$$\int_{-\infty}^{\infty} \left(a_n\left(1+Ln^{-2/3}\right) - x\right) \left(P_{n-1}W\right)^2(x)\,dx \geq 0.$$

Then the extremal property of x_{1n} gives

$$\begin{aligned} & a_n\left(1+Ln^{-2/3}\right) - x_{1n} \\ = & \inf_{P_{n-1}} \int_{-\infty}^{\infty} \left(a_n\left(1+Ln^{-2/3}\right) - x\right) \left(P_{n-1}W\right)^2(x)\,dx \Big/ \int_{-\infty}^{\infty} \left(P_{n-1}W\right)^2(x) \geq 0. \end{aligned}$$

■

Remark
In [9], the estimation of the analogous integral I_1 was incomplete; the error is corrected above.

Proof of (1.17) of Corollary 1.4
We use the fact [12, Theorem 1, p. 299] that there is an even entire function G with all non-negative Maclaurin series coefficients such that

$$G \sim W^{-2} \text{ in } \mathbb{R}. \tag{6.5}$$

Then setting

$$\lambda_{jn} = \lambda_n\left(W^2, x_{jn}\right),$$

we may apply the Posse-Markov-Stieltjes inequalities [3, p. 33], to deduce that

$$\begin{aligned}\lambda_{jn}G(x_{jn}) &= \frac{1}{2}\left[\sum_{k:|x_{kn}|<|x_{j-1,n}|}\lambda_{kn}G(x_{kn}) - \sum_{k:|x_{kn}|<|x_{j,n}|}\lambda_{kn}G(x_{kn})\right] \\ &\leq \frac{1}{2}\left[\int_{-x_{j-1,n}}^{x_{j-1,n}} - \int_{-x_{j+1,n}}^{x_{j+1,n}}\right] G(t)\,W^2(t)\,dt \\ &= \int_{x_{j+1,n}}^{x_{j-1,n}} G(t)\,W^2(t)\,dt.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_{jn}G(x_{jn}) + \lambda_{j+1,n}G(x_{j+1,n}) &= \frac{1}{2}\left[\sum_{k:|x_{kn}|<|x_{j-1,n}|}\lambda_{kn}G(x_{kn}) - \sum_{k:|x_{kn}|<|x_{j+1,n}|}\lambda_{kn}G(x_{kn})\right] \\ &\geq \frac{1}{2}\left[\int_{-x_{jn}}^{x_{jn}} - \int_{-x_{j+1,n}}^{x_{j+1,n}}\right] G(t)\,W^2(t)\,dt \\ &= \int_{x_{j+1,n}}^{x_{jn}} G(t)\,W^2(t)\,dt.\end{aligned}$$

Then (6.5) and our bounds for Christoffel functions yield

$$\Lambda_n(x_{jn}) \leq C(x_{j-1,n} - x_{j+1,n}); \tag{6.6}$$

$$\Lambda_n(x_{jn}) + \Lambda_n(x_{j+1,n}) \geq C(x_{jn} - x_{j+1,n}). \tag{6.7}$$

The proof will be complete if we show that uniformly in j and n,

$$\Lambda_n(x_{jn}) \sim \Lambda_n(x_{j+1,n}). \tag{6.8}$$

Note that in the overlap region $\left[\frac{a_n}{4}, \frac{3a_n}{4}\right]$, $\Lambda_n \sim \frac{a_n}{n}$. So for $x_{jn}, x_{j+1,n}$ in this overlap region, (6.8) is immediate. Suppose next that $0 \leq x_{j+1,n} \leq x_{jn} \leq a_n/4$. Recall from (3.17) that for $t \in \left[0, \frac{1}{4}a_n\right]$,

$$\rho_n(t) \sim \rho_n(2t).$$

Although this was proved for $W \in \mathcal{SF}^+$, it actually holds for $W \in \mathcal{SF}$, since Q' is positive and continuous in any compact subinterval of $(0,\infty)$ (and ρ_n involves values of $Q'(x), x \geq 1$) and is identical to its modification outside a finite interval. We also use that ρ_n is decreasing. Then if

$$x_{jn} \leq 2x_{j+1,n} \leq \frac{1}{4}a_n,$$

we see that

$$\rho_n\left(x_{j+1,n}\right) \geq \rho_n\left(x_{jn}\right) \sim \rho_n\left(\frac{x_{jn}}{2}\right) \geq \rho_n\left(x_{j+1,n}\right)$$

so

$$\Lambda_n\left(x_{jn}\right) = \frac{1}{\rho_n\left(x_{jn}\right)} \sim \frac{1}{\rho_n\left(x_{j+1,n}\right)} = \Lambda_n\left(x_{j+1,n}\right).$$

If $0 \leq x_{jn}, x_{j+1,n} \leq \frac{1}{4}a_n$ but $x_{jn} > 2x_{j+1,n}$, then our spacing gives

$$x_{jn} \sim x_{jn} - x_{j+1,n} \leq C/\rho_n\left(x_{jn}\right).$$

Here

$$\rho_n\left(x_{jn}\right) = \int_{\max\{1,x_{jn}\}}^{2\max\{1,x_{jn}\}} \frac{Q'(s)}{s} ds \geq CQ'\left(x_{jn}\right),$$

again by (3.1) applied to the modification $\widetilde{Q}$ of Q and as the two are identical outside a bounded interval. Combining these two inequalities gives

$$x_{jn}Q'\left(x_{jn}\right) \leq C.$$

As $tQ'(t) \to \infty, t \to \infty$, we deduce that $x_{jn} \leq C$ and hence

$$\frac{x_{jn}}{a_n}, \frac{x_{j+1,n}}{a_n} \leq \frac{C}{a_n}.$$

Combining (3.9), (3.8) (if necessary applied to the modified weight) gives

$$\rho_n\left(x_{jn}\right) \sim \rho_n\left(x_{j+1,n}\right)$$

and hence (6.8) follows again. For $x_{jn} \geq \frac{a_n}{4}$, we proceed as follows: choose L such that

$$x_{1n} \leq a_n\left(1 + \frac{L}{2}n^{-2/3}\right).$$

Then

$$\begin{aligned} 1 &\leq \frac{1 - x_{j+1,n}/\left(a_n\left(1+Ln^{-2/3}\right)\right)}{1 - x_{jn}/\left(a_n\left(1+Ln^{-2/3}\right)\right)} \\ &= 1 + \frac{x_{jn} - x_{j+1,n}}{a_n\left(1+Ln^{-2/3}\right)\left[1 - x_{jn}/\left(a_n\left(1+Ln^{-2/3}\right)\right)\right]} \\ &\leq 1 + C\frac{1}{n\left[1 - x_{jn}/\left(a_n\left(1+Ln^{-2/3}\right)\right)\right]^{3/2}} \leq C_1, \end{aligned}$$

by our bounds on the largest zero, the Christoffel functions, and (6.7), (6.8). We have thus shown that for $x_{jn} \geq \frac{a_n}{4}$,

$$1 - x_{jn}/\left(a_n\left(1+Ln^{-2/3}\right)\right) \sim 1 - x_{j+1,n}/\left(a_n\left(1+Ln^{-2/3}\right)\right)$$

or equivalently,

$$\max\left\{n^{-2/3}, 1-\frac{x_{jn}}{a_n}\right\} \sim \max\left\{n^{-2/3}, 1-\frac{x_{j+1,n}}{a_n}\right\} \tag{6.9}$$

and hence, taking account of the fact that $1/\rho_n \sim \varphi_n$ in the overlap region $\left[\frac{1}{4}a_{n,}\frac{3}{4}a_n\right]$,

$$\Lambda_n(x_{jn}) = \varphi_n(x_{jn}) \sim \varphi_n(x_{j+1,n}) = \Lambda_n(x_{j+1,n}).$$

■

7. Orthogonal Polynomials

We follow the treatment in [9, p. 246 ff.]. Define

$$\overline{Q}(x,t) = \frac{xQ'(x) - tQ'(t)}{x^2 - t^2} \tag{7.1}$$

and

$$A_n(x) = 2\frac{\gamma_{n-1}}{\gamma_n}\int_{-\infty}^{\infty} p_n^2(t) W^2(t) \overline{Q}(x,t)\, dt. \tag{7.2}$$

(Recall here that γ_n is the leading coefficient of p_n). Let $K_n(x,t)$ denote the nth reproducing kernel, so that

$$\begin{aligned} K_n(x,t) &= K_n\left(W^2, x, t\right) = \sum_{j=0}^{n-1} p_j(x) p_j(t) \\ &= \frac{\gamma_{n-1}}{\gamma_n}\frac{p_n(x) p_{n-1}(t) - p_{n-1}(x) p_n(t)}{x-t}. \end{aligned}$$

As in the previous section, we let

$$\lambda_{jn} = \lambda_n\left(W^2, x_{jn}\right).$$

Some key identities are recorded in:

Lemma 7.1
(a)

$$p_n'(x_{jn}) = A_n(x_{jn}) p_{n-1}(x_{jn}). \tag{7.3}$$

(b)

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1}}{\gamma_n} A_n(x_{jn}) p_{n-1}^2(x_{jn}) = \frac{\gamma_{n-1}}{\gamma_n} A_n^{-1}(x_{jn}) p_n'(x_{jn})^2. \tag{7.4}$$

Proof
See for example [10, Lemma 12.2, p. 327 and p. 328], and use evenness of Q. ■

Next, we bound $A_n(x)$. We shall use the following consequence of (1.5) and (1.6): we may choose $A^{\#} \leq 1$ and $C^{\#} > 0$ such that

$$x \geq C^{\#} \Rightarrow A^{\#} \leq \frac{(xQ'(x))'}{Q'(x)} \leq 2 \tag{7.5}$$

and hence

$$\frac{Q'(x)}{x} \text{ is decreasing in } [C^{\#}, \infty). \tag{7.6}$$

The latter follows from the identity

$$\frac{d}{dx}\left(\frac{Q'(x)}{x}\right) = \frac{Q'(x)}{x^2}\left[\frac{(xQ'(x))'}{Q'(x)} - 2\right].$$

We shall also use

$$\left(\frac{y}{x}\right)^{1-A^{\#}} \leq \frac{Q'(y)}{Q'(x)} \leq \left(\frac{y}{x}\right)^2, y \geq x \geq C^{\#}, \tag{7.7}$$

which follows by integrating (7.5) as in Lemma 3.1. In the rest of this section, $A^{\#}$ and $C^{\#}$ have the meaning just described.

Lemma 7.2
Assume that $W \in \mathcal{SF}$. For $n \geq 1$ and $2C^{\#} \leq x \leq a_n\left(1 + Ln^{-2/3}\right)$,

$$C_1 \frac{n}{a_n^2} \leq A_n(x) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right) \leq C_2 \frac{Q'(x)}{x}. \tag{7.8}$$

Proof
We claim first that for $x \geq C^{\#}, t > 0$,

$$\overline{Q}(x,t) \sim \frac{Q'(\max\{x,t\})}{\max(x,t)}, \tag{7.9}$$

To see this, observe first that since $tQ'(t)$ is increasing in t, then for $t \geq 2x$,

$$\overline{Q}(x,t) \leq \frac{tQ'(t)}{t^2\left(1 - \frac{1}{4}\right)} = \frac{4}{3}\frac{Q'(\max\{x,t\})}{\max(x,t)}.$$

Moreover, using (7.7) which is applicable as $t \geq C^{\#}$,

$$\overline{Q}(x,t) \geq \frac{tQ'(t)\left(1 - 2^{A^{\#}-2}\right)}{t^2} = C\frac{Q'(\max\{x,t\})}{\max(x,t)}.$$

The case $x \leq \frac{t}{2}$ is similar. Finally, if $\frac{x}{2} < t < 2x$, then for some $u \in \left[\frac{x}{2}, 2x\right]$, and hence having $u \geq C^{\#}$,

$$\overline{Q}(x,t) = \frac{(uQ'(u))'}{x+t} \sim \frac{Q'(x)}{x} \sim \frac{Q'(\max\{x,t\})}{\max(x,t)}$$

by (7.6) and (7.7). So we have (7.9). Then for $x \in [C^{\#}, \infty)$,

$$A_n(x) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right) \sim \frac{Q'(x)}{x} \int_0^{\min\{x,a_n\}} (p_n W)^2(t)\, dt$$
$$+ \int_{\min\{x,a_n\}}^{a_n} \frac{Q'(t)}{t} (p_n W)^2(t)\, dt + \int_{a_n}^{\infty} \frac{Q'(t)}{t} (p_n W)^2(t)\, dt. \tag{7.10}$$

In view of (7.6), we obtain

$$A_n(x) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right) \leq \frac{Q'(x)}{x} \int_0^{\infty} (p_n W)^2(t)\, dt.$$

In the other direction, we obtain for $x \in \left[C^{\#}, a_n\left(1 + Ln^{-2/3}\right)\right]$,

$$A_n(x) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right) \geq \frac{Q'\left(a_n\left(1 + Ln^{-2/3}\right)\right)}{a_n\left(1 + Ln^{-2/3}\right)} \int_0^{a_n} (p_n W)^2(t)\, dt \geq C\frac{n}{a_n^2},$$

by the evenness of $(p_n W)^2$, the restricted range inequality Lemma 3.4(b), and (3.2) (applied if necessary to the modified weight). ■

Proof of Theorem 1.5(a)

We use a form of the Christoffel-Darboux formula and then Cauchy-Schwarz to deduce

$$\begin{aligned} p_n^2(x) &= K_n^2(x, x_{kn})(x - x_{kn})^2 / \left[\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn})\right]^2 \\ &\leq \lambda_n^{-1}\left(W^2, x\right) \lambda_n^{-1}\left(W^2, x_{kn}\right)(x - x_{kn})^2 / \left[\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn})\right]^2 \\ &= \lambda_n^{-1}\left(W^2, x\right)\left[A_n(x) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right)\right](x - x_{kn})^2. \end{aligned}$$

by Lemma 7.1(b). Let $x \in \left[0, a_n\left(1 + Ln^{-2/3}\right)\right]$ and x_{kn} be the zero of p_n closest to x. Applying Lemma 7.2, the lower bounds for Christoffel functions in Theorem 1.2, and the spacing of zeros in Corollary 1.4, as well as (6.8), gives

$$(p_n W)^2(x) \leq C\Lambda_n(x_{kn})\left[A_n(x_{kn}) / \left(\frac{\gamma_{n-1}}{\gamma_n}\right)\right], x \in \left[0, a_n\left(1 + Ln^{-2/3}\right)\right]. \tag{7.11}$$

We deduce that

$$(p_n W)^2(x) \leq C\Lambda_n(x_{kn})\frac{Q'(x_{kn})}{x_{kn}}, x \in \left[C^{\#}, a_n\left(1 + Ln^{-2/3}\right)\right]. \tag{7.12}$$

Now let us assume in addition that $x \geq \varepsilon a_n$. Our spacing and (3.2), (7.7) give

$$\frac{Q'(x_{kn})}{x_{kn}} \sim \frac{Q'(x)}{x} \sim \frac{n}{a_n^2}.$$

Moreover Λ_n is given by (1.8 - 1.10), and as noted there, since $1/\rho_n$ and φ_n agree in the overlap region,

$$\Lambda_n(x_{kn}) \sim \frac{a_n}{n} \max\left\{n^{-2/3}, 1 - \frac{|x_{kn}|}{a_n}\right\}^{-1/2}.$$

Finally, (6.9) allows us to replace x_{kn} by x in the last right-hand side. So we obtain for $\varepsilon a_n \leq x \leq a_n\left(1 + Ln^{-2/3}\right)$,

$$(p_n W)^2(x) \leq C a_n^{-1} \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{-1/2}.$$

■

We record also:

Lemma 7.3
Assume that $W \in \mathcal{SF}$. Then for $C^{\#} \leq x \leq \frac{1}{2}a_n$,

$$(p_n W)^2(x) \leq C\frac{Q'(x)}{x} \Big/ \int_{\max\{1,x\}}^{a_n} \frac{Q'(s)}{s} ds. \tag{7.13}$$

Moreover, if in (7.7), $A^{\#} < 1$,

$$(p_n W)^2(x) \leq \frac{C}{x}; \tag{7.14}$$

and if $A = 1$,

$$(p_n W)^2(x) \leq \frac{C}{x}\left(\log \frac{a_n}{x}\right)^{-1}. \tag{7.15}$$

Proof
From (7.12) and (1.11), we obtain (7.13). Next, by (7.7),

$$\int_x^{a_n} \frac{Q'(s)}{s} ds \geq \frac{Q'(x)}{x^{A^{\#}-1}} \int_x^{a_n} s^{A^{\#}-2} ds.$$

Then (7.14) and (7.15) follow. ■

For Theorem 1.5(b), we need:

Lemma 7.4

Assume the hypotheses of Theorem 1.5(b).
(a) Let $\eta \in (0,1)$. There exists C_η such that for $y \geq x \geq C_\eta$,

$$\left(\frac{y}{x}\right)^{-\eta} \leq \frac{Q'(y)}{Q'(x)} \leq \left(\frac{y}{x}\right)^{\eta}. \tag{7.16}$$

(b) For $n \geq 1, \varepsilon \in \left[0, \frac{1}{e}\right], x \in [C_\varepsilon, \varepsilon a_n]$,

$$\rho_n(x) \geq \frac{3}{4} Q'(x) |\log \varepsilon|. \tag{7.17}$$

(c) Let $K, M > 0$. There exists n_0 such that for $n \geq n_0$ and $x \in [0, M]$,

$$\rho_n(x) \geq K. \tag{7.18}$$

Proof
(a) By (1.20), there exists C_ε such that for $y \geq x \geq C_\varepsilon$,

$$\frac{1-\eta}{x} \leq \frac{(xQ'(x))}{Q'(x)} \leq \frac{1+\eta}{x}.$$

Integrating this over $[x, y]$ where $y \geq x \geq C_\eta$ gives the result.
(b) From (a), if $\varepsilon a_n \geq x \geq C_\varepsilon$,

$$\begin{aligned} \rho_n(x) &= \int_x^{a_n} \frac{Q'(y)}{y} dy \\ &\geq Q'(x) x^\varepsilon \int_x^{a_n} y^{-1-\varepsilon} dy \\ &= \frac{Q'(x)}{\varepsilon}\left(1 - \left(\frac{x}{a_n}\right)^\varepsilon\right) \\ &\geq \frac{Q'(x)}{\varepsilon}(1-\varepsilon^\varepsilon). \end{aligned}$$

Now if $\varepsilon \in (0, e^{-1}]$,

$$1 - \varepsilon^\varepsilon = 1 - \exp(-\varepsilon |\log \varepsilon|) \geq \frac{3}{4} \varepsilon |\log \varepsilon|,$$

and then (7.17) follows.
(c) This follows directly from the divergence of the integral in (1.21). ■

Proof of Theorem 1.5(b)
Let us fix $\varepsilon, \beta \in (0,1)$ and let

$$h_n(x) = a_n x^\beta (p_n W)^2(x), x \in [0, \infty).$$

We use some of the ideas used for Theorem 1.5(a). First if $x \in (0, 2C^{\#}]$,

$$\overline{Q}(x,t) \leq \begin{cases} \frac{Q'(x)}{x}, & x \geq 2t \\ \frac{Q'(t)}{t}, & t \geq 2x \end{cases}.$$

If $t \in \left[\frac{x}{2}, 2x\right]$, we obtain for some u between t, x,

$$\overline{Q}(x,t) = \frac{(uQ'(u))'}{x+t} \leq \frac{C}{x},$$

recall that $Q'(u)$ and $uQ''(u)$ are bounded in $(0, 2C^{\#}]$. Combining all the above, we obtain

$$\overline{Q}(x,t) \leq \begin{cases} \frac{C}{x}, & t \leq 2x \\ \frac{Q'(t)}{t}, & t \geq 2x \end{cases}.$$

Then from the definition (7.2) of A_n, we see that for $x \in \left[0, 2C^{\#}\right]$,

(7.19)

$$A_n(x)/\frac{\gamma_{n-1}}{\gamma_n} \leq \frac{C}{x}\int_0^{2x} h_n(t)\, t^{-\beta}dt + \frac{C}{a_n}\int_{2x}^{\varepsilon a_n} h_n(t)\frac{Q'(t)}{t^{1+\beta}}dt + C\int_{\varepsilon a_n}^{\infty}\frac{Q'(t)}{t}(p_nW)^2(t)\,dt.$$

Here using (7.16) with ε replaced by $\beta/2$, we obtain for $x \in \left[0, 2C^{\#}\right]$,

$$\begin{aligned} & \frac{C}{a_n}\int_x^{\varepsilon a_n} h_n(t)\frac{Q'(t)}{t^{1+\beta}}dt \\ \leq\ & C\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}\left\{\int_x^{C^{\#}}\frac{dt}{t^{1+\beta}} + \frac{Q'\left(C^{\#}\right)}{C^{\#\beta/2}}\int_{C^{\#}}^{\varepsilon a_n}\frac{dt}{t^{1+\beta/2}}\right\} \\ \leq\ & C\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}x^{-\beta}, \end{aligned}$$

with C independent of ε, n, x. Next from (3.2),

$$C\int_{\varepsilon a_n}^{a_n}\frac{Q'(t)}{t}(p_nW)^2(t)\,dt \leq C_2\frac{n}{a_n^2}.$$

Here C_2 does depend on ε. Then substituting in (7.19),

$$A_n(x)/\frac{\gamma_{n-1}}{\gamma_n} \leq C_1\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}x^{-\beta} + C_2\frac{n}{a_n^2}, x \in [0, 2C^{\#}], \tag{7.20}$$

with C_1 independent of ε, and C_2 depending on ε. If $x \in \left[2C^{\#}, \varepsilon a_n\right]$, the estimation is easier: we continue (71.0) as

$$\begin{aligned} A_n(x)/\left(\frac{\gamma_{n-1}}{\gamma_n}\right) \leq\ & C\frac{Q'(x)}{a_n x}\int_0^x h_n(t)\, t^{-\beta}dt \\ & + \frac{C}{a_n}\int_x^{\varepsilon a_n}\frac{Q'(t)}{t^{1+\beta}}h_n(t)\,dt + C\frac{n}{a_n^2}. \end{aligned}$$

Here using (7.16) and assuming $2C^{\#} \geq C_{\beta/2}$, as we may, we obtain

$$
\begin{aligned}
&\frac{C}{a_n}\int_x^{\varepsilon a_n} \frac{Q'(t)}{t^{1+\beta}} h_n(t)\, dt \\
\leq\ & C\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}\frac{Q'(x)}{x^{\beta/2}}\int_x^{\varepsilon a_n}\frac{dt}{t^{1+\beta/2}} \leq C\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}\frac{Q'(x)}{x^{\beta}}.
\end{aligned}
$$

Hence

(7.21)

$$
A_n(x)/\left(\frac{\gamma_{n-1}}{\gamma_n}\right) \leq C_1\frac{\|h_n\|_{L_\infty[0,\varepsilon a_n]}}{a_n}Q'(x)\,x^{-\beta} + C_2\frac{n}{a_n^2}, x \in \left[2C^{\#}, \varepsilon a_n\right],
$$

with C_1 independent of ε, and C_2 depending on ε. Next, we use (7.11) to deduce

$$
\begin{aligned}
h_n(x) &= a_n x^\beta (p_n W)^2(x) \\
&\leq C a_n x^\beta \Lambda_n(x_{kn})\, A_n(x_{kn})/\frac{\gamma_{n-1}}{\gamma_n}.
\end{aligned}
$$

For $x \geq 2C^{\#}$, we continue (7.21) using the bound from Lemma 3.4,

$$
\Lambda_n(x_{kn}) = 1/\rho_n(x_{kn}) \leq 1/\rho_n(\varepsilon a_n) \leq C\frac{a_n}{n}
$$

and the bound from Lemma 7.4,

$$
\Lambda_n(x) = 1/\rho_n(x) \leq \frac{4}{3Q'(x)\left|\log\varepsilon\right|}.
$$

This yields

$$
h_n(x) \leq C_3\varepsilon \|h_n\|_{L_\infty[0,\varepsilon a_n]} + C_2 a_n^\beta.
$$

As C_3 is independent of ε, we may choose $\varepsilon = \frac{1}{2C_2}$, so

$$
\|h_n\|_{L_\infty\left[2C^{\#},\varepsilon a_n\right]} \leq \frac{1}{2}\|h_n\|_{L_\infty[0,\varepsilon a_n]} + C_2 a_n^\beta.
$$

For $x \in [0, 2C^*]$, we obtain instead from (7.20) and Lemma 7.4(c) that for $n \geq n_0(\varepsilon, \beta)$,

$$
\begin{aligned}
h_n(x) &\leq C/\rho_n(x)\|h_n\|_{L_\infty[0,\varepsilon a_n]} + C_2 x^\beta \\
&\leq \frac{1}{2}\|h_n\|_{L_\infty[0,\varepsilon a_n]} + C_2 a_n^\beta.
\end{aligned}
$$

Combining the two norm bounds on h_n gives

$$
\|h_n\|_{L_\infty[0,\varepsilon a_n]} \leq \frac{1}{2}\|h_n\|_{L_\infty[0,\varepsilon a_n]} + C_2 a_n^\beta
$$

and hence

$$
\|h_n\|_{L_\infty[0,\varepsilon a_n]} \leq 2C_2 a_n^\beta.
$$

Thus

$$|p_nW|^2(x) \le Ca_n^{-1}\left(\frac{a_n}{x}\right)^{\beta}, x \in [0, \varepsilon a_n].$$

Here C depends on ε, β but β and ε are independent of one another. Let $\delta \in (0,1)$. Choosing $\beta = \beta(\delta)$ small enough, we deduce that

$$|p_nW|^2(x) \le Ca_n^{-1}n^{\delta}, x \in \left[\frac{1}{n}, \varepsilon a_n\right].$$

To fill in the bound in $\left[-\frac{1}{n}, \frac{1}{n}\right]$, we use a standard Schur type inequality: there exists $C > 0$ such that for $n \ge 2$ and polynomials P of degree$\le n$,

$$\|P\|_{L_\infty[-1,1]} \le \|P\|_{L_\infty[-1,1]\setminus\left[-\frac{1}{n},\frac{1}{n}\right]}.$$

Applying this to $P = p_n$, and using that $W^{\pm 1}$ is bounded in $[-1,1]$ gives

$$|p_nW|^2(x) \le Ca_n^{-1}n^{\delta}, x \in [-\varepsilon a_n, \varepsilon a_n].$$

For $\varepsilon a_n \le |x| \le a_n$, we instead have

$$|p_nW|^2(x) \le a_n^{-1}\max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{-1/2} \le Ca_n^{-1}n^{1/3}.$$

If $\delta < \frac{1}{3}$, we can combine these bounds as

$$\left|p_nW^2\right|(x) \le a_n^{-1}n^{1/6}, |x| \le a_n.$$

The restricted range inequality Lemma 3.4(b) shows that this bound persists throughout the real line.

We proceed to establish the lower bound. For this, we use (7.3) and (7.8) to deduce that if $|x_{jn}| \ge \varepsilon a_n$,

$$\begin{aligned}(p_n'W)(x_{jn})^2 &\sim \lambda_{jn}^{-1}\frac{A_n(x_{jn})}{\gamma_{n-1}/\gamma_n} \\ &\sim \varphi_n(x_{jn})^{-1}\frac{Q'(a_n)}{a_n} \sim \left(\frac{n}{a_n}\right)^2 a_n^{-1}\max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{1/2}\end{aligned}$$

so

$$\left|(p_nW)'(x_{jn})\right| \sim \frac{n}{a_n^{3/2}}\max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{1/4}. \tag{7.22}$$

But by the Markov-Bernstein inequality Theorem 1.3 in [7, p. 1067],

$$\left|(p_nW)'(x_{jn})\right| \le C\frac{n}{a_n}\max\left\{n^{-2/3}, 1 - \frac{|x_{jn}|}{a_n}\right\}^{1/2}\|p_nW\|_{L_\infty(\mathbb{R})}$$

so

$$\|p_n W\|_{L_\infty(\mathbb{R})} \geq C a_n^{-1/2} \max\left\{n^{-2/3}, 1 - \frac{|x_{jn}|}{a_n}\right\}^{-1/4},$$

and choosing $j = 1$ and using our estimate for the largest zero x_{1n} gives

$$\|p_n W\|_{L_\infty(\mathbb{R})} \geq C a_n^{-1/2} n^{1/6}.$$

■

We record:

Corollary 7.5
Assume the hypotheses of Theorem 1.5(b).
(a) There exists $\varepsilon \in (0,1)$ with the following property: given $\delta > 0$, we have for $n \geq n_0(\delta)$,

$$\left|p_n\left(W^2, x\right)\right| \leq C a_n^{-1} n^{\delta}, |x| \leq \varepsilon a_n. \tag{7.23}$$

(b) Let $\varepsilon \in (0,1)$. For $n \geq n_0$ and $|x_{jn}| \geq \varepsilon a_n$,

$$\left|(p_n W)'(x_{jn})\right| \sim \frac{n}{a_n^{3/2}} \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{1/4} \tag{7.24}$$

and

$$|(p_{n-1} W)(x_{jn})| \sim a_n^{-1} \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{1/4}. \tag{7.25}$$

Proof
(a) This was proved in the course of the proof of Theorem 1.5(b).
(b) We must prove (7.25). From (7.3), and then (7.8), (7.24)

$$\begin{aligned} |(p_{n-1} W)(x_{jn})| &= \left|(p_n W)'(x_{jn})\right| A_n(x_{jn})^{-1} \\ &\sim \frac{n}{a_n^{3/2}} \max\left\{n^{-2/3}, 1 - \frac{|x|}{a_n}\right\}^{1/4} \left(\frac{n}{a_n^2} \frac{\gamma_{n-1}}{\gamma_n}\right). \end{aligned}$$

It remains to show that

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n. \tag{7.26}$$

The upper bound implicit in this relation follows from

$$\begin{aligned} \frac{\gamma_{n-1}}{\gamma_n} &= \int_{-\infty}^{\infty} x p_{n-1}(x) p_n(x) W^2(x)\, dx \\ &\leq C a_n \int_{-a_n}^{a_n} |p_{n-1}(x) p_n(x)| W^2(x)\, dx \leq C, \end{aligned}$$

by the restricted range inequality Lemma 3.4(b) and Cauchy-Schwarz. For the lower bound, we can use (7.4) in the form

$$\begin{aligned} 1 &= \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x_{jn})}{\frac{\gamma_{n-1}}{\gamma_n}} \lambda_{jn} p_{n-1}^2(x_{jn}) \\ &\leq C \frac{n}{a_n^2} \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \lambda_{jn} p_{n-1}^2(x_{jn}), \end{aligned}$$

for $|x_{jn}| \geq \varepsilon a_n$. It is an easy consequence of the spacing in Corollary 1.4 that there are at least Cn zeros $x_{jn} \in \left[\frac{1}{2}a_n, a_n\right]$. Adding over these gives

$$\begin{aligned} Cn &\leq C \frac{n}{a_n^2} \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \sum_{j=1}^{n} \lambda_{jn} p_{n-1}^2(x_{jn}) \\ &= C \frac{n}{a_n^2} \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2, \end{aligned}$$

by the Gauss quadrature formulae. So we have the lower bound implicit in (7.26). ■

References

[1] C. Berg, Y. Chen, M. Ismail, Small eigenvalues of large Hankel matrices: the indeterminate case. Math. Scand. 91 (2002), 67–81.

[2] Y. Chen, M. Ismail, Some indeterminate moment problems and Freud-like weights. Constr. Approx. 14 (1998), 439–458.

[3] G. Freud, Orthogonal Polynomials, Akademiai Kiado/Pergamon, Budapest, 1971

[4] G. Freud, A. Giroux, Q.I. Rahman, *On Approximation by Polynomials with Weight* $\exp(-|x|)$, Canad. J. Math., 30(1978), 358-372.

[5] P. Koosis, *The Logarithmic Integral I*, Cambridge University Press, Cambridge, 1988.

[6] T. Kriecherbauer and K. T.-R. McLaughlin, *Strong Asymptotics of Polynomials Orthogonal with Respect to Freud Weights*, International Math. Research Notes, 6(1999), 299-333.

[7] Eli Levin and D.S. Lubinsky, L_∞ *Markov and Bernstein Inequalities for Freud Weights*, SIAM J. Math. Anal., 21(1990), 1065-1082.

[8] Eli Levin and D.S. Lubinsky, *Christoffel Functions, Orthogonal Polynomials, and Nevai's Conjecture for Freud Weights,* Constr. Approx., 8(1992) 463-535.

[9] Eli Levin and D.S. Lubinsky, *Orthogonal Polynomials and Christoffel Functions for* $\exp(-|x|^\alpha), \alpha \leq 1$, J. Approx. Theory, 80(1995) 219-252.

[10] Eli Levin and D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.

[11] Eli Levin and D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights* $x^{2\rho}\exp(-2Q(x))$ *on* $[0,d)$, J. Approx. Theory, 134(2005), 199-256.

[12] *Gaussian Quadrature, Weights on the Whole Real Line, and Even Entire Functions with Nonnegative Even Order Derivatives*, J. Approx. Theory, 46(1986), 297-313.

[13] D.S. Lubinsky, *Which Weights on* $\mathbb{R}$ *admit Jackson Theorems*?, to appear in Israel Journal of Mathematics.

[14] D.S. Lubinsky, *Jackson and Bernstein Theorems for the Weight* $\exp(-|x|)$ *on* $\mathbb{R}$, to appear in Israel Journal of Mathematics.

[15] H.N. Mhaskar, *Introduction to the Theory Of Weighted Polynomial Approximation*, World Scientific, Singapore, 1996.

[16] H.N. Mhaskar and E.B. Saff, *Extremal Problems Associated with Exponential Weights*, Trans. Amer. Math. Soc., 285(1984), 223-234.

[17] P. Nevai, Geza Freud, *Orthogonal Polynomials and Christoffel Functions: A Case Study,* J. Approx. Theory, 48(1986), 3-167.

[18] P. Nevai, V. Totik, *Weighted Polynomial Inequalities*, Constructive Approximation, 2(1986), 113-127.

[19] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer, New York, 1997.

[20] J.A. Shohat and J.D. Tamarkin, *The Problem of Moments,* American Mathematical Society, New York, 1943.

[21] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol. 23, 3rd. edn., American Mathematical Society, Providence, 1975.

[22] V. Totik, *Weighted Approximation with Varying Weight,* Springer Lecture Notes in Mathematics, Vol. 1569, Springer, Berlin, 1994.

[1]Mathematics Department, The Open University of Israel, P.O. Box 808, Raanana 43107, Israel., [2]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., [1] lubinsky@math.gatech.edu, [2] elile@oumail.ac.il