UNIVERSALITY LIMITS INVOLVING ORTHOGONAL POLYNOMIALS ON A SMOOTH CLOSED CONTOUR

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Abstract. We establish universality limits for measures on a smooth closed contour Γ in the plane. Assume that μ is a regular measure on Γ, in the sense of Stahl, Totik, and Ullmann. Let Γ_1 be a closed subarc of Γ, such that μ is absolutely continuous in an open arc containing Γ_1, and μ' is positive and continuous in that open subarc. Then universality for μ holds in Γ_1, in the sense that the reproducing kernels $\{K_n(z,t)\}$ for μ satisfy

$$\lim_{n\to\infty}\frac{K_n\left(z_0+\frac{2\pi is}{n}\frac{\Phi(z_0)}{\Phi'(z_0)},z_0+\frac{2\pi i\bar{t}}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}\right)}{K_n(z_0,z_0)}=e^{i\pi(s-t)}S(s-t),$$

uniformly for $z_0\in\Gamma_1$, and s,t in compact subsets of the complex plane. Here $S(z)=\frac{\sin\pi z}{\pi z}$ is the sinc kernel, and Φ is a conformal map of the exterior of Γ onto the exterior of the unit ball.

1. Introduction and Results[1]

In the theory of random Hermitian matrices, arising from scattering theory in physics, universality limits play an important role. They can be reduced to scaling limits for reproducing kernels involving orthogonal polynomials, which makes the analysis feasible. This has been completed in a very wide array of settings [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [15], [16], [19]. In particular, for the unit circle, universality limits have been investigated in [7], and for subarcs in [12]. A common feature is the appearance of the sinc kernel

$$S(z)=\frac{\sin\pi z}{\pi z}.$$

In this paper, we investigate measures on a smooth closed contour

$$\Gamma=\{\gamma(s):s\in[0,L]\},$$

where $L>0$. Γ is assumed to be "smooth" in the following sense: γ'' exists and is continuous on $[0,L]$, and satisfies a Lipschitz condition of some positive order $\beta>0$. Thus, for some $C>0$,

$$\left|\gamma''(s)-\gamma''(t)\right|\le C\left|t-s\right|^{\beta},\ s,t\in[0,L].$$

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In addition, we assume that γ is periodic on $[0, L]$, so that $\gamma^{(j)}(0) = \gamma^{(j)}(L)$, $j = 0, 1, 2$. These smoothness assumptions are needed to apply asymptotics of orthogonal polynomials proved by P.K. Suetin [18]. In Suetin's terminology, $\Gamma \in C(2, \beta)$.

We denote the exterior of Γ by D, and denote the conformal map of D onto the exterior of the unit ball by Φ, normalized by $\Phi(\infty) = \infty$, and $\Phi'(\infty) > 0$. We denote its inverse by Ψ. The assumption that $\Gamma \in C(2, \beta)$ ensures that Φ extends continuously to Γ, and moreover Φ'' is continuous on Γ, satisfying there a Lipschitz condition of order β [18]. Similar statements apply to Ψ and the unit circle. In addition, $|\Phi'|$ and $|\Psi'|$ are bounded above and below on Γ and the unit circle resepctively.

The equilibrium density associated with Γ, is denoted by $\omega_\Gamma(t), t \in \Gamma$. It is a positive continuous function, satisfying

$$\int_\Gamma \log|z - t| \, \omega_\Gamma(t) \, |dt| = \log \ \mathrm{cap}(\Gamma), \ z \in \Gamma,$$

where log cap(Γ) is the logarithmic capacity of Γ.

We assume that μ is a finite positive Borel measure on Γ, and $\{p_n\}$ are orthonormal polynomials for μ, so that p_n is a polynomial of degree n, with positive leading coefficient, and

$$\frac{1}{2\pi} \int_\Gamma p_n(z) \overline{p_m(z)} d\mu(z) = \delta_{mn}.$$

We let

$$K_n(z, w) = \sum_{j=0}^{n-1} p_j(z) \overline{p_j(w)}$$

denote the nth reproducing kernel for μ.

One of the key concepts in asymptotics of orthogonal polynomials, is the notion of regularity, (in the sense of Stahl, Totik, and Ullman) [17]. This is not to be confused with the notion of a regular Borel measure. In the setting of this paper, μ is regular if

$$\sup_{\deg(P) \le n} \left[\frac{\|P\|_{L_\infty(\Gamma)}}{\left(\int_\Gamma |P|^2 d\mu \right)^{1/2}} \right]^{1/n} \to 1 \text{ as } n \to \infty.$$

A sufficient condition for regularity is that the Radon-Nikodym derivative μ' (with respect to arclength) is positive a.e. on Γ. However, there are pure discrete and singularly continuous measures that are regular. Our main result is:

Theorem 1.1
Let Γ be a simple closed curve in the complex plane, of class $C(2, \beta)$, for some $\beta \in (0, 1)$. Let μ be a finite positive Borel measure on Γ that is regular. Let Γ_1 be a closed subarc of Γ, such that μ is absolutely continuous with

respect to arclength, in an open arc containing Γ_1, and the Radon-Nikodym derivative μ' (with respect to arclength) is positive and continuous in that open subarc. Then uniformly for $z_0 \in \Gamma_1$ and s, t in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(z_0 + \frac{2\pi is}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}, z_0 + \frac{2\pi i\bar{t}}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}\right)}{K_n\left(z_0, z_0\right)} = e^{i\pi(s-t)} S\left(s-t\right). \tag{1.1}$$

Remarks

(a) In the case where Γ is the unit circle, this reduces to a special case of the result in [7].

(b) The assumption of continuity of μ' in a neighborhood of Γ_1 is severe. We use it to apply uniform asymptotics of Totik for Christoffel functions [20]. It could be replaced by the more implicit assumption that

$$\lim_{n\to\infty} \frac{1}{n}\left[K_n\left(\gamma\left(x_0 + \frac{s}{n}\right), \gamma\left(x_0 + \frac{s}{n}\right)\right) - K_n\left(\gamma\left(x_0\right), \gamma\left(x_0\right)\right)\right] = 0,$$

uniformly for s in compact subsets of the real line and $z_0 = \gamma\left(x_0\right) \in \Gamma_1$. This most likely follows under the weaker condition that μ' is positive and continuous at z_0 (rather than in a neighborhood), and may well follow from the proofs in [20], but is not formally stated there.

(c) It is a classic result that $\omega_\Gamma\left(z_0\right) = \frac{1}{2\pi}\left|\Phi'\left(z_0\right)\right|$ [1, p. 21, eqn. (2.3)], so we can also express the universality limit as

$$\lim_{n\to\infty} \frac{K_n\left(z_0 + \frac{is}{n\omega_\Gamma(z_0)}e^{i\Theta(z_0)}, z_0 + \frac{i\bar{t}}{n\omega_\Gamma(z_0)}e^{i\Theta(z_0)}\right)}{K_n\left(z_0, z_0\right)} = e^{i\pi(s-t)} S\left(s-t\right),$$

where, for some determination of the argument,

$$\Theta\left(z_0\right) = \arg\left(\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right).$$

One can think of $\omega_\Gamma\left(z_0\right) e^{-i\Theta(z_0)}$ as the complex form of the equilibrium density.

Theorem 1.1 will follow partly from the following simple general result:

Proposition 1.2

Assume that $L > 0$ and $\gamma : [0, L] \to \mathbb{C}$ is a differentiable function with γ' continuous. Let $\Gamma = \gamma\left([0, L]\right)$. Assume that g, θ are functions defined on γ, with g continuous and complex valued and non-vanishing, while θ is real valued and differentiable, and θ' is continuous. Assume that for $n \geq 0$, $f_n : \Gamma \to \mathbb{C}$ is a function satisfying

$$f_n\left(z\right) = g\left(z\right) e^{2\pi in\theta(z)}\left(1 + o\left(1\right)\right),\ n \to \infty, \tag{1.2}$$

uniformly for $z \in \Gamma$. Let

$$K_n^*\left(z, w\right) = \sum_{j=0}^{n-1} f_j\left(z\right)\overline{f_j\left(w\right)},\ n \geq 1. \tag{1.3}$$

Then uniformly for x_0 in compact subsets of $(0, L)$, and uniformly for s, t in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{1}{n} K_n^* \left(\gamma \left(x_0 + \frac{s}{n} \right), \gamma \left(x_0 + \frac{t}{n} \right) \right)$$
$$= |g(\gamma(x_0))|^2 e^{i\pi(s-t)(\theta\circ\gamma)'(x_0)} S\left((s-t)(\theta\circ\gamma)'(x_0) \right), \tag{1.4}$$

and

$$\lim_{n\to\infty} \frac{K_n^* \left(\gamma \left(x_0 + \frac{s}{n} \right), \gamma \left(x_0 + \frac{t}{n} \right) \right)}{K_n^* \left(\gamma (x_0), \gamma (x_0) \right)} = e^{i\pi(s-t)(\theta\circ\gamma)'(x_0)} S\left((s-t)(\theta\circ\gamma)'(x_0) \right). \tag{1.5}$$

This also holds for x_0 in compact subsets of $[0, L]$ if γ is periodic on $[0, L]$ so that $\gamma^{(j)}(0) = \gamma^{(j)}(L)$, $j = 0, 1$.

There are a number of easy consequences of Theorem 1.1:

Corollary 1.3
Assume the hypotheses of Theorem 1.1. Let k, ℓ be non-negative integers and

$$K_n^{(k,\ell)}(z, z) = \sum_{j=0}^{n-1} p_j^{(k)}(z) \overline{p_j^{(\ell)}(z)}. \tag{1.6}$$

Then uniformly for $z_0 \in \Gamma_1$,

$$\lim_{n\to\infty} \frac{1}{n^{k+\ell}} \left(\frac{\Phi(z_0)}{\Phi'(z_0)} \right)^k \left(\frac{\overline{\Phi(z_0)}}{\overline{\Phi'(z_0)}} \right)^\ell \frac{K_n^{(k,\ell)}(z_0, z_0)}{K_n(z_0, z_0)} = \frac{1}{k+\ell+1}. \tag{1.7}$$

Corollary 1.4
Assume the hypotheses of Theorem 1.1. Let $M \geq 1$ and $z_0 \in \Gamma_1$. There exist, for large enough n, simple zeros $\zeta_{n,j}$ of $K_n(z_0, \cdot)$, $j = \pm 1, \pm 2, ..., \pm M$, with

$$\lim_{n\to\infty} n \left(\zeta_{n,j} - z_0 \right) = 2\pi i j \frac{\Phi(z_0)}{\Phi'(z_0)}. \tag{1.8}$$

Moreover, for large enough n, these are the only possible zeros of $K_n(z_0, \cdot)$ in $\left\{ z : |z - z_0| \leq \frac{M+\frac{1}{2}}{n} 2\pi \left| \frac{\Phi(z_0)}{\Phi'(z_0)} \right| \right\}$.

We prove Proposition 1.2, as well as the special case of Theorem 1.1, where $d\mu(z) = |dz|$, in Section 2. The general form of Theorem 1.1 is proved in Section 3. Corollary 1.3 and 1.4 are proved in Section 4. In the sequel, K_n denotes the nth reproducing kernel for μ. For other measures, such as ν, their nth reproducing kernel is denoted by K_n^ν. Sometimes we'll add the superscript μ as well, to distinguish $K_n = K_n^\mu$ from K_n^ν.

2. Proof of Proposition 1.2 and a special case of Theorem 1.1

Proof of Proposition 1.2

Write $z = z(x_0, s, n) = \gamma\left(x_0 + \frac{s}{n}\right)$ and $w = w(x_0, t, n) = \gamma\left(x_0 + \frac{t}{n}\right)$. Then, as g is continuous,

$$\begin{aligned}
&\frac{1}{n}K_n^*\left(\gamma\left(x_0+\frac{s}{n}\right),\gamma\left(x_0+\frac{t}{n}\right)\right)\\
=\ &\frac{1}{n}\sum_{j=0}^{n-1} f_j(z)\,\overline{f_j(w)}\\
=\ &\frac{1}{n}\left|g(\gamma(x_0))\right|^2\sum_{j=0}^{n-1} e^{2\pi ij[\theta(z)-\theta(w)]}(1+o(1))\\
=\ &\frac{1}{n}\left|g(\gamma(x_0))\right|^2\left[\frac{1-e^{2\pi in[\theta(z)-\theta(w)]}}{1-e^{2\pi i[\theta(z)-\theta(w)]}}\right]+o(1)\\
=\ &\left|g(\gamma(x_0))\right|^2 e^{\pi i(n-1)[\theta(z)-\theta(w)]}\frac{\sin(\pi n[\theta(z)-\theta(w)])}{n\sin(\pi[\theta(z)-\theta(w)])}+o(1).
\end{aligned}$$

Here,

$$\begin{aligned}
&\theta(z)-\theta(w)\\
=\ &\theta\left(\gamma\left(x_0+\frac{s}{n}\right)\right)-\theta\left(\gamma\left(x_0+\frac{t}{n}\right)\right)\\
=\ &\int_{x_0+\frac{t}{n}}^{x_0+\frac{s}{n}}(\theta\circ\gamma)'(x)\,dx\\
=\ &(\theta\circ\gamma)'(x_0)\frac{s-t}{n}+o\left(\frac{1}{n}\right),
\end{aligned}$$

uniformly for s, t in compact subsets of the real line, and x_0 in compact subsets of $(0, L)$, by continuity of θ', ϕ'. If we have periodicity on $[0, L]$, then we may also allow x_0 in compact subsets of $[0, L]$. Then, as $n \to \infty$,

$$\begin{aligned}
\frac{\sin(\pi n[\theta(z)-\theta(w)])}{n\sin(\pi[\theta(z)-\theta(w)])} &= \frac{\sin\left(\pi\left[(\theta\circ\gamma)'(x_0)(s-t)\right]\right)}{\pi\left[(\theta\circ\gamma)'(x_0)(s-t)\right]}+o(1)\\
&= S\left((\theta\circ\gamma)'(x_0)(s-t)\right)+o(1),
\end{aligned}$$

with obvious modifications when $s = t$. Then (1.4) follows. Setting $s, t = 0$ in (1.4), we also obtain

$$\lim_{n\to\infty}\frac{1}{n}K_n^*(\gamma(x_0),\gamma(x_0)) = \left|g(\gamma(x_0))\right|^2. \tag{2.1}$$

Then (1.5) also follows.■

Proof of Theorem 1.1 for $d\mu(z) = |dz|$

Let $d\mu(z) = |dz|$ on Γ, so that $\mu' = 1$ on Γ. We use Suetin's asymptotic for p_n [18, Theorem 2.3, p. 50]: as $n \to \infty$,

$$p_n(z) = g(z)\,\Phi(z)^n\,(1 + o(1)), \tag{2.2}$$

uniformly for $z \in \Gamma$. Here g is a a function analytic in D, and continuous in $\bar{D}$, that is non-vanishing on D. It is closely related to a Szegő function for the exterior of Γ. Note that we choose the parameters $p = 0, p' = 1, \alpha' = \beta$ and $\frac{1}{2} < \alpha < \beta + \frac{1}{2}$ in Suetin's formulation. We may assume that our $\beta < \frac{1}{2}$, so that $0 < \alpha < 1$. Moreover, we can choose any positive absolutely continuous measure μ that is sufficiently smooth, but for us, the special case $d\mu(z) = |dz|$ will do. Write for $z \in \Gamma$,

$$\Phi(z) = e^{2\pi i\theta(z)}.$$

We then have that θ is real valued and continuously differentiable, and can assume that $\theta : [0, 1] \to \mathbb{R}$. Our asymptotic (2.2) becomes

$$p_n(z) = g(z)\,e^{2\pi in\theta(z)}\,(1 + o(1))$$

uniformly on Γ. Let $x_0 \in [0, L]$ and $z_0 = \gamma(x_0)$. Then Proposition 1.2 gives

$$\lim_{n\to\infty} \frac{K_n\left(\gamma\left(x_0 + \frac{s}{n}\right), \gamma\left(x_0 + \frac{t}{n}\right)\right)}{K_n(\gamma(x_0), \gamma(x_0))} = e^{i\pi(s-t)(\theta\circ\gamma)'(x_0)} S\left((s - t)(\theta \circ \gamma)'(x_0)\right), \tag{2.3}$$

uniformly for s, t in compact subsets of the real line. This is also uniform for x_0 in a compact subset of $[0, L]$. Here if $x_0 = 0$ or L, we use the periodicity of γ. This last limit holds for any given smooth parametrization γ of Γ. In particular, it holds for

$$\gamma(t) = \Psi\left(e^{2\pi it}\right), t \in [0, 1].$$

Recall here that Ψ is the conformal map of the exterior of the unit ball onto the exterior of Γ. With this parametrization, we see that

$$e^{2\pi it} = \Phi \circ \Psi\left(e^{2\pi it}\right) = \Phi \circ \gamma(t) = e^{2\pi i\theta\circ\gamma(t)}. \tag{2.4}$$

By continuity of $\theta \circ \gamma$, it follows that for some integer m independent of t, $\theta \circ \gamma(t) = t + m$, so

$$(\theta \circ \gamma)'(t) = 1, t \in [0, 1]. \tag{2.5}$$

Next, as Ψ' is continuous, and as $|\gamma'| = |\Psi'|$ is bounded below,

$$\gamma\left(x_0 + \frac{s}{n}\right) = \gamma(x_0) + \frac{s + \varepsilon_n(s)}{n}\gamma'(x_0), \tag{2.6}$$

where $\varepsilon_n(s) \to 0$ as $n \to \infty$, uniformly for s in compact subsets of the real line, and $x_0 \in [0, 1]$. Moreover, from (2.4), and the chain rule,

$$\gamma'(x_0) = \frac{2\pi ie^{2\pi ix_0}}{\Phi'(\gamma(x_0))} = \frac{2\pi i\Phi(z_0)}{\Phi'(z_0)}. \tag{2.7}$$

Taking account of this, (2.6) and (2.7), we see that our asymptotic (2.3) becomes
(2.8)

$$\lim_{n\to\infty}\frac{K_n\left(z_0+\frac{s+\varepsilon_n(s)}{n}\frac{2\pi i\Phi(z_0)}{\Phi'(z_0)},z_0+\frac{t+\varepsilon_n(t)}{n}\frac{2\pi i\Phi(z_0)}{\Phi'(z_0)}\right)}{K_n\left(z_0,z_0\right)}=e^{i\pi(s-t)}S\left(s-t\right).$$

We would like to drop $\varepsilon_n(s), \varepsilon_n(t)$ and allow s, t to be complex. For this, we use normality. Let

$$f_n(s,t)=\frac{K_n\left(z_0+\frac{s}{n}\frac{2\pi i\Phi(z_0)}{\Phi'(z_0)},z_0+\frac{\bar{t}}{n}\frac{2\pi i\Phi(z_0)}{\Phi'(z_0)}\right)}{K_n\left(z_0,z_0\right)}.$$

This is a polynomial in s, t. We'll show that $\{f_n\}$ are uniformly bounded for s, t in compact subsets of $\mathbb{C}$, and hence are a normal family. It then follows that they are equicontinuous, so we can indeed drop the $\varepsilon_n(s)$ and $\varepsilon_n(t)$ above. The extension to complex s, t follows from analytic continuation as the right-hand side of (2.8) is entire in s, t.

Finally, we establish the uniform boundedness. Let Γ_1 be as in Theorem 1.1. First, our asymptotics above (see (2.1)) show that in some subarc Γ_2 of Γ, containing Γ_1 as an interior arc, we have

$$\sup_{n\geq 1,z\in\Gamma_2}\frac{1}{n}\left|K_n\left(z,z\right)\right|<\infty.$$

Cauchy-Schwarz gives

$$\sup_{n\geq 1,z,w\in\Gamma_2}\frac{1}{n}\left|K_n\left(z,w\right)\right|<\infty.$$

Let Γ_3 be a proper subarc of Γ_2, containing Γ_1 as an interior arc. Since Γ_3 is smooth, we can apply the Bernstein-Walsh lemma separately in z, w, and elementary estimates for Green's functions (or equilibrium measures), to show that given $R > 0$, there exists C_R such that

$$\sup_{n\geq 1;z,w\in\Gamma_3;|s|,|t|\leq R}\frac{1}{n}\left|K_n\left(z+\frac{s}{n},w+\frac{\bar{t}}{n}\right)\right|\leq C_R.$$

We skip the technical details - see [7, p. 556, Lemma 6.1] for a similar situation. This gives the desired normality of $\{f_n\}$, using also (2.1). Finally, for uniformity in $z_0\in\Gamma_1$, the above bounds hold uniformly in z_0, so the $\{f_n\}$ are uniformly normal in z_0. ■

3. Proof of Theorem 1.1 in the General Case

We begin with a comparison inequality [10], which has been widely used in universality proofs:

Lemma 3.1
Let $c > 0$ and ν, ω be finite positive Borel measures on Γ with $d\nu \leq c\,d\omega$.

Denote their reproducing kernels respectively by K_n^{ν} and K_n^{ω}. Then for $u, v \in \mathbb{C}$,

(3.1)
$$\left|\left(K_n^{\nu} - \frac{1}{c}K_n^{\omega}\right)(u,v)\right| / K_n^{\nu}(u,u) \leq \left(\frac{K_n^{\nu}(v,v)}{K_n^{\nu}(u,u)}\right)^{1/2}\left[1 - \frac{K_n^{\omega}(u,u)}{cK_n^{\nu}(u,u)}\right]^{1/2}.$$

Proof

Now

$$\begin{aligned} & \frac{1}{2\pi}\int_{\Gamma}\left|\left(cK_n^{\nu} - K_n^{\omega}\right)(u,z)\right|^2 d\nu(z) \\ = & \frac{c^2}{2\pi}\int_{\Gamma}\left|K_n^{\nu}(u,z)\right|^2 d\nu(z) - \frac{2c}{2\pi}\operatorname{Re}\int_{\Gamma}\left(\overline{K_n^{\nu}}K_n^{\omega}\right)(u,z)\,d\nu(z) \\ & + \frac{1}{2\pi}\int_{\Gamma}\left|K_n^{\omega}(u,z)\right|^2 d\nu(z) \\ = & c^2K_n^{\nu}(u,u) - 2cK_n^{\omega}(u,u) + \frac{1}{2\pi}\int_{\Gamma}\left|K_n^{\omega}(u,z)\right|^2 d\nu(z), \end{aligned}$$

by the reproducing kernel property. As $d\nu \leq c\,d\omega$, we also have

$$\frac{1}{2\pi}\int_{\Gamma}\left|K_n^{\omega}(u,z)\right|^2 d\nu(z) \leq \frac{c}{2\pi}\int_{\Gamma}\left|K_n^{\omega}(u,z)\right|^2 d\omega(z) = cK_n^{\omega}(u,u).$$

So

$$\frac{1}{2\pi}\int_{\Gamma}\left|\left(cK_n^{\nu} - K_n^{\omega}\right)(u,z)\right|^2 d\nu(z) \leq c\left(cK_n^{\nu}(u,u) - K_n^{\omega}(u,u)\right). \tag{3.2}$$

Next for any polynomial P of degree $\leq n-1$, we have the Christoffel function estimate

$$|P(v)| \leq K_n^{\nu}(v,v)^{1/2}\left(\frac{1}{2\pi}\int_{\Gamma}|P(z)|^2 d\nu(z)\right)^{1/2}.$$

Applying this to $P(z) = \left(cK_n^{\nu} - K_n^{\omega}\right)(u,z)$ and using (3.2) gives, for all complex u, v

$$\begin{aligned} & \left|\left(cK_n^{\nu} - K_n^{\omega}\right)(u,v)\right| \\ \leq & K_n^{\nu}(v,v)^{1/2}\left[c\left(cK_n^{\nu}(u,u) - K_n^{\omega}(u,u)\right)\right]^{1/2}. \end{aligned}$$

■

The next ingredient is asymptotics for Christoffel functions. As mentioned before, here we impose unnecessarily severe hypotheses on our measure, so that we can use results from [20], whereas we only need asymptotics for $K_n\left(\gamma\left(x_0 + \frac{s}{n}\right), \gamma\left(x_0 + \frac{s}{n}\right)\right)$. These probably follow from the proofs there under weaker hypotheses, though they are not explicitly stated. Recall that we denote the equilibrium density of the curve Γ with respect to arclength by ω_{Γ}.

Lemma 3.2

Let ν be a finite positive Borel measure on Γ that is regular. Let Γ_1 be a *closed subarc of Γ, such that ν is absolutely continuous with respect to arclength in an open arc containing Γ_1, and ν' is positive and continuous in that open subarc. Then, uniformly for s in compact subsets of the real line, and $z_0 = \gamma(x_0) \in \Gamma_1$,*

$$\lim_{n\to\infty} \frac{1}{n} K_n^{\nu}\left(\gamma\left(x_0 + \frac{s}{n}\right), \gamma\left(x_0 + \frac{s}{n}\right)\right) = \frac{\omega_\Gamma(z_0)}{\nu'(z_0)}. \tag{3.3}$$

Proof

Theorem 1.2 in [20, p. 2056] establishes the stronger statement that

$$\lim_{n\to\infty} \frac{1}{n} K_n(\gamma(x), \gamma(x)) = \frac{\omega_\Gamma(\gamma(x))}{\nu'(\gamma(x))}$$

uniformly for $\gamma(x)$ in an open subarc of Γ containing Γ_1. Since ω_Γ, ν' and γ are continuous, the result follows. ■

Proof of Theorem 1.1

We already have the desired universality for the measure with density 1 on Γ, which we denote by ν. Thus $\nu' = 1$ on Γ. Let $\varepsilon \in (0,1)$, and choose $\delta > 0$ such that for $z_0 = \gamma(x_0) \in \Gamma_1$,

$$(1+\varepsilon)^{-1} \le \frac{\mu'(\gamma(s))}{\mu'(z_0)} \le 1+\varepsilon,\ s \in J(x_0,\delta) = [x_0-\delta, x_0+\delta]. \tag{3.4}$$

Here δ is independent of x_0. This is possible as μ', γ' are continuous in an open arc containing Γ_1. For a given x_0, let $\Gamma_0(x_0) = \Gamma(J(x_0,\delta))$. Let μ_s denote the singular part of μ, and define a measure ω by

$$d\omega(z) = \begin{cases} \mu'(z_0)\, d\nu(z), & \text{in } \Gamma(x_0,\delta) \\ \max\{\mu'(z_0), \mu'(z)\}\, d\nu(z) + d\mu_s(z) & \text{in } \Gamma\backslash\Gamma(x_0,\delta) \end{cases}.$$

This ensures that $(1+\varepsilon)\, d\omega \ge d\mu$ and $d\omega \ge \mu'(z_0)\, d\nu$ on Γ. Now ω is regular, as it is regular in both $\Gamma\backslash\Gamma(x_0,\delta)$ and $\Gamma(x_0,\delta)$ [17, p. 148, Theorem 5.3.3]. Then the previous lemma shows that

$$\lim_{n\to\infty} \frac{K_n^{\nu}\left(\gamma\left(x_0+\frac{s}{n}\right), \gamma\left(x_0+\frac{s}{n}\right)\right)}{K_n^{\omega}\left(\gamma\left(x_0+\frac{s}{n}\right), \gamma\left(x_0+\frac{s}{n}\right)\right)} = \mu'(z_0) \tag{3.5}$$

and

$$\lim_{n\to\infty} \frac{K_n^{\mu}\left(\gamma\left(x_0+\frac{s}{n}\right), \gamma\left(x_0+\frac{s}{n}\right)\right)}{K_n^{\omega}\left(\gamma\left(x_0+\frac{s}{n}\right), \gamma\left(x_0+\frac{s}{n}\right)\right)} = 1 \tag{3.6}$$

uniformly for s in compact subsets of the real line. Let $u_n = u_n(s) = \gamma\left(x_0+\frac{s}{n}\right)$ and $v_n = v_n(s) = \gamma\left(x_0+\frac{t}{n}\right)$. By Lemma 3.1, with $c = \mu'(z_0)^{-1}$,

$$\begin{aligned} &\left|\left(K_n^{\nu} - \mu'(z_0)\, K_n^{\omega}\right)(u_n, v_n)\right| / K_n^{\nu}(u_n, u_n) \\ \le\ &\left(\frac{K_n^{\nu}(v_n, v_n)}{K_n^{\nu}(u_n, u_n)}\right)^{1/2} \left[1 - \mu'(z_0)\frac{K_n^{\omega}(u_n, u_n)}{K_n^{\nu}(u_n, u_n)}\right]^{1/2}. \end{aligned} \tag{3.7}$$

Here, using that $K_n^{\nu}(v_n, v_n) \sim n$ and $K_n^{\nu}(v_n, v_n) \sim n$, uniformly in s, t, and x_0, as follows from Lemma 3.2, and the limit (3.5), we obtain

$$\lim_{n\to\infty} \left|\left(K_n^{\nu} - \mu'(z_0) K_n^{\omega}\right)(u_n, v_n)\right| / n = 0, \tag{3.8}$$

uniformly for s, t in compact subsets of the real line. Next, Lemma 3.2 gives $K_n^{\mu}(v_n, v_n) \sim n$ and $K_n^{\mu}(v_n, v_n) \sim n$, and then Lemma 3.1 with $c = 1 + \varepsilon$ gives

$$\begin{aligned} & \left|\left(K_n^{\mu} - \frac{1}{1+\varepsilon} K_n^{\omega}\right)(u_n, v_n)\right| / K_n^{\mu}(u_n, u_n) \\ \leq & \left(\frac{K_n^{\mu}(v_n, v_n)}{K_n^{\mu}(u_n, u_n)}\right)^{1/2} \left[1 - \frac{1}{1+\varepsilon} \frac{K_n^{\omega}(u_n, u_n)}{K_n^{\mu}(u_n, u_n)}\right]^{1/2} \end{aligned}$$

and letting $n \to \infty$, and using (3.6), gives

$$\limsup_{n\to\infty} \left|\left(K_n^{\mu} - \frac{1}{1+\varepsilon} K_n^{\omega}\right)(u_n, v_n)\right| / n \leq C\varepsilon^{1/2},$$

uniformly for s, t in compact subsets of the real line, and $z_0 \in \Gamma_1$, with C independent of s, t. Combining this and (3.8), gives

$$\limsup_{n\to\infty} \left|\left(K_n^{\mu} - \frac{1}{(1+\varepsilon)\mu'(z_0)} K_n^{\nu}\right)(u_n, v_n)\right| / n \leq C\varepsilon^{1/2}.$$

Again, using $K_n^{\nu}(u_n, v_n) = O(n)$, and that μ' is bounded below in Γ_1, gives

$$\limsup_{n\to\infty} \left|\left(K_n^{\mu} - \frac{1}{\mu'(z_0)} K_n^{\nu}\right)(u_n, v_n)\right| / n \leq C\varepsilon^{1/2}.$$

As the left-hand side is independent of ε, we deduce that

$$\lim_{n\to\infty} \left|\left(K_n^{\mu} - \frac{1}{\mu'(z_0)} K_n^{\nu}\right)(u_n, v_n)\right| / n = 0$$

uniformly for s, t in compact subsets of the real line, and $z_0 \in \Gamma_1$. As we already have the universality limit for K_n^{ν}, that for K_n^{μ} follows, in a form similar to (2.3). The extension to complex s, t, may be completed as in the proof of the special case of Theorem 1.1 in Section 2. ■

4. Proof of Corollaries 1.3 and 1.4

Proof of Corollary 1.3
Taylor series expansion shows that

$$\begin{aligned}
&\frac{K_n\left(z_0+\frac{2\pi is}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}, z_0+\frac{2\pi i\bar{t}}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}\right)}{K_n\left(z_0,z_0\right)} \\
=\ & \frac{1}{K_n\left(z_0,z_0\right)}\sum_{j=0}^{n-1} p_j\left(z_0+\frac{2\pi is}{n}\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right)\overline{p_j\left(z_0+\frac{2\pi i\bar{t}}{n}\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right)} \\
=\ & \frac{1}{K_n\left(z_0,z_0\right)}\sum_{j=0}^{n-1}\left(\sum_{k=0}^{\infty}\frac{p_j^{(k)}\left(z_0\right)}{k!}\left(\frac{2\pi is}{n}\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right)^k\right)\overline{\left(\sum_{\ell=0}^{\infty}\frac{p_j^{(\ell)}\left(z_0\right)}{\ell!}\left(\frac{2\pi i\bar{t}}{n}\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right)^\ell\right)} \\
=\ & \sum_{k,\ell=0}^{\infty}\frac{1}{k!\ell!}\left(2\pi is\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)}\right)^k\left(-2\pi it\frac{\overline{\Phi\left(z_0\right)}}{\overline{\Phi'\left(z_0\right)}}\right)^\ell\frac{K_n^{(k,\ell)}\left(z_0,z_0\right)}{K_n\left(z_0,z_0\right)n^{k+\ell}}.
\end{aligned}$$

The interchanges are justified, since the series all terminate. We know that this converges uniformly for s,t in compact subsets of the plane to

$$e^{i\pi(s-t)}\frac{\sin\pi\left(s-t\right)}{\pi\left(s-t\right)}=\sum_{k,\ell=0}^{\infty}\frac{1}{k!\ell!}\left(2\pi is\right)^k\left(-2\pi it\right)^\ell\frac{1}{k+\ell+1}.$$

This last double series identity follows by straightforward manipulation, cf. [7, p. 547, eqn. (2.6)]. Recall that when sequences of analytic functions converge uniformly, their Taylor series coefficients converge to those of the limit function. Then comparing the coefficients in the two double series gives the result. The uniformity in z_0, may be established by a normality argument. ■

Proof of Corollary 1.4
This is a consequence of Hurwitz' theorem on zeros of uniformly convergent sequences of analytic functions. Note that $S\left(z\right)$ has zeros only at the non-0 integers. Then Theorem 1.1 implies that $K_n\left(z_0,z_0+\frac{2\pi i\bar{t}}{n}\frac{\Phi(z_0)}{\Phi'(z_0)}\right)$ has simple zeros $t_{n,j}$, with

$$\lim_{n\to\infty}t_{n,j}=j,\ \ j=\pm1,\pm2,...\pm M,$$

and moreover, these are the only zeros in some neighborhood of $\left[-M-\frac{1}{2},M+\frac{1}{2}\right]$. We can then set

$$\zeta_{n,j}=z_0+\frac{2\pi i\bar{t}_{n,j}}{n}\frac{\Phi\left(z_0\right)}{\Phi'\left(z_0\right)},\ \ j=\pm1,\pm2,...\pm M.$$

■

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