ON RECURRENCE COEFFICIENTS FOR RAPIDLY DECREASING EXPONENTIAL WEIGHTS

E. LEVIN [1], D. S. LUBINSKY[2]

ABSTRACT. Let, for example,

$$W(x) = \exp\left(-\exp_k\left(1-x^2\right)^{-\alpha}\right),\ x \in [-1,1]$$

where $\alpha > 0,\ k \geq 1$, and $\exp_k = \exp(\exp(\ldots\exp()))$ denotes the kth iterated exponential. Let $\{A_n\}$ denote the recurrence coefficients in the recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + A_{n-1}p_{n-1}(x)$$

for the orthonormal polynomials $\{p_n\}$ associated with W^2. We prove that as $n \to \infty$,

$$\frac{1}{2} - A_n = \frac{1}{4}(\log_k n)^{-1/\alpha}(1+o(1)),$$

where $\log_k = \log(\log(\ldots\log()))$ denotes the kth iterated logarithm. This illustrates the relationship between the rate of convergence to $\frac{1}{2}$ of the recurrence coefficients, and the rate of decay of the exponential weight at ± 1. More general non-even exponential weights on a non-symmetric interval (a,b) are also considered.

1. Introduction and Results[1]

Let $-\infty \leq a < 0 < b \leq \infty$, $Q : (a,b) \to [0,\infty)$ be continuous, and $W = \exp(-Q)$. Then, provided all power moments exist, we may define orthonormal polynomials

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + \ldots, \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int_a^b p_n p_m W^2 = \delta_{mn}.$$

These orthonormal polynomials satisfy a recurrence relation of the form

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + A_{n-1}p_{n-1}(x),$$

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where

$$A_n = \frac{\gamma_n}{\gamma_{n+1}} > 0 \text{ and } B_n \in \mathbb{R},\, n \geq 1.$$

(We use uppercase for A_n rather than the more common lowercase, since we want to use the lower case for the Mhaskar-Rakhmanov-Saff numbers.) In the case when $(a,b) = (-1,1)$, a classical result of Rakhmanov [8] implies that

$$\lim_{n\to\infty} A_n = \frac{1}{2} \text{ and } \lim_{n\to\infty} B_n = 0,$$

and hence W^2 is a member of the Nevai-Blumenthal class.

The rate of convergence of A_n to $\frac{1}{2}$ and B_n to 0 has been studied for decades. Many properties of the weight W^2 (or more generally a measure) can be formulated in terms of series involving $\left|A_n - \frac{1}{2}\right|$ and $|B_n|$. For example, it is known [8] that Szegö's condition

$$\int_{-1}^{1} \frac{\log W(x)}{\sqrt{1-x^2}} dx > -\infty,$$

is satisfied iff

$$\inf_{n\geq 1} 2^n A_1 A_2 ... A_n > 0.$$

In recent years, Barry Simon and his collaborators have formulated results of this type that go way beyond Szegö's condition [1], [3], [8], [9]. For example, they consider weights satisfying the weaker condition

$$\int_{-1}^{1} \log W(x) \sqrt{1-x^2}\, dx > -\infty.$$

In this paper, we shall consider weights that vanish so rapidly at the endpoints of the interval that all of these conditions are violated. When (a,b) is unbounded, the situation is more complicated - see for example, [4].

In analyzing exponential weights $W = e^{-Q}$, an important role is played by the Mhaskar-Rakhmanov-Saff numbers $a < a_{-n} < a_n < b$, the roots of the equations

$$n = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n - x)}} dx; \tag{1.1}$$

$$0 = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n - x)}} dx. \tag{1.2}$$

For example, when Q is convex and $Q(0) = 0$, $a_{\pm n}$ are well defined and unique, and $a_{-n} < 0 < a_n$. One important feature of the $a_{\pm n}$ is the Mhaskar-Saff identity

$$\|PW\|_{L_\infty[-1,1]} = \|PW\|_{L_\infty[a_{-n},a_n]}, \tag{1.3}$$

valid for all polynomials P of degree $\leq n$ [5], [6], [7].

We define the center of the Mhaskar-Rakhmanov-Saff interval

$$\beta_n = \frac{1}{2}(a_n + a_{-n}) \tag{1.4}$$

and its half-length

$$\delta_n = \frac{1}{2}(a_n - a_{-n}). \tag{1.5}$$

In the special case that Q is even, we have $a_{-n} = -a_n = \delta_n$; $\beta_n = 0$ and a_n is the root of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt.$$

In this paper, we show for a large class of exponential weights that

$$\frac{A_n}{\delta_n} - \frac{1}{2} = O\left(n^{-C}\right) \text{ and } \frac{B_n - \beta_n}{\delta_n} = O\left(n^{-C}\right)$$

for some $C > 0$. When $(a, b) = (-1, 1)$ and δ_n approaches 1 with a rate slower than any negative power of n, this leads to A_n approaching $\frac{1}{2}$ with a rate slower than any negative power of n. In this case, we can give the exact rate of approach of A_n to $\frac{1}{2}$.

One special case of our results deals with exponential weights on $[-1, 1]$ that decay rapidly at the endpoints, though with possibly differing rates. In the sequel, $\exp_0(x) = x$ and for $k \geq 1$

$$\exp_k = \exp(\exp(\ldots\exp()))$$

denotes the kth iterated exponential. Moreover, $\log_0 x = x$ and for $k \geq 1$

$$\log_k = \log(\log(\ldots\log()))$$

denotes the kth iterated logarithm.

Theorem 1.1
Let $k, \ell \geq 0$, with at least one positive; let $\alpha, \beta > 0$ and

$$W(x) = \begin{cases} \exp\left(\exp_k(1) - \exp_k\left(1-x^2\right)^{-\alpha}\right), & x \in [0,1) \\ \exp\left(\exp_\ell(1) - \exp_\ell\left(1-x^2\right)^{-\beta}\right), & x \in (-1,0] \end{cases}. \tag{1.6}$$

Then

$$\frac{1}{2}-A_n = \frac{1}{8}\left[(\log_k n)^{-1/\alpha}+(\log_\ell n)^{-1/\beta}\right](1+o(1));$$

$$B_n = O\left((\log_k n)^{-1/\alpha}+(\log_\ell n)^{-1/\beta}\right). \tag{1.7}$$

Note that the factors $\exp_k(1)$ and $\exp_\ell(1)$ are inserted to ensure continuity of the exponent at 0. When $k=\ell$, they can be factored out and dispensed with. Thus for the even weight

$$W(x)=\exp\left(-\exp_k\left(1-x^2\right)^{-\alpha}\right), x\in(-1,1),$$

where $k\geq 1$ and $\alpha>0$, the theorem gives

$$\frac{1}{2}-A_n=\frac{1}{4}(\log_k n)^{-1/\alpha}(1+o(1)).$$

Our general class of weights is given in:

Definition 1.2
Let $-\infty\leq a<0<b\leq\infty$, *and* $W=e^{-Q}$, *where* $Q:(a,b)\to[0,\infty)$ *satisfies the following properties:*
(a) Q' *is continuous in* $(a,b)\setminus\{0\}$ *and* $Q(0)=0$.
(b) Q'' *exists and is positive in* $(a,b)\setminus\{0\}$.
(c)

$$\lim_{t\to a+\text{ or }b-} Q(t)=\infty.$$

(d) The function

$$T(t)=\frac{tQ'(t)}{Q(t)},\ t\neq 0,$$

satisfies, for some $C_1>0$ *and* $\Lambda>1$

$$1<\Lambda\leq T(s)\leq C_1T(t),\ 0<s/t<1,$$

provided $s,t\in(a,b)\setminus\{0\}$.
(e) There exists $C_2>0$ *such that*

$$\frac{Q''(x)}{|Q'(x)|}\leq C_2\frac{|Q'(x)|}{Q(x)}\text{ a.e. } x\in(a,b)\setminus\{0\}.$$

Then we write $W\in\mathcal{F}(C^2)$.
(f) Suppose in addition, that for each $\varepsilon>0$,

$$T(x)=O\left(Q(x)^{\varepsilon}\right),\ x\to a+, \tag{1.8}$$

or

$$T(x)=O\left(Q(x)^{\varepsilon}\right),\ x\to b-, \tag{1.9}$$

or both these hold. Then we write $W \in \mathcal{E}(C^2)$.

Remarks
(a) As examples of $W \in \mathcal{E}(C^2)$, we mention the weights of Theorem 1.1. Other examples are $W = e^{-Q}$, where

$$Q(x) = \begin{cases} \exp_\ell(|x|^\alpha) - \exp_\ell(0), & x \in [0, \infty), \\ \exp_k\left(|x|^\beta\right) - \exp_k(0), & x \in (-\infty, 0] \end{cases},$$

where $k, \ell \geq 0$ with at least one positive, and $\alpha, \beta > 1$, and (as above) $\exp_k$ denotes the kth iterated exponential. In the case $k = \ell = 0$, $W \in \mathcal{F}(C^2) \backslash \mathcal{E}(C^2)$. See [4, pp. 8-9] for further orientation.
(b) On a finite interval, the weight $W = e^{-Q}$, where $\alpha, \beta > 0$,

$$Q(x) = \begin{cases} (1 - x^2)^{-\alpha} - 1, & x \in [0, 1), \\ (1 - x^2)^{-\beta} - 1, & x \in (-1, 0] \end{cases}$$

belongs to $\mathcal{F}(C^2) \backslash \mathcal{E}(C^2)$, since both (1.8) and (1.9) are violated. However, if $k \geq 1$, and

$$Q(x) = \begin{cases} \exp_k\left((1 - x^2)^{-\alpha}\right) - \exp_k(1), & x \in [0, 1), \\ (1 - x^2)^{-\beta} - 1, & x \in (-1, 0] \end{cases},$$

then $W = e^{-Q} \in \mathcal{E}(C^2)$, since (1.8) is fulfilled. Basically on $[-1, 1]$, (1.8) and (1.9) are satisfied only for weights whose exponent Q grows faster than any positive power of $(1 - x^2)^{-1}$ as $|x| \to 1$.

Our most general result is:

Theorem 1.3
Let $W \in \mathcal{F}(C^2)$. *Then for some* $C > 0$,

$$\frac{A_n}{\delta_n} - \frac{1}{2} = O\left(n^{-C}\right) \text{ and } \frac{B_n - \beta_n}{\delta_n} = O\left(n^{-C}\right). \tag{1.10}$$

Remarks
(a) We note that in [4, Thm. 15.2, p. 402], we proved this for a more general class of weights, with $o(1)$ instead of $O\left(n^{-C}\right)$.
(b) The same proof works for a larger class of weights, namely the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ in [4]. However, that class has a less explicit definition, so is omitted.

When the interval is finite, and we have information on the rate of approach of δ_n to $\frac{b-a}{2}$, then we can turn this order relation into an asymptotic. This requires a little more notation: throughout this paper, we use the notation $\sim$ in the following sense: given sequences of real numbers $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if for some positive constants C_1, C_2 independent of n, we have

$$C_1 \leq c_n/d_n \leq C_2.$$

Corollary 1.4
Let $W \in \mathcal{E}(C^2)$, and (a,b) be finite. Then

$$\frac{b-a}{4} - A_n = \left(\frac{b-a}{4} - \frac{\delta_n}{2}\right)(1+o(1)); \tag{1.11}$$

$$\frac{a+b}{2} - B_n = O\left(\frac{b-a}{4} - \frac{\delta_n}{2}\right). \tag{1.12}$$

Moreover, if we define $a_{\pm t}$ for all t by (1.1-1.2), then

$$\frac{b-a}{4} - \frac{\delta_n}{2} \sim \left(\int_n^\infty + \int_{-\infty}^{-n}\right) \frac{dt}{tT(a_t)}. \tag{1.13}$$

To make this asymptotic more explicit, we need further hypotheses. Since $Q(x)\, sign(x)$ is strictly increasing on (a,b), and maps that interval onto $(-\infty,\infty)$, it has an inverse, which we denote by $Q^{[-1]} : (-\infty,\infty) \to (a,b)$.

Corollary 1.5
Let $W \in \mathcal{E}(C^2)$, and (a,b) be finite. Assume also that for each $\eta \in (0,1)$, there exists A_η and $\varepsilon \in (0,1)$ such that

$$u \geq A_\eta \Rightarrow \frac{b - Q^{[-1]}(u^{1-\varepsilon})}{b - Q^{[-1]}(u)} \leq 1 + \eta, \tag{1.14}$$

with a similar assertion for negative u. Then

$$\begin{aligned} \frac{b-a}{4} - A_n &= \frac{1}{4}\left(b - Q^{[-1]}(n) + Q^{[-1]}(-n) - a\right)(1+o(1)); \\ \frac{b+a}{4} - B_n &= O\left(b - Q^{[-1]}(n) + Q^{[-1]}(-n) - a\right). \end{aligned}$$

(1.15)

Remarks

(a) In the special case $(a,b)=(-1,1)$, the result becomes

$$\begin{aligned} \frac{1}{2}-A_n &= \frac{1}{4}\left(2-Q^{[-1]}(n)-Q^{[-1]}(-n)\right)(1+o(1)); \\ B_n &= O\left(2-Q^{[-1]}(n)-Q^{[-1]}(-n)\right). \end{aligned}$$

(b) The condition (1.14) is not always true for $W\in\mathcal{E}(C^2)$. For example, if $A>1$ and $W=\exp(-Q)$, where

$$Q(x)=\exp\left(\left|\log\left(1-x^2\right)\right|^A\right)-1,\ x\in(-1,1),$$

then $W\in\mathcal{E}(C^2)$, but (1.14) fails. For this weight one can check that the conclusion of Corollary 1.5 is still true provided $A>2$. It is not, however true for $1<A<2$, since $1-Q^{[-1]}(n)$ and $1-a_n$ decay at different rates in this case. We shall discuss this example in more detail in Section 5. So when one does not assume something like (1.14), one cannot always reformulate Corollary 1.4 as Corollary 1.5.

We give the idea of proof in the next section, and the technical details in Section 3. Throughout this paper, $C, C_1, C_2 ...$ denote positive constants independent of $n, x, ...$ The same symbol does not necessarily denote the same constant in different occurrences.

2. The Idea of Proof

We use well known representations of A_n, B_n in the form

$$\begin{aligned} \frac{A_n}{\delta_n} &= \int_a^b\left(\frac{x-\beta_n}{\delta_n}\right)p_n(x)\,p_{n+1}(x)\,W^2(x)\,dx; \\ \frac{B_n-\beta_n}{\delta_n} &= \int_a^b\left(\frac{x-\beta_n}{\delta_n}\right)p_n^2(x)\,W^2(x)\,dx. \end{aligned} \tag{2.1}$$

We split the interval (a,b) as

$$(a,b)=\mathcal{I}_n\cup\mathcal{J}_n\cup\mathcal{K}_n \tag{2.2}$$

where

$$\mathcal{K}_n=(a,b)\setminus\left(a_{-n}\left(1+n^{-C_1}\right),a_n\left(1+n^{-C_1}\right)\right); \tag{2.3}$$

is the main "tail" interval;

(2.4)

$$\mathcal{J}_n=\left(a_{-n}\left(1+n^{-C_1}\right),a_n\left(1+n^{-C_1}\right)\right)\setminus\left(a_{-n}+\delta_n n^{-\varepsilon},a_n-\delta_n n^{-\varepsilon}\right);$$

consists of small intervals near $a_{\pm n}$; and

$$\mathcal{I}_n=\left(a_{-n}+\delta_n n^{-\varepsilon},a_n-\delta_n n^{-\varepsilon}\right) \tag{2.5}$$

is the "main part" of (a,b). Here $C_1>0$ and $\varepsilon \in \left(0,\frac{1}{20}\right)$ are constants independent of n.

Using restricted range inequalities, we show that for some $C_1, C_2 > 0$

$$\int_{\mathcal{K}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n(x)\, p_{n+1}(x)\, W^2(x)\, dx = O\left(\exp\left(-n^{C_2}\right)\right), \tag{2.6}$$

with a similar tail estimate for the integral for B_n. Next, we can use global bounds on $p_n W$ and $p_{n+1}W$, namely

$$|p_n W|(x)\, |(x-a_{-n})(a_n - x)|^{1/4} \le C,\ \ x \in (a,b)$$

that were established in [4] to show that

$$\int_{\mathcal{J}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n(x)\, p_{n+1}(x)\, W^2(x)\, dx = O\left(n^{-C_3}\right), \tag{2.7}$$

for some $C_3 > 0$, with similar estimates for integrals arising from B_n.

Then it remains to deal with the integrals

$$I = \int_{\mathcal{I}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n(x)\, p_{n+1}(x)\, W^2(x)\, dx$$

and

$$J = \int_{\mathcal{I}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n^2(x)\, W^2(x)\, dx.$$

We make the substitution

$$u = L_n(x) = \frac{x-\beta_n}{\delta_n} \Leftrightarrow x = L_n^{[-1]}(u) = \beta_n + \delta_n u$$

that maps $[a_{-n}, a_n]$ onto $[-1,1]$, and $\mathcal{J}_n$ onto $[-1+n^{-\varepsilon}, 1-n^{-\varepsilon}]$ so that

$$I = \delta_n \int_{-1+n^{-\varepsilon}}^{1-n^{-\varepsilon}} u \left(p_n p_{n+1} W^2\right)\left(L_n^{[-1]}(u)\right) du; \tag{2.8}$$

$$J = \delta_n \int_{-1+n^{-\varepsilon}}^{1-n^{-\varepsilon}} u \left(p_n^2 W^2\right)\left(L_n^{[-1]}(u)\right) du. \tag{2.9}$$

If ε is small enough, asymptotics in [4] show that for $x = \cos\theta \in [-1+n^{-\varepsilon}, 1-n^{-\varepsilon}]$, and $m = n, n+1$

$$\begin{aligned} &\delta_n^{1/2} (p_m W)\left(L_n^{[-1]}(\cos\theta)\right)(\sin\theta)^{1/2} \\ &= \sqrt{\frac{2}{\pi}} \cos\left((m-n)\theta + \Gamma\right) + O\left(n^{-C_4}\right). \end{aligned} \tag{2.10}$$

Here $\Gamma = \Gamma(n, \theta)$ is an explicitly given function. Then

$$\begin{aligned} I &= \frac{2}{\pi} \int_{n^{-C_5}}^{\pi - n^{-C_5}} \cos\theta \cos\Gamma \cos(\theta + \Gamma)\ d\theta + O\left(n^{-C_6}\right) \\ &= \frac{1}{\pi} \int_{n^{-C_5}}^{\pi - n^{-C_5}} \cos\theta \left[\cos(\theta + 2\Gamma) + \cos\theta\right]\ d\theta + O\left(n^{-C_6}\right) \\ &= \frac{1}{\pi} \int_{n^{-C_5}}^{\pi - n^{-C_5}} \left(\cos\theta \cos(\theta + 2\Gamma) + \frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\ d\theta + O\left(n^{-C_6}\right) \\ &= \frac{1}{2} + \frac{1}{\pi} \int_{n^{-C_5}}^{\pi - n^{-C_5}} \cos\theta \cos(\theta + 2\Gamma)\ d\theta + O\left(n^{-C_6}\right). \end{aligned} \tag{2.11}$$

We show that the integral in the last right-hand side is $O\left(n^{-C}\right)$ for some $C > 0$, by using a Riemann-Lebesgue type lemma. To do this, one uses the fact that $\Gamma = n f_n(\theta)$ for some smooth increasing f_n, makes a substitution $t = f_n^{[-1]}(\theta)$, and then applies results on the degree of trigonometric polynomial approximation. This is the most technical part of the proof. Similar considerations apply to J. Combining the above estimates (2.6), (2.7) and (2.11) gives the result.

3. Proof of Theorem 1.3

In this section, we flesh out the technical details for the ideas given in the previous section. We begin with the tail integrals in (2.6), employing a restricted range inequality. Throughout we assume that $W \in \mathcal{F}(C^2)$ and use the notation $a_{\pm n}, \delta_n, \beta_n, T$ introduced in Section 1, as well as that from Section 2 - in particular, $\mathcal{I}_n, \mathcal{J}_n, \mathcal{K}_n$ of (2.2) to (2.5). Indeed our main task is to rigorously estimate the integrals over $\mathcal{I}_n, \mathcal{J}_n$ and $\mathcal{K}_n$. We also let

$$\eta_{\pm n} = \left\{ nT(a_{\pm n}) \frac{|a_{\pm n}|}{\delta_n} \right\}^{-2/3}.$$

We note that in the case of a finite interval $[a, b]$, the factor $\frac{|a_{\pm n}|}{\delta_n} \sim 1$, so can be dropped.

Lemma 3.1
Let $0 < p \le \infty$ and $W \in \mathcal{F}(C^2)$. Then for some $C_1, C_2, C_3 > 0$, for all $n \ge 1$, and all polynomials P of degree $\le n$,

$$\|PW\|_{L_p(\mathcal{K}_n)} \le C_2 \exp\left(-n^{C_3}\right) \|PW\|_{L_p(a_{-n}, a_n)}.$$

Proof
This is an easy consequence of Theorem 4.2(a) and (b) in [4, p. 96]. Taking in (b) there $\kappa_\pm = n^C \eta_{\pm n}$, with $C < \frac{2}{3}$ so small that

$$\kappa_\pm < T\left(a_{\pm n}\right)^{-1},$$

which is possible by Lemma 3.7 there [4, p. 76, eqn. (3.39)], we obtain

$$\|PW\|_{L_p((a,b)\setminus[a_{-n}(1+\kappa_-),a_n(1+\kappa_+)])} \leq C_2 \exp\left(-C_4 n^{3C/2}\right) \|PW\|_{L_p(a,b)}.$$

Since [4, eqn. (3.39), p. 76], if $C > 0$ is small enough,

$$\kappa_\pm = n^C \left\{ nT\left(a_{\pm n}\right) \sqrt{\frac{|a_{\pm n}|}{\delta_n}} \right\}^{-2/3} = O\left(n^{-C_1}\right),$$

for some $C_1 > 0$, the result follows, using also Theorem 4.2(a) in [4, p. 96]. ■

Lemma 3.2
For some $C_1, C_2, C_3 > 0$,

$$\int_{\mathcal{K}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n(x)\, p_{n+1}(x) \right| W^2(x)\, dx \leq C_2 \exp\left(-n^{C_3}\right); \tag{3.1}$$

$$\int_{\mathcal{K}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n^2(x) \right| W^2(x)\, dx \leq C_2 \exp\left(-n^{C_3}\right). \tag{3.2}$$

Proof
We note first that

$$\int_{\mathcal{I}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n(x)\, p_{n+1}(x) \right| W^2(x)\, dx \leq 1 + n^{-\varepsilon}.$$

Indeed, $\left|\frac{x-\beta_n}{\delta_n}\right| \leq 1 + n^{-\varepsilon}$ for $x \in \mathcal{I}_n$ and we can apply the Cauchy-Schwarz inequality. Similarly,

$$\int_{\mathcal{I}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n^2(x)\, W^2(x) \right| dx \leq 1 + n^{-\varepsilon}.$$

We now apply Lemma 3.1 with the weight $W^2 = e^{-2Q}$, and $p = 1$; we also use the fact that a_{n+1} for W is the same as a_{2n+2} for W^2, while [4, (3.51), p. 81]

$$a_{n+1}/a_n = 1 + O\left(\frac{1}{nT\left(a_n\right)}\right) = 1 + O\left(\frac{1}{n}\right),$$

with a similar relation for a_{-n-1}/a_{-n}. This gives the result. ■

Next we go a little inside the Mhaskar-Rakhmanov-Saff interval:

Lemma 3.3
There exist C_1, C_2 such that

$$\int_{\mathcal{J}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n(x)\, p_{n+1}(x) \right| W^2(x)\, dx \leq C_1 n^{-C_2}; \tag{3.3}$$

$$\int_{\mathcal{J}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n^2(x) \right| W^2(x)\, dx \leq C_1 n^{-C_2}. \tag{3.4}$$

Proof
We use the bound

$$|p_m(x)|\, W(x)\, |(x-a_{-n-1})(a_{n+1}-x)|^{1/4} \leq C,\ \ x \in (a,b).$$

It is valid for $m = n$ and $n+1$. See [4, (15.41), p. 413], and replace n by $n+1$ there. Then we see that

$$\begin{aligned} & \int_{\mathcal{J}_n} \left| \left(\frac{x-\beta_n}{\delta_n} \right) p_n(x)\, p_{n+1}(x) \right| W^2(x)\, dx \\ \leq\ & C \int_{\mathcal{J}_n} \frac{dx}{\sqrt{|x-a_{-n-1}|\,|a_{n+1}-x|}} \\ \leq\ & C \max\left\{ n^{-\varepsilon/2}, n^{-C_1/2} \right\}, \end{aligned}$$

since [4, eqn. (3.51), p. 81]

$$\frac{1-a_{\pm n}}{1-\alpha_{\pm(n+1)}} = 1 + O\left(\frac{1}{n}\right).$$

■

The most difficult part is the next dealing with the integral over $\mathcal{I}_n$. We use:

Lemma 3.4
Let

$$L_n(x) = \frac{x-\beta_n}{\delta_n} \Leftrightarrow L_n^{[-1]}(u) = \beta_n + \delta_n u. \tag{3.5}$$

There exists $\varepsilon > 0$ such that uniformly for $n \geq m \geq n - \frac{1}{2} n^{1/3}$, and uniformly for $|x| \leq 1 - n^{-\varepsilon}$, $x = \cos\theta$,

$$\begin{aligned} & \delta_n^{1/2} (p_m W)\left(L_n^{[-1]}(x) \right) \left(1-x^2\right)^{1/4} \\ = & \sqrt{\frac{2}{\pi}} \cos\left(\left(m - n + \frac{1}{2} \right) \theta + n f_n(\theta) - \frac{\pi}{4} \right) + O\left(n^{-\varepsilon}\right). \end{aligned}$$

(3.6)

Here

$$f_n(\theta) = \pi \int_{\cos\theta}^{1} \sigma_n^*(t)\,dt; \tag{3.7}$$

$$\sigma_n^*(t) = \frac{\delta_n}{n}\sigma_n\left(L_n^{[-1]}(t)\right),\ t \in [-1,1], \tag{3.8}$$

and

$$\sigma_n(x) = \frac{\sqrt{(x-a_{-n})(a_n-x)}}{\pi^2}\int_{a_{-n}}^{a_n}\frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}.$$

Proof
This is Theorem 15.3 in [4, p. 403]. ■

We shall also need some estimates on σ_n^* :

Lemma 3.5
(a)

$$\int_{-1}^{1}\sigma_n^*(t)\,dt = 1. \tag{3.9}$$

(b) Uniformly for $n \geq 1$ and $t \in (-1,1)$,

$$\sigma_n^*(t) \sim \frac{\sqrt{1-t^2}}{(1-t+\chi_t)\left(1+t+\chi_{-t}\right)}, \tag{3.10}$$

where

$$\chi_{\pm t} = \frac{|a_{\pm t}|}{\delta_t T(a_{\pm t})}. \tag{3.11}$$

(c) For some $C > 0$ and for all $u, v \in [-1,1]$,
(3.12)

$$\left|\sigma_n^*(u)\sqrt{1-u^2} - \sigma_n^*(v)\sqrt{1-v^2}\right| \leq C\left|\frac{u-v}{(1-u+\chi_t)\left(1+u+\chi_{-t}\right)}\right|^{1/4}.$$

Proof
(a) This is (1.75) in [4, p. 17].
(b) This is Theorem 1.10(IV) in [4, p. 17]. Note that our class $\mathcal{F}(C^2)$ is contained in the class $\mathcal{F}\left(Lip\frac{1}{2}\right)$ there.
(c) This is Theorem 6.3(a) in [4, p. 148] with $\psi(x) = |x|^{1/2}$. ■

Lemma 3.6
Let $\varepsilon > 0$ *and*

$$\begin{aligned} I_n &= \int_{\mathcal{I}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n(x)\, p_{n+1}(x)\, W^2(x)\, dx; \\ J_n &= \int_{\mathcal{I}_n} \left(\frac{x-\beta_n}{\delta_n}\right) p_n^2(x)\, W^2(x)\, dx. \end{aligned}$$

Then

$$\begin{aligned} I_n &= \frac{1}{2} + \frac{1}{\pi} K_n + O\left(n^{-C}\right); \\ J_n &= \frac{1}{\pi}\left(L_n - M_n\right) + O\left(n^{-C}\right), \end{aligned}$$

where

$$K_n = \int_{n^{-\varepsilon}}^{\pi - n^{-\varepsilon}} \cos\theta \sin\left((2n+2) f_{n+1}(\theta)\right) d\theta; \tag{3.13}$$

$$L_n = \int_{n^{-\varepsilon}}^{\pi - n^{-\varepsilon}} \cos^2\theta \sin\left((2n+2) f_{n+1}(\theta)\right) d\theta; \tag{3.14}$$

$$M_n = \int_{n^{-\varepsilon}}^{\pi - n^{-\varepsilon}} \cos\theta \sin\theta \cos\left((2n+2) f_{n+1}(\theta)\right) d\theta. \tag{3.15}$$

Proof
The substitution $x = L_n^{[-1]}(\cos\theta)$ gives

$$\begin{aligned} I_n &= \delta_n \int_{\cos^{-1}(1-n^{-\varepsilon})}^{\cos^{-1}\left(-1+n^{-\varepsilon}\right)} \cos\theta \left(p_n p_{n+1} W^2\right)\left(L_n^{[-1]}(\cos\theta)\right) \sin\theta \; d\theta \\ &= \frac{2}{\pi} \int_{\cos^{-1}(1-n^{-\varepsilon})}^{\cos^{-1}\left(-1+n^{-\varepsilon}\right)} \cos\theta \cos\left(-\frac{\theta}{2} + (n+1) f_{n+1}(\theta) - \frac{\pi}{4}\right) \times \\ &\quad \times \cos\left(\frac{\theta}{2} + (n+1) f_{n+1}(\theta) - \frac{\pi}{4}\right) d\theta + O\left(n^{-C}\right), \end{aligned}$$

by Lemma 3.4, applied with n replaced by $n+1$ and $m = n, n+1$. Absorbing part of the integral into the order term, we continue this as

$$\begin{aligned} I_n &= \frac{1}{\pi} \int_{n^{-\varepsilon}}^{\pi - n^{-\varepsilon}} \cos\theta \left[\cos\left((2n+2) f_{n+1}(\theta) - \frac{\pi}{2}\right) + \cos\theta\right] d\theta + O\left(n^{-C}\right) \\ &= \frac{1}{2} + \frac{1}{\pi} \int_{n^{-\varepsilon}}^{\pi - n^{-\varepsilon}} \cos\theta \sin\left((2n+2) f_{n+1}(\theta)\right) d\theta + O\left(n^{-C}\right). \end{aligned}$$

Similarly,

$$\begin{aligned} J_n &= \delta_n \int_{\cos^{-1}(1-n^{-\varepsilon})}^{\cos^{-1}\left(-1+n^{-\varepsilon}\right)} \cos\theta \left(p_n^2 W^2\right)\left(L_n^{[-1]}(\cos\theta)\right)\sin\theta \; d\theta \\ &= \frac{2}{\pi}\int_{\cos^{-1}(1-n^{-\varepsilon})}^{\cos^{-1}\left(-1+n^{-\varepsilon}\right)} \cos\theta\cos^2\left(-\frac{\theta}{2}+(n+1)f_{n+1}(\theta)-\frac{\pi}{4}\right)d\theta + O\left(n^{-C}\right) \\ &= \frac{1}{\pi}\int_{n^{-\varepsilon}}^{\pi-n^{-\varepsilon}} \left[\begin{array}{c} -\cos\theta\sin\theta\cos\left((2n+2)f_{n+1}(\theta)\right) \\ +\cos^2\theta\sin\left((2n+2)f_{n+1}(\theta)\right)\end{array}\right] d\theta + O\left(n^{-C}\right). \end{aligned}$$

■

Now we study properties of the function f_n defined by (3.7), with a view to showing $K_n, L_n, M_n \to 0$ as $n \to \infty$.

Lemma 3.7
(a) f_n is a strictly increasing continuous function that maps $[0,\pi]$ onto $[0,\pi]$.
(b) For $n \geq 1$ and $\theta \in [0,\pi]$,

$$C_1 \sin^2\theta \leq f_n'(\theta) \leq C_2. \tag{3.16}$$

(c) For $n \geq 1$ and $\theta \in [0,\pi]$,

$$\left|f_n'(\theta) - f_n'(\phi)\right| \leq C\left(\frac{|\theta-\phi|}{\max\left\{\sin^2\theta, \sin^2\phi\right\}}\right)^{1/4}. \tag{3.17}$$

(d)

$$C_1\theta \geq f_n(\theta) \geq C_2\theta^3,\ \theta \in \left[0, \frac{\pi}{2}\right]; \tag{3.18}$$

$$C_1(\pi-\theta) \geq \pi - f_n(\theta) \geq C_2(\pi-\theta)^3,\ \theta \in \left[\frac{\pi}{2}, \pi\right]. \tag{3.19}$$

(e) Let $g_n = f_n^{[-1]}$ denote the inverse function of f_n. Then

$$C_1 t \leq g_n(t) \leq C_2 t^{1/3},\ t \in \left[0, \frac{\pi}{2}\right];$$

$$C_1(\pi-t) \leq \pi - g_n(t) \leq C_2(\pi-t)^{1/3},\ t \in \left[\frac{\pi}{2}, \pi\right]. \tag{3.20}$$

(f) For $n \geq 1$,

$$C_2 \leq g_n'(t) \leq C_1 t^{-2},\ t \in \left[0, \frac{\pi}{2}\right]. \tag{3.21}$$

$$C_2 \leq g_n'(t) \leq C_1(\pi-t)^{-2},\ t \in \left[\frac{\pi}{2}, \pi\right]. \tag{3.22}$$

(g) For $n \geq 1$ and $0 < s < t < \pi$,

$$|g_n'(s) - g_n'(t)| \leq C\,|s-t|^{1/4} \max\left\{s^{-5}, (\pi - t)^{-5}\right\}. \tag{3.23}$$

Proof

(a) The normalization (3.9) shows that f_n maps $[0,\pi]$ onto $[0,\pi]$.

(b) We see from (3.7) and Lemma 3.5(b) that for $\theta \in [0,\pi]$,

$$f_n'(\theta) = \pi\sigma_n^*(\cos\theta)\sin\theta \geq C_1 \sin^2\theta,$$

and

$$f_n'(\theta) \leq C_2.$$

(c) This follows from Lemma 3.5(c).

(d) Now for $\theta \in \left[0, \frac{\pi}{2}\right]$,

$$\begin{aligned} f_n(\theta) &= \int_0^\theta f_n'(t)\,dt \\ &\geq C\int_0^\theta \sin^2 t\ dt \\ &\geq C_2\theta^3. \end{aligned}$$

Similarly, our upper bound on f_n' gives

$$f_n(\theta) \leq C_1\theta.$$

The bound near π is proved similarly.

(e) Firstly setting $\theta = g_n(t)$ in the bounds in (d) gives

$$\begin{aligned} C_1 g_n(t) &\geq t \geq C_2 g_n^3(t)\,,\ g_n(t) \in \left[0, \frac{\pi}{2}\right]; \\ C_1(\pi - g_n(t)) &\geq \pi - t \geq C_2(\pi - g_n(t))^3\,,\ g_n(t) \in \left[\frac{\pi}{2}, \pi\right]. \end{aligned}$$

Here the constants are independent of n, t. Moreover, for each fixed $\varepsilon > 0$, $g_n \sim 1$ in $[\varepsilon, \pi - \varepsilon]$, uniformly in n. Then the result follows.

(f) For $t \in \left[0, \frac{\pi}{2}\right]$, we know that $g_n(t) \leq \pi - C$, and hence the bounds of (a), (e) give

$$g_n'(t) = \frac{1}{f_n'(g_n(t))} \leq C\sin^{-2} g_n(t) \leq Ct^{-2}.$$

Similarly, the bounds of (a) give for $t \in [0,\pi]$,

$$g_n'(t) = \frac{1}{f_n'(g_n(t))} \geq C.$$

For $t \in \left[\frac{\pi}{2}, \pi\right]$, we know that $g_n(t) \geq C$ and hence the bounds of (b), (e) give

$$g_n'(t) = \frac{1}{f_n'(g_n(t))} \leq C \sin^{-2} g_n(t) \leq C(\pi - g_n(t))^{-2} \leq C(\pi - t)^{-2}.$$

(g) For $0 < s < t \leq \frac{\pi}{2}$,

$$\begin{aligned} |g_n'(t) - g_n'(s)| &= \left| \frac{1}{f_n'(g_n(t))} - \frac{1}{f_n'(g_n(s))} \right| \\ &\leq \frac{|f_n'(g_n(s)) - f_n'(g_n(t))|}{|f_n'(g_n(t)) f_n'(g_n(s))|} \\ &\leq \left(\frac{|g_n(s) - g_n(t)|}{\max\left\{\sin^2 g_n(s), \sin^2 g_n(t)\right\}} \right)^{1/4} g_n'(t) g_n'(s) \\ &\leq C \left(\frac{|s-t| s^{-2}}{s^2} \right)^{1/4} t^{-2} s^{-2} \leq C |s-t|^{1/4} s^{-5}, \end{aligned}$$

by the Mean Value Theorem and the bounds of (f). Similarly for $\frac{\pi}{2} \leq s < t < \pi$,

$$|g_n'(t) - g_n'(s)| \leq C |s-t|^{1/4} (\pi - t)^{-5}.$$

Combining these two estimates in an obvious way gives the result. ■

Lemma 3.8

For some $C > 0$,

$$K_n, \; L_n, \; M_n = O\left(n^{-C}\right). \tag{3.24}$$

Proof

We estimate K_n; the proof for L_n, M_n is very similar. By a substitution $t = 2f_{n+1}(\theta)$ in (3.13) and the properties of g_{n+1} in the previous lemma, we see that

$$K_n = \int_{n^{-\varepsilon}}^{2\pi - n^{-\varepsilon}} \cos g_{n+1}\left(\frac{t}{2}\right) \sin(n+1)t \; g_{n+1}'\left(\frac{t}{2}\right) \frac{1}{2} \, dt + O\left(n^{-C}\right).$$

Define

$$h_n(t) = \frac{1}{2} \left\{ \begin{array}{cc} \left[\cos g_{n+1}\left(\frac{1}{2} n^{-\varepsilon}\right)\right] g_{n+1}'\left(\frac{1}{2} n^{-\varepsilon}\right), & t \in [0, n^{-\varepsilon}] \\ \left[\cos g_{n+1}\left(\frac{t}{2}\right)\right] g_{n+1}'\left(\frac{t}{2}\right), & t \in [n^{-\varepsilon}, 2\pi - n^{-\varepsilon}] \\ \left[\cos g_{n+1}\left(\pi - \frac{1}{2} n^{-\varepsilon}\right)\right] g_{n+1}'\left(\pi - \frac{1}{2} n^{-\varepsilon}\right), & t \in [2\pi - n^{-\varepsilon}, 2\pi] \end{array} \right. .$$

Then still

$$K_n = \int_0^{2\pi} h_n(t) \sin(n+1)t \, dt + O\left(n^{-C}\right).$$

Indeed by Lemma 3.7(f),

$$\begin{aligned}&\int_0^{n^{-\varepsilon}} h_n(t)\sin(n+1)t\ dt \\ =\ & \frac{1}{2}\left[\cos g_{n+1}\left(\frac{1}{2}n^{-\varepsilon}\right)\right] g'_{n+1}\left(\frac{1}{2}n^{-\varepsilon}\right)\int_0^{n^{-\varepsilon}} \sin(n+1)t\ dt \\ =\ & O\left(n^{2\varepsilon-1}\right),\end{aligned}$$

and we assumed $0 < \varepsilon < \frac{1}{20}$. Now we use the orthogonality of $\sin(n+1)t$ to trigonometric polynomials T of degree $\leq n$ to deduce that

$$|K_n| \leq \inf_T \int_0^{2\pi} |h_n - T| + O\left(n^{-C}\right),$$

where the inf is taken over all T of degree $\leq n-1$. We continue this using Jackson estimates [2, Thm. 2.3, p. 205] as

$$\begin{aligned}|K_n| &\leq \sup_{x,y\in[0,2\pi],|x-y|\leq\frac{1}{n}} |h_n(x) - h_n(y)| + O\left(n^{-C}\right) \\ &\leq Cn^{-\frac{1}{4}+5\varepsilon} + O\left(n^{-C}\right),\end{aligned}$$

by the estimates of Lemma 3.7 (f), (g). We assumed in Section 2 that $\varepsilon < \frac{1}{20}$ and so we are done. ■

Proof of Theorem 1.3

In (2.1), we split the integrals as in (2.2),

$$\int_a^b = \int_{\mathcal{I}_n} + \int_{\mathcal{J}_n} + \int_{\mathcal{K}_n}.$$

We showed that

$$\int_{\mathcal{J}_n} + \int_{\mathcal{K}_n} = O\left(n^{-C}\right)$$

in Lemmas 3.2 and 3.3. The remaining integrals over $\mathcal{I}_n$ were handled in Lemma 3.6 and Lemma 3.8, where we showed for the first integral in (2.1),

$$\int_{\mathcal{I}_n} = \frac{1}{2} + O\left(n^{-C}\right).$$

The second integral in (2.1) is similar. ■

4. Proof of Corollaries 1.4, 1.5 and Theorem 1.1

Throughout this section, we assume at least the hypotheses of Corollary 1.4 - in particular that (a,b) is finite. By Theorem 1.3 and finiteness of (a,b), which forces boundedness of $\{\delta_n\}$,

$$A_n - \frac{\delta_n}{2} = O\left(n^{-C}\right) \text{ and } B_n - \beta_n = O\left(n^{-C}\right),$$

so

$$\begin{aligned} \frac{b-a}{4} - A_n &= \frac{b-a}{4} - \frac{\delta_n}{2} + O\left(n^{-C}\right); \\ \frac{b+a}{4} - B_n &= \frac{b+a}{4} - \beta_n + O\left(n^{-C}\right) = O\left(\frac{b-a}{4} - \delta_n\right) + O\left(n^{-C}\right), \end{aligned} \tag{4.1}$$

recall $a < 0 < b$ and $a_{-n} < 0 < a_n$. The first two conclusions of Corollary 1.4 will then follow if we can show that for each fixed $\varepsilon > 0$, and large enough n,

$$\frac{b-a}{4} - \delta_n > n^{-\varepsilon}. \tag{4.2}$$

We first gather some technical estimates:

Lemma 4.1
(a) Uniformly for $t \neq 0$,

$$\frac{a_t'}{a_t} \sim \frac{1}{tT(a_t)}. \tag{4.3}$$

(b) Uniformly for $t \neq 0$,

$$Q(a_t) \sim |t|\, T(a_t)^{-1/2}. \tag{4.4}$$

(c) $T(x) \to \infty$ as $x \to a+$ or $b-$.
Proof
(a) See [4, p. 79, Thm. 3.10(a)].
(b) See (3.18) in [4, p. 69, Lemma 3.4] and note that $\delta_n \sim |a_{\pm n}| \sim 1$ in this case.
(c) See Lemma 3.2(f) in [4, p. 65] and note that there the interval is (c,d) rather than (a,b). ■

Lemma 4.2

Assume that $W \in \mathcal{E}(C^2)$. *Then as* $n \to \infty$,

$$b - a_n \sim \int_n^\infty \frac{dt}{tT(a_t)}; \tag{4.5}$$

$$a_{-n} - a \sim \int_{-\infty}^{-n} \frac{dt}{tT(a_t)}. \tag{4.6}$$

Moreover, given $\varepsilon > 0$, *we have for large enough* n,

$$\int_n^\infty \frac{dt}{tT(a_t)} + \int_{-\infty}^{-n} \frac{dt}{tT(a_t)} > n^{-\varepsilon}. \tag{4.7}$$

Proof

If $m > n$,

$$\log \frac{a_m}{a_n} = \int_n^m \frac{a_t'}{a_t}\, dt \sim \int_n^m \frac{dt}{tT(a_t)},$$

by Lemma 4.1(a). The constants implicit in $\sim$ are independent of m, n. Since $a_m \to b$ as $m \to \infty$, we obtain

$$1 - \frac{a_n}{b} \sim \log \frac{b}{a_n} \sim \int_n^\infty \frac{dt}{tT(a_t)}.$$

Then (4.5) follows and (4.6) is similar. Next, we assume (1.8), (the case where (1.9) holds is similar),

$$T(u) = O(Q(u)^\varepsilon),\ u \to b-.$$

Then as $t \to \infty$, Lemma 4.1(b) gives

$$T(a_t) = O(Q(a_t)^\varepsilon) = O(t^\varepsilon). \tag{4.8}$$

This has the consequence that

$$\int_n^\infty \frac{dt}{tT(a_t)} \geq C \int_n^\infty \frac{dt}{t^{1+\varepsilon}} \geq Cn^{-\varepsilon},$$

as stressed. Then (4.7) follows. ■

Proof of Corollary 1.4

We add (4.5) , (4.6) and divide by 4: for given $\varepsilon > 0$, as $n \to \infty$,

$$\frac{b-a}{4} - \frac{\delta_n}{2} \sim \int_n^\infty \frac{dt}{tT(a_t)} + \int_{-\infty}^{-n} \frac{dt}{tT(a_t)} > n^{-\varepsilon}. \tag{4.9}$$

Now the result follows immediately from (4.1). ■

Lemma 4.3
Under the hypotheses of Corollary 1.5,

$$\lim_{n\to\infty}\frac{b-a_n}{b-Q^{[-1]}(n)}=1 \tag{4.10}$$

and

$$\lim_{n\to\infty}\frac{a_{-n}-a}{Q^{[-1]}(-n)-a}=1. \tag{4.11}$$

Proof
By Lemma 4.1(b), (c), and (4.8), we have for large enough n,

$$n^{1-\varepsilon}\leq Q(a_n)\leq n$$
$$\Rightarrow b-Q^{[-1]}\left(n^{1-\varepsilon}\right)\geq b-a_n\geq b-Q^{[-1]}(n).$$

By hypothesis, given $\eta\in(0,1)$, there exists $\varepsilon>0$ so that for large enough n,

$$1\leq\frac{b-Q^{[-1]}\left(n^{1-\varepsilon}\right)}{b-Q^{[-1]}(n)}\leq 1+\eta.$$

Then (4.10) follows. The other relation is similar. ■

Proof of Corollary 1.5
By the lemma,

$$\begin{aligned}\frac{b-a}{4}-\frac{\delta_n}{2}&=\frac{1}{4}\left(b-Q^{[-1]}(n)+Q^{[-1]}(-n)-a\right)(1+o(1));\\ \frac{b+a}{4}-\frac{\beta_n}{2}&=O\left(b-Q^{[-1]}(n)+Q^{[-1]}(-n)-a\right).\end{aligned}$$

(4.12)

Now (4.1) and the fact that $\frac{b-a}{4}-\frac{\delta_n}{2}$ decays slower than any negative power of n (recall Lemma 4.2) give the result. ■

Proof of Theorem 1.1
We first show that these weights satisfy the hypotheses of Corollary 1.4 and 1.5. Let us assume $k\geq 1$ (the case where $\ell\geq 1$ is similar). In (1.37) of [4, p. 9], it is shown that as $x\to 1-$,

$$T(x)=\frac{2\alpha}{(1-x^2)^{\alpha+1}}\left[\prod_{j=1}^{k-1}\exp_j\left(\left(1-x^2\right)^{-\alpha}\right)\right](1+o(1)).$$

From this follows that for each $\varepsilon>0$,

$$T(x)=O\left(\log Q(x)^{1+\varepsilon}\right), x\to 1-,$$

which is much stronger than (1.8). The remaining hypotheses to belong to $\mathcal{F}(C^2)$ and $\mathcal{E}(C^2)$ follow easily, and were outlined in [4, p. 9]. We next show that the hypotheses of Corollary 1.5 are satisfied with $(a,b) = (-1,1)$. We have

$$\log_k Q(x) = \left(1-x^2\right)^{-\alpha}$$

$$\Rightarrow 1 - Q^{[-1]}(u)^2 = (\log_k u)^{-1/\alpha}$$

$$\Rightarrow 1 - Q^{[-1]}(u) = \frac{1}{2}(\log_k u)^{-1/\alpha}(1+o(1)) \tag{4.13}$$

as $u \to \infty$. Then

$$\begin{aligned} \frac{1-Q^{[-1]}\left(u^{1-\varepsilon}\right)}{1-Q^{[-1]}(u)} &= \frac{\left(\log_k u^{1-\varepsilon}\right)^{-\frac{1}{\alpha}}}{(\log_k u)^{-\frac{1}{\alpha}}}(1+o(1)) \\ &\leq (1-\varepsilon)^{-\frac{1}{\alpha}}(1+o(1)), \end{aligned}$$

even if $k > 1$. For given $\eta > 0$ and correspondingly small ε, this is no larger than $1+\eta$ so we can satisfy (1.14). If $\ell \geq 1$, we obtain a similar relation for $Q^{[-1]}(-n)$. Then Corollary 1.5, (4.13) and its analogue for negative u, give the conclusion of Theorem 1.1. When $\ell = 0$, $a_{-n}+1$ decays like a negative power of n (cf. [4, p. 31]), and $1-Q^{[-1]}(-n)$ also decays like a negative power of n. Then the dominant term in (1.7) is that involving $(\log_k n)^{-1/\alpha}$, and the term $(\log_\ell n)^{-1/\beta} = n^{-1/\beta}$ is much smaller, and can be absorbed into the order term. Again the result follows. ∎

5. An Example

In this section, we let $(a,b) = (-1,1)$, $A > 1$ and

$$Q(x) = \exp\left(\left|\log\left(1-x^2\right)\right|^A\right) - 1,\ x \in (-1,1).$$

Lemma 5.1
$W = e^{-Q} \in \mathcal{E}(C^2)$.
Proof
We see that

$$Q'(x) = [Q(x)+1]\,A\left|\log\left(1-x^2\right)\right|^{A-1}\frac{2x}{1-x^2} \tag{5.1}$$

and hence

$$T(x) = \frac{xQ'(x)}{Q(x)} = \left(1+\frac{1}{Q(x)}\right)A\left|\log\left(1-x^2\right)\right|^{A-1}\frac{2x^2}{1-x^2}. \tag{5.2}$$

We first show that

$$T(x) \geq 2A > 1,\ x \in (0,1). \tag{5.3}$$

Now

$$1 + \frac{1}{Q(x)} = \frac{1}{1 - \exp\left(-\left|\log\left(1-x^2\right)\right|^A\right)} \geq \left|\log\left(1-x^2\right)\right|^{-A},$$

using the elementary inequality

$$1 - e^{-u} \leq u,\ u \geq 0.$$

Hence

$$T(x) \geq 2A \frac{x^2}{\left|\log\left(1-x^2\right)\right|\left(1-x^2\right)}.$$

Using the elementary inequality

$$-\log(1-t)(1-t) \leq t,\ t \in (0,1),$$

we then obtain (5.3), the most difficult part of Definition 1.2(d). The relation

$$T(s) \leq C_1 T(t),\ 0 < s/t < 1,$$

follows for small s, t since $2A \leq T \leq C$ there. For s, t a little larger, we use the fact that if $C \in (0,1)$, we have

$$T(x) \sim \left|\log\left(1-x^2\right)\right|^{A-1}\left(1-x^2\right)^{-1} \text{ in } (C,1)$$

and the function on the right-hand side is increasing in x. So we have (d) of Definition 1.2. The requirement (e) is easy. Finally, we prove (f). Let $\varepsilon > 0$ and $K > 1/\varepsilon$. For x close enough to 1,

$$Q(x) \geq \exp\left(K\left|\log\left(1-x^2\right)\right|\right) = \left(1-x^2\right)^{-K}.$$

Then as $x \to 1-$,

$$T(x)/Q(x)^{\varepsilon} = O\left(\left(1-x^2\right)^{K\varepsilon-1}\left|\log\left(1-x^2\right)\right|^{A-1}\right) = o(1).$$

■

Lemma 5.2
(a) The limit (1.14) of Corollary 1.5 fails. More precisely, given $\varepsilon \in (0,1)$, we have

$$\lim_{t\to\infty} \frac{1 - Q^{[-1]}\left(t^{1-\varepsilon}\right)}{1 - Q^{[-1]}(t)} = \infty.$$

(b) If $A < 2$, the conclusion of Corollary 1.5 fails. More precisely,

$$\lim_{n\to\infty} \frac{1 - a_n}{1 - Q^{[-1]}(n)} = \infty.$$

Proof

(a) If $t = Q(x)$, then

$$1 + t = \exp\left(\left|\log\left(1 - x^2\right)\right|^A\right)$$

and hence

$$\exp\left(-\left(\log(1+t)\right)^{1/A}\right) = 1 - x^2.$$

Then as $x \to 1-$,

$$\begin{aligned} 1 - x &= \frac{1}{2}\exp\left(-\left(\log(1+t)\right)^{1/A}\right)(1 + o(1)) \\ &= \frac{1}{2}\exp\left(-(\log t)^{1/A}\right)(1 + o(1)). \end{aligned}$$

That is,

$$1 - Q^{[-1]}(t) = \frac{1}{2}\exp\left(-(\log t)^{1/A}\right)(1 + o(1)).$$

Then as $t \to \infty$,

$$\begin{aligned} \frac{1 - Q^{[-1]}\left(t^{1-\varepsilon}\right)}{1 - Q^{[-1]}(t)} &= \exp\left((\log t)^{1/A}\left\{1 - (1-\varepsilon)^{1/A}\right\}\right)(1 + o(1)) \\ &\to \infty. \end{aligned}$$

(b) We use the relation (4.4). Then

$$\log Q(a_n) = \log n - \frac{1}{2}\log T(a_n) + O(1)$$

(5.4)

$$\Rightarrow \left|\log\left(1 - a_n^2\right)\right|^A = \log n - \frac{1}{2}(A-1)\log\left|\log\left(1 - a_n^2\right)\right| - \frac{1}{2}\log\left|1 - a_n^2\right| + O(1).$$

Writing

$$\left|\log\left(1 - a_n^2\right)\right| = (\log n)^{1/A} - \eta,$$

we obtain

$$\left|\log\left(1 - a_n^2\right)\right|^A = \log n - \eta A(\log n)^{1-1/A} + O\left(\eta^2(\log n)^{1-2/A}\right).$$

Substituting this in (5.4) gives

$$\begin{aligned} &-\eta A(\log n)^{1-1/A} + O\left(\eta^2(\log n)^{1-2/A}\right) \\ =\ &O(\log\log n) - \frac{1}{2}(\log n)^{1/A} + \frac{\eta}{2} \end{aligned}$$

and hence

$$\frac{1}{2} = \eta\left\{A(\log n)^{1-2/A} + \frac{1}{2}(\log n)^{-1/A}\right\} + O\left(\eta^2(\log n)^{1-2/A}\right) + O\left((\log n)^{-1/A}(\log\log n)\right).$$

Here as $1-2/A>-1/A$, we obtain

$$\eta=\frac{1}{2A}(\log n)^{-1+2/A}(1+o(1)),$$

so

$$\left|\log\left(1-a_n^2\right)\right|=(\log n)^{1/A}-\frac{1}{2A}(\log n)^{-1+2/A}(1+o(1))$$

$$\Rightarrow 1-a_n=\frac{1}{2}\exp\left(-(\log n)^{1/A}+\frac{1}{2A}(\log n)^{-1+2/A}(1+o(1))\right)(1+o(1))$$

$$\begin{aligned}\Rightarrow \quad & \frac{1-a_n}{1-Q^{[-1]}(n)}=\exp\left(\frac{1}{2A}(\log n)^{-1+2/A}(1+o(1))\right)\\ \rightarrow \quad & \infty,\end{aligned}$$

as $n\to\infty$. Then the conclusion of Corollary 1.4 does not translate into the conclusion of Corollary 1.5. ■

Note that for $A<2$, we obtain from Corollary 1.4,

$$\begin{aligned}\frac{1}{2}-A_n &= \frac{1}{2}(1-a_n)(1+o(1))\\ &= \frac{1}{4}\exp\left(-(\log n)^{1/A}+\frac{1}{2A}(\log n)^{-1+2/A}(1+o(1))\right).\end{aligned}$$

Of course, with a little more work, this may be made more precise.

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[1]Mathematics Department, The Open University of Israel, P.O. Box 808, Raanana 43107, Israel., [2]School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., [1] lubinsky@math.gatech.edu, [2] elile@openu.ac.il