LOCAL LIMITS FOR ORTHOGONAL POLYNOMIALS FOR VARYING MEASURES VIA UNIVERSALITY

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Abstract. We consider orthogonal polynomials $\left\{p_n\left(e^{-2nQ_n},x\right)\right\}$ for varying measures and use universality limits to prove "local limits"

$$\lim_{n\to\infty}\frac{p_n\left(e^{-2nQ_n},y_{jn}+\frac{z}{\tilde{K}_n\left(y_{jn},y_{jn}\right)}\right)}{p_n\left(e^{-2nQ_n},y_{jn}\right)}e^{-\frac{nQ_n'\left(y_{jn}\right)}{\tilde{K}_n\left(y_{jn},y_{jn}\right)}z}=\cos\pi z.$$

Here y_{jn} is a local maximum point of $|p_n|\,e^{-nQ_n}$ in the "bulk" of the support, $\tilde{K}_n\left(y_{jn},y_{jn}\right)$ is the normalized reproducing kernel, and the limit holds uniformly for z in compact subsets of the plane. We also consider local limits at the "soft edge" of the spectrum, which involve the Airy function.

1. Introduction[1]

For $n\geq 1$, let μ_n be a finite positive Borel measure with support supp$[\mu_n]$ and infinitely many points in the support, and all finite power moments

$$\int x^j d\mu_n\left(x\right),\ j=0,1,2,...\ .$$

Then we may define orthonormal polynomials

$$p_m\left(\mu_n,x\right)=\gamma_m\left(\mu_n\right)x^m+...,\gamma_m\left(\mu_n\right)>0,$$

$m\geq 0$, satisfying the orthonormality conditions

$$\int p_j\left(\mu_n,\cdot\right)p_k\left(\mu_n,\cdot\right)d\mu_n=\delta_{jk}.$$

Throughout we use μ_n' to denote the Radon-Nikodym derivative of μ_n. The nth reproducing kernel for μ_n is

$$\begin{aligned}K_n\left(x,y\right)&=K_n\left(\mu_n,x,y\right)=\sum_{k=0}^{n-1}p_k\left(\mu_n,x\right)p_k\left(\mu_n,y\right)\\&=\frac{\gamma_{n-1}\left(\mu_n\right)}{\gamma_n\left(\mu_n\right)}\frac{p_n\left(\mu_n,x\right)p_{n-1}\left(\mu_n,y\right)-p_{n-1}\left(\mu_n,x\right)p_n\left(\mu_n,y\right)}{x-y}\qquad(1.1)\end{aligned}$$

and the normalized kernel is

$$\widetilde{K}_n\left(x,y\right)=\mu_n'\left(x\right)^{1/2}\mu_n'\left(y\right)^{1/2}K_n\left(x,y\right).\qquad(1.2)$$

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The nth Christoffel function is

$$\lambda_n(x) = \lambda_n(\mu_n, x) = 1/K_n(x,x).$$

The zeros of $p_n(\mu_n, x)$ are denoted by

$$x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n}.$$

We emphasize that K_n, λ_n, and $\{x_{jn}\}_{1\leq j\leq n}$ correspond to the nth measure μ_n.

The *universality limit in the bulk* asserts that

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \mathbb{S}(a-b), \tag{1.3}$$

where

$$\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}.$$

Typically, this is established uniformly for a, b in compact subsets of the real line. In many of the most important applications,

$$d\mu_n(x) = e^{-2nQ(x)}dx.$$

There are several methods to establish universality limits, and an extensive literature. See for example [1], [2], [3], [4], [5], [6], [7], [9], [10], [13], [17], [18], [19], [20], [22], [23], [24], [26], [28], [29]. One method is to pass from asymptotics for orthonormal polynomials to universality limits.

In recent papers [14], [15], [16] it was shown that one can partially proceed in the opposite direction, by deducing local ratio asymptotics from universality limits. A related observation appears in [30]. Perhaps the most impressive such result involves asymptotics at an endpoint of the interval of orthogonality. Let J_α be the usual Bessel function of the first kind and order α,

$$J_\alpha(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\alpha}}{n!\Gamma(\alpha+n+1)},$$

and J_α^* be the normalized Bessel function $J_\alpha^*(z) = J_\alpha(z)/z^\alpha$.

Theorem A [14]
Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho > 0$, μ is absolutely continuous in $J = [1-\rho, 1]$, and in J, its absolutely continuous component has the form $w(x) = h(x)(1-x)^\alpha$, where $\alpha > -1$ and

$$\lim_{x\to 1-} h(x) = 1.$$

Then uniformly for z in compact subsets of $\mathbb{C}$, we have

$$\lim_{n\to\infty} \frac{p_n\left(\mu, 1 - \frac{z^2}{2n^2}\right)}{p_n(\mu, 1)} = \frac{J_\alpha^*(z)}{J_\alpha^*(0)}. \tag{1.4}$$

See [25] for the definition of regular measures on $[-1,1]$. Here we note that if μ' exists and is positive a.e. in $[-1,1]$, then μ is regular.

In a subsequent paper, the same method was used to establish local asymptotics inside the interval of orthogonality:

Theorem B [15]
Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Then if $y_{jn} \in J$ satisfies $p_n'(y_{jn}) = 0$,

$$\lim_{n\to\infty} \frac{p_n\left(\mu, y_{jn} + \frac{z}{n\omega(y_{jn})}\right)}{p_n(\mu, y_{jn})} = \cos \pi z$$

uniformly for $y_{jn} \in J$ and z in compact subsets of $\mathbb{C}$. Here ω is the equilibrium density for the support of μ.

Analogues for measures on the unit circle were explored in [16]. In this paper, we shall establish local asymptotics from universality limits in the setting of varying measures. We note that because of the extra factors in the limits, even in the bulk case, the results cannot be deduced from earlier ones.

We shall state our results in the bulk in Section 2, and those at the soft edge in Section 3. We prove the results of Section 2 in Section 4, and those of Section 3 in Section 5. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurences.

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2. Local Limits in the Bulk

We need some concepts from potential theory for external fields [21]. Let Σ be a closed set on the real line, and e^{-Q} be a continuous function on Σ. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} (Q(x) - \log|x|) = \infty.$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu} \left(\int \int \log \frac{1}{|x-t|} d\nu(x)\, d\nu(t) + 2 \int Q\, d\nu \right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ω_Q, characterized by the following conditions: let

$$V^{\omega_Q}(z) = \int \log \frac{1}{|z-t|} d\omega_Q(t)$$

denote the potential for ω_Q. Then

$$\begin{aligned} V^{\omega_Q} + Q &\geq F_Q \text{ on } \Sigma; \\ V^{\omega_Q} + Q &= F_Q \text{ in } \operatorname{supp}[\omega_Q]. \end{aligned}$$

Here the number F_Q is a constant. We let $\sigma_Q(x) = \frac{d\omega_Q}{dx}$.

Our first theorem is based on results in [10], [27].

Theorem 2.1

Let e^{-Q} be a continuous non-negative function on the set Σ, which is assumed to consist of at most finitely many intervals. If Σ is unbounded, we assume also

$$\lim_{|x|\to\infty, x\in\Sigma} (Q(x) - \log|x|) = \infty.$$

Let h be a bounded positive continuous function on Σ, and for $n \geq 1$, let

$$d\mu_n(x) = h(x)\, e^{-2nQ(x)} dx. \tag{2.1}$$

Let J be a closed interval lying in the interior of supp$[\omega_Q]$, where ω_Q denotes the equilibrium measure for Q. Assume that ω_Q is absolutely continuous in a neighborhood of J, and that σ_Q and Q' are continuous in that neighborhood, while $\sigma_Q > 0$ there.

(a) Let $c > 0$, and assume that for $n \geq 1$, we are given $\xi_n \in J$ such that

$$\min_{1\leq k\leq n} |\xi_n - x_{kn}| \geq \frac{c}{n}. \tag{2.2}$$

Then uniformly for z in compact subsets of the plane, and also uniformly in ξ_n satisfying (2.2) we have

$$\lim_{n\to\infty} \frac{p_n\left(\mu_n, \xi_n + \frac{z}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{p_n(\mu_n, \xi_n)} e^{-\frac{z}{\tilde{K}_n(\xi_n,\xi_n)} \frac{p_n'(\xi_n)}{p_n(\xi_n)}} = \cos \pi z. \tag{2.3}$$

(b) In particular, uniformly for $y_{jn} \in J$ that is a local maximum of $|p_n(\mu_n, \cdot)|\, e^{-nQ(\cdot)}$ and for z in compact subsets of the plane, we have

$$\lim_{n\to\infty} \frac{p_n\left(\mu_n, y_{jn} + \frac{z}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{p_n(\mu_n, y_{jn})} e^{-\frac{nQ'\left(y_{jn}\right)}{\tilde{K}_n\left(y_{jn},y_{jn}\right)} z} = \cos \pi z. \tag{2.4}$$

We note that there exists such a y_{jn} between any two successive zeros $x_{j+1,n}, x_{jn}$ of $p_n(\mu_n, \cdot)$. Our next result allows varying Q_n, but with support consisting of one interval, rather than finitely many intervals. It is based on results from [12]:

Theorem 2.2

For $n \geq 1$, let $I_n = (c_n, d_n)$, where $-\infty \leq c_n < d_n \leq \infty$. Assume that for some $r^* > 1$, $[-r^*, r^*] \subset I_n$, for all $n \geq 1$. Assume that

$$\mu_n'(x) = e^{-2nQ_n(x)}, x \in I_n, \tag{2.5}$$

where
(i) $Q_n(x)/\log(2+|x|)$ *has limit* ∞ *at* c_n+ *and* d_n-.
(ii) Q_n' *is strictly increasing and continuous in* I_n.
(iii) There exists $\alpha \in (0,1)$, $C>0$ *such that for* $n \geq 1$ *and* $x,y \in [-r^*, r^*]$,

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C|x-y|^{\alpha}. \tag{2.6}$$

(iv) There exists $\alpha_1 \in \left(\frac{1}{2}, 1\right)$, $C_1 > 0$, *and an open neighborhood* I_0 of 1 *and* -1, *such that for* $n \geq 1$ *and* $x, y \in I_n \cap I_0$,

$$\left|Q_n'(x) - Q_n'(y)\right| \leq C_1|x-y|^{\alpha_1}. \tag{2.7}$$

(v) $[-1,1]$ *is the support of the equilibrium distribution* ω_{Q_n} *for* Q_n.
Then the assertions (a), (b) of Theorem 2.1 hold, where in (b), y_{jn} *is a local maximum of* $|p_n| e^{-nQ_n}$ *in any compact subinterval* J *of* $(-1,1)$.

We shall deduce Theorems 2.1 and 2.2 from a general proposition for a sequence of measures $\{\mu_n\}$.

Theorem 2.3
Assume that for $n \geq 1$ *we have a measure* μ_n *supported on the real line with infinitely many points in its support, and all finite power moments. Let* $\{\xi_n\}$ *be a bounded sequence of real numbers, and* $\{\tau_n\}$ *be a sequence of positive numbers that is bounded above and below by positive constants, while* $\{\Psi_n\}$ *is a sequence of real numbers. Assume that uniformly for* a, b *in compact subsets of* $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(\mu_n, \xi_n + \frac{a\tau_n}{n}, \xi_n + \frac{b\tau_n}{n}\right)}{K_n(\mu_n, \xi_n, \xi_n)} e^{\Psi_n(a+b)} = \mathbb{S}(a-b). \tag{2.8}$$

Let us be given some infinite sequence of integers $\mathcal{T}$. *The following are equivalent:*
(I)

$$\sup_{n\in\mathcal{T}} \left| \frac{\tau_n}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn}} + \Psi_n \right| < \infty \text{ and } \sup_{n\in\mathcal{T}} \frac{1}{n^2} \sum_{j=1}^{n} \frac{1}{(\xi_n - x_{jn})^2} < \infty. \tag{2.9}$$

(II) For each $R > 0$, *there exists* C_R *such that*

$$\sup_{n\in\mathcal{T}} \sup_{|z|\leq R} \left| \frac{p_n\left(\mu_n, \xi_n + \frac{\tau_n z}{n}\right)}{p_n(\mu_n, \xi_n)} e^{\Psi_n z} \right| \leq C_R. \tag{2.10}$$

(III) From every subsequence of $\mathcal{T}$, *there is a further subsequence* $\mathcal{S}$ *such that*

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n\left(\mu_n, \xi_n + \frac{z\tau_n}{n}\right)}{p_n(\mu_n, \xi_n)} e^{\Psi_n z} = \cos(\pi z) + \frac{\alpha}{\pi} \sin \pi z, \tag{2.11}$$

uniformly for z *in compact subsets of* $\mathbb{C}$, *where*

$$\alpha = \lim_{n\to\infty, n\in\mathcal{S}} \left[\frac{\tau_n}{n} \frac{p_n'(\mu_n, \xi_n)}{p_n(\mu_n, \xi_n)} + \Psi_n \right] \tag{2.12}$$

and α is bounded independently of $\mathcal{S}$.

Remarks

Theorem 2.3 is similar to Theorem 1.3 in [15], which deals with fixed weights, but there the factor $e^{\Psi_n z}$ that enables us to deal with varying exponential weights is missing.

Corollary 2.4

Under the hypotheses of Theorem 2.1(b) or 2.2,

$$\lim_{n\to\infty} \frac{\left[(p_n\mu_n')\left(y_{jn}+\frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)\right]^2+\left[\frac{1}{\pi\tilde{K}_n(y_{jn},y_{jn})}(p_n\mu_n')'\left(y_{jn}+\frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)\right]^2}{(p_n\mu_n')(y_{jn})^2}=1,$$

uniformly for x in compact subsets of the real line.

3. Local Limits at the Soft Edge

For the classical Hermite weight $\exp\left(-x^2\right)$ on $\mathbb{R}$, universality limits at the "soft" edge of the spectrum take the form [31, p. 152]

$$\lim_{n\to\infty}\frac{1}{\sqrt{2}n^{1/6}}\tilde{K}_n\left(\sqrt{2n}\left(1+\frac{a}{2n^{2/3}}\right),\sqrt{2n}\left(1+\frac{b}{2n^{2/3}}\right)\right)=\mathbb{A}i\left(a,b\right),$$

and for the scaled (or contracted) Hermite weight $\exp\left(-2nx^2\right)$, the form is

$$\lim_{n\to\infty}\frac{1}{2n^{2/3}}\tilde{K}_n\left(1+\frac{a}{2n^{2/3}},1+\frac{b}{2n^{2/3}}\right)=\mathbb{A}i\left(a,b\right), \tag{3.1}$$

where $\mathbb{A}i\left(\cdot,\cdot\right)$ is the Airy kernel, defined by

$$\mathbb{A}i\left(a,b\right)=\begin{cases}\frac{Ai(a)Ai'(b)-Ai'(a)Ai(b)}{a-b}, & a\neq b,\\ Ai'\left(a\right)^2-aAi\left(a\right)^2, & a=b,\end{cases} \tag{3.2}$$

and Ai is the Airy function, defined on the real line by [32]

$$Ai\left(x\right)=\frac{1}{\pi}\int_0^\infty\cos\left(\frac{1}{3}t^3+xt\right)dt. \tag{3.3}$$

The Airy function satisfies the differential equation

$$Ai''\left(z\right)-zAi\left(z\right)=0. \tag{3.4}$$

For $a=b=0$, (3.1) gives

$$\lim_{n\to\infty}\frac{1}{2n^{2/3}}\tilde{K}_n\left(1,1\right)=\mathbb{A}i\left(0,0\right), \tag{3.5}$$

so we may reformulate (3.1) as

$$\lim_{n\to\infty}\frac{\tilde{K}_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a,1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}b\right)}{\tilde{K}_n\left(1,1\right)}=\frac{\mathbb{A}i\left(a,b\right)}{\mathbb{A}i\left(0,0\right)}. \tag{3.6}$$

It is this formulation of the universality limit that was studied in [11]. There it was also shown that the limit for real a, b gives

$$\lim_{n\to\infty} \frac{K_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}v\right)}{K_n(1,1)} e^{-\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}nQ_n'(1)(u+v)} = \frac{\mathbb{A}i(u,v)}{\mathbb{A}i(0,0)}, \tag{3.7}$$

uniformly for u, v in compact subsets of the complex plane. The limit (3.6) has been established (with slightly different formulations) for varying exponential weights, using the Riemann-Hilbert method and $\overline{\partial}$ techniques by Miller and McLaughlin for a general class of non-analytic varying weights [19].

We prove

Theorem 3.1

Assume that for $n \geq 1$ we have a measure μ_n supported on the real line with infinitely many points in its support, and all finite power moments. Let $\{\rho_n\}$ be a sequence of positive numbers with limit 0, while $\{\Phi_n\}$ is a sequence of real numbers, such that uniformly for u, v in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n(1+\rho_n u, 1+\rho_n v)}{K_n(1,1)} e^{-\Phi_n(u+v)} = \frac{\mathbb{A}i(u,v)}{\mathbb{A}i(0,0)}. \tag{3.8}$$

Let us be given some infinite sequence of integers $\mathcal{T}$. The following are equivalent:

(I)

$$\sup_{n\in\mathcal{T}} \left| \rho_n \sum_{j=1}^{n} \frac{1}{1-x_{jn}} + \Phi_n \right| < \infty \text{ and } \sup_{n\in\mathcal{T}} \rho_n^2 \sum_{j=1}^{n} \frac{1}{(1-x_{jn})^2} < \infty. \tag{3.9}$$

(II) For each $R > 0$, there exists C_R such that

$$\sup_{n\in\mathcal{T}} \sup_{|z|\leq R} \left| \frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z} \right| \leq C_R. \tag{3.10}$$

(III) From every subsequence of $\mathcal{T}$, there is a further subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z} = \frac{Ai'(z)}{Ai'(0)} + c_0 \left\{ Ai(z)\, Ai'(0) - Ai'(z)\, Ai(0) \right\}, \tag{3.11}$$

uniformly for z in compact subsets of $\mathbb{C}$, where

$$c_0 = \frac{1}{Ai'(0)^2} \lim_{n\to\infty, n\in\mathcal{S}} \left\{ \rho_n \frac{p_n'(1)}{p_n(1)} + \Phi_n \right\} \tag{3.12}$$

and c_0 is bounded independently of $\mathcal{S}$.

Remarks
(a) Note that if we choose

$$\Phi_n = -\rho_n \sum_{j=1}^{n} \frac{1}{1-x_{jn}} = -\rho_n \frac{p_n'(1)}{p_n(1)},$$

then the limit in (3.11) simplifies to

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z} = \frac{Ai'(z)}{Ai'(0)},$$

and the right-hand side is independent of the subsequence, so we can take $\mathcal{S}$ to be the full sequence of positive integers if $\mathcal{T}$ also is.
(b) The universality limit (3.6) has been thus far only established for weights for which asymptotics are also known for the orthogonal polynomials at the soft edge, typically via the Riemann-Hilbert method [18], [19]. Thus Theorem 3.1 will be more useful when universality limits have been established at the soft edge without the much deeper asymptotics for the orthonormal polynomials.
(c) A natural question is whether there are analogous results at the "hard edge", which arises when one considers varying weights on $[0,\infty)$, with a Laguerre type factor at 0. If we consider varying weights of the form $x^{2\alpha}e^{-nQ(x)}$ on $[0,\infty)$, where $\alpha > -\frac{1}{2}$, then the universality limit at the hard edge 0 takes the form

$$\lim_{n\to\infty} \frac{1}{nc} K_n\left(\frac{u}{nc}, \frac{v}{nc}\right) = \mathbb{J}_\alpha^0(u,v)$$

where c is an appropriate constant, u, v lie in bounded subsets of $(0,\infty)$, and $\mathbb{J}_\alpha^0$ is a slightly unusual form of the Bessel kernel. See [8, Theorem 1.1]. Note that the scaling factor is $\frac{1}{n}$ rather than the $\frac{1}{n^2}$ in Theorem A above. This is evidently because the varying term e^{-nQ} largely overrides the fixed factor $x^{2\alpha}$. A second case to consider would be varying weights of the form $x^{n\alpha}e^{-nQ_n(x)}$, with α necessarily nonnegative, which would lead to a different universality limit at 0, probably more like Theorem A above. The local limits in these disparate cases seem worthy of investigation.

4. Proof of Theorems 2.1-3 and Corollary 2.4

In this section we abbreviate $p_n(\mu_n, z)$ as $p_n(z)$, $p_{n-1}(\mu_n, z)$ as $p_{n-1}(z)$, $K_n(\mu_n, z, w)$ as $K_n(z,w)$, and $\frac{\gamma_{n-1}}{\gamma_n}(\mu_n)$ as $\frac{\gamma_{n-1}}{\gamma_n}$.

Lemma 4.1
Assume the hypotheses of Theorem 2.3, and in particular, (2.8). Assume also that through the subsequence $\mathcal{S}$, uniformly for z in compact subsets of $\mathbb{C}$, and some finite valued function $f(z)$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n\left(\xi_n + \frac{z\tau_n}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} = f(z). \tag{4.1}$$

Then

$$f(z) = \cos \pi z + \frac{1}{\pi} f'(0) \sin \pi z, \tag{4.2}$$

where

$$f'(0) = \lim_{n \to \infty, n \in \mathcal{S}} \left[\frac{\tau_n}{n} \frac{p_n'(\xi_n)}{p_n(\xi_n)} + \Psi_n \right]. \tag{4.3}$$

Proof

This is similar to that of Lemma 2.1 in [15], but because of the extra factors here, we provide full details. From

$$\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(w) = \left[\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(u) \right] + \left[\frac{p_{n-1}}{p_n}(u) - \frac{p_{n-1}}{p_n}(w) \right]$$

and the Christoffel-Darboux formula, we deduce that

$$\frac{K_n(z,w)}{p_n(z)\, p_n(w)}(w-z) = \frac{K_n(u,z)}{p_n(z)\, p_n(u)}(u-z) + \frac{K_n(w,u)}{p_n(u)\, p_n(w)}(w-u).$$

Replace z, w, u respectively by $\xi_n + \frac{z\tau_n}{n}$, $\xi_n + \frac{w\tau_n}{n}, \xi_n + \frac{u\tau_n}{n}$. Divide each denominator by $p_n(\xi_n)^2$ and each numerator by $K_n(\xi_n, \xi_n)$ as well as $\frac{\tau_n}{n}$. Take limits through the subsequence $\mathcal{S}$. Observe that the first term on the left becomes

$$\frac{K_n\left(\xi_n + \frac{z\tau_n}{n}, \xi_n + \frac{w\tau_n}{n}\right) e^{\Psi_n(z+w)}}{K_n(\xi_n, \xi_n)} \frac{w-z}{\left(\frac{p_n\left(\xi_n + \frac{z\tau_n}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} \right) \left(\frac{p_n\left(\xi_n + \frac{w\tau_n}{n}\right)}{p_n(\xi_n)} e^{\Psi_n w} \right)}$$

and that this has the subsequential limit $\frac{\mathbb{S}(z-w)}{f(z)f(w)}(w-z)$. Similar considerations hold for the two terms on the right, so we obtain, if $f(u)\, f(w) f(z) \neq 0$, that

$$\frac{\mathbb{S}(z-w)}{f(z)\, f(w)}(w-z) = \frac{\mathbb{S}(u-z)}{f(z)\, f(u)}(u-z) + \frac{\mathbb{S}(w-u)}{f(u)\, f(w)}(w-u)$$

and hence

$$\frac{\sin \pi(w-z)}{f(z)\, f(w)} = \frac{\sin \pi(u-z)}{f(z)\, f(u)} + \frac{\sin \pi(w-u)}{f(u)\, f(w)}$$

Multiplying by $f(u)\, f(z)\, f(w)$ and using analytic continuation, gives for all u, z, w,

$$f(u) \sin \pi(w-z) = f(w) \sin \pi(u-z) + f(z) \sin \pi(w-u). \tag{4.4}$$

The double angle formula for trigonometric functions yields the elementary identity

$$\cos \pi u \sin \pi(w-z) = \cos \pi w \sin \pi(u-z) + \cos \pi z \sin \pi(w-u).$$

Then we can recast (4.4) as

$$[f(u) - \cos \pi u] \sin \pi(w-z) = [f(w) - \cos \pi w] \sin \pi(u-z) + [f(z) - \cos \pi z] \sin \pi(w-u). \tag{4.5}$$

Note that the definition (4.1) of f ensures that $f(0) = 1$. Setting $u = 0$ gives

$$0 = -[f(w) - \cos \pi w] \sin \pi z + [f(z) - \cos \pi z] \sin \pi w$$

so if $(\sin \pi z)(\sin \pi w) \neq 0$, we have

$$\frac{f(z) - \cos \pi z}{\sin \pi z} = \frac{f(w) - \cos \pi w}{\sin \pi w}.$$

So both sides are necessarily constant. Fix any such w, and call the right-hand side c. We have at first for all non-integer z, and then for all z,

$$f(z) - \cos \pi z = c \sin \pi z.$$

We see that

$$f'(0) = c\pi,$$

so

$$f(z) = \cos \pi z + \frac{1}{\pi} f'(0) \sin \pi z.$$

Finally, because of the uniform convergence, we can differentiate the asymptotic relation (4.1), so that

$$f'(z) = \lim_{n\to\infty, n\in\mathcal{S}} \left[\frac{\tau_n}{n} \frac{p_n'\left(\xi_n + \frac{z\tau_n}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} + \frac{p_n\left(\xi_n + \frac{z\tau_n}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} \Psi_n \right] \tag{4.6}$$

and hence also obtain (4.3). ■

Proof of Theorem 2.3
(I) ⇒ (II)

$$\begin{aligned}
\log \left| \frac{p_n\left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} \right| &= \sum_{j=1}^{n} \log \left| 1 + \frac{\tau_n z}{n(\xi_n - x_{jn})} \right| + \Psi_n \operatorname{Re} z \\
&= \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{2\tau_n \operatorname{Re}(z)}{n(\xi_n - x_{jn})} + \frac{\tau_n^2 |z|^2}{(n(\xi_n - x_{jn}))^2} \right) + \Psi_n \operatorname{Re} z \\
&\leq \left[\frac{\tau_n}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn}} + \Psi_n \right] \operatorname{Re} z + \frac{\tau_n^2 |z|^2}{2n^2} \sum_{j=1}^{n} \frac{1}{(\xi_n - x_{jn})^2}.
\end{aligned}$$

(4.7)

Then our hypotheses (2.9) give the uniform boundedness.
(II)⇒(I)
Suppose we have the uniform boundedness (2.10). Then by normality, from every subsequence of $\mathcal{T}$, we can choose a further subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n\left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n(\xi_n)} e^{\Psi_n z} = f(z),$$

where f is an entire function. Then also from (2.10), with $R = 1$,

$$\sup_{|z|\leq 1} |f(z)| \leq C_1.$$

Because of the uniform convergence for z in compact sets, the differentiated sequence also converges, so (cf. (4.3))

$$\lim_{n\to\infty, n\in\mathcal{S}} \left| \frac{\tau_n}{n} \frac{p_n'(\xi_n)}{p_n(\xi_n)} + \Psi_n \right| = \left| f'(0) \right|.$$

By Cauchy's inequalities for derivatives of analytic functions, $|f'(0)|$ is bounded above by C_1 independently of the subsequence $\mathcal{S}$, so

$$\sup_{n\in\mathcal{T}} \left| \frac{\tau_n}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn}} + \Psi_n \right| < \infty.$$

So we have the first requirement in (2.9). Next, setting $z = iy$, we have for real y,

$$C_1 \geq \log \left| \frac{p_n\left(\xi_n + \frac{i\tau_n y}{n}\right)}{p_n(\xi_n)} e^{\Psi_n iy} \right| = \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{\tau_n^2 y^2}{(n(\xi_n - x_{jn}))^2} \right).$$

Let us assume that $\tau_n \geq d > 0$ for all n and set $y = 1$. Then also for each j

$$\begin{aligned} C_1 &\geq \frac{1}{2} \log \left(1 + \frac{d^2}{(n(\xi_n - x_{jn}))^2} \right) \\ &\Rightarrow e^{2C_1} \geq 1 + \frac{d^2}{(n(\xi_n - x_{jn}))^2} \\ &\Rightarrow C_2 := e^{2C_1} - 1 \geq \frac{d^2}{(n(\xi_n - x_{jn}))^2}. \end{aligned}$$

Now there exists C_3 depending only on C_2 such that

$$\log(1+t) \geq C_3 t \text{ for } t \in [0, C_2].$$

Then

$$\begin{aligned} C_1 &\geq \log \left| \frac{p_n\left(\xi_n + \frac{i\tau_n}{n}\right)}{p_n(\xi_n)} \right| \\ &\geq \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{d^2}{(n(\xi_n - x_{jn}))^2} \right) \\ &\geq \frac{C_3}{2} d^2 \sum_{j=1}^{n} \frac{1}{(n(\xi_n - x_{jn}))^2}. \end{aligned}$$

Here C_1, C_3, d are independent of n, so we have also

$$\sup_{n\in\mathcal{T}} \sum_{j=1}^{n} \frac{1}{\left(n\left(\xi_n - x_{jn}\right)\right)^2} < \infty.$$

Thus we also have the second requirement in (2.9).

(II)⇒(III)

Because of the uniform boundedness, we can extract a subsequence $\mathcal{S}$ of $\mathcal{T}$ such that

$$\lim_{n\in\mathcal{S}} \frac{p_n\left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n\left(\xi_n\right)} e^{\Psi_n z} = f\left(z\right)$$

uniformly for z in compact subsets of $\mathbb{C}$. Then Lemma 4.1 shows that f has the form (4.2-4.3), which is the same as that in (2.11) and (2.12).

(III)⇒(II)

Since α is bounded independently of the subsequence, we obtain the uniform boundedness in (2.10). ■

We next list some results we need for the proof of Theorem 2.1:

Lemma 4.2

Assume the hypotheses of Theorem 2.1 on Q. Let J be a closed interval lying in the interior of supp$[\omega_Q]$, where ω_Q denotes the equilibrium measure for Q. Assume that ω_Q is absolutely continuous in a neighborhood of J, and that σ_Q and Q' are continuous in that neighborhood, while $\sigma_Q > 0$ there.

(a) Uniformly for $\xi \in J$ and u, v in compact subsets of the plane,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi,\xi\right)} e^{-\frac{n}{\tilde{K}_n(\xi,\xi)} Q'(\xi)(u+v)} = \mathbb{S}\left(u - v\right).$$

(b) Uniformly for $\xi \in J$

$$\lim_{n\to\infty} \frac{1}{n} \tilde{K}_n\left(\xi,\xi\right) = \sigma_Q\left(\xi\right).$$

(c) Uniformly for $x_{jn}, x_{j+1,n} \in J$,

$$\tilde{K}_n\left(x_{jn}, x_{jn}\right)\left(x_{jn} - x_{j+1,n}\right) = 1 + o\left(1\right).$$

(d) If $y_{jn} \in \left(x_{j+1,n}, x_{jn}\right)$ is a local maximum of $\left|p_n\left(\mu_n, \cdot\right) e^{-nQ}\right|$, then for $k = j, j+1$,

$$\left|x_{kn} - y_{jn}\right| \geq C/n.$$

Here C is independent of j, depending only on J.

Proof

(a) This was stated in [10, p. 749] in (1.13) and established in the proof of Theorem 1.2 [10, p. 766].

(b) This was proved by Totik [27, Theorem 1.2, p. 326].

(c) From the Christoffel-Darboux formula, $K_n\left(x_{jn}, x_{j+1,n}\right) = 0$. Thus if

$x_{j+1,n} = x_{jn} - \frac{\varepsilon_n}{\tilde{K}_n(x_{jn},x_{jn})}$, where $\varepsilon_n \to \varepsilon_0$ as $n \to \infty$ through a subsequence $\mathcal{T}$ of integers, we have

$$\begin{aligned} 0 &= \lim_{n\to\infty, n\in\mathcal{T}} \frac{K_n\left(x_{jn}, x_{j+1,n}\right)}{K_n\left(x_{jn}, x_{jn}\right)} e^{\frac{n}{\tilde{K}_n\left(x_{jn},x_{jn}\right)} Q'(x_{jn})\varepsilon_n} \\ &= \lim_{n\to\infty, n\in\mathcal{T}} \frac{K_n\left(x_{jn}, x_{jn} - \frac{\varepsilon_n}{\tilde{K}_n(x_{jn},x_{jn})}\right)}{K_n\left(x_{jn}, x_{jn}\right)} e^{\frac{n}{\tilde{K}_n\left(x_{jn},x_{jn}\right)} Q'(x_{jn})\varepsilon_n} = \mathbb{S}\left(\varepsilon_0\right). \end{aligned}$$

Since $\mathbb{S}(0) = 1$, we cannot have $\varepsilon_0 = 0$. It also follows that ε_0 is a non-zero integer. As $x_{j+1,n}$ is the closest zero on the left, and $\mathbb{S}(t)$ vanishes at all non-zero integers, it follows from Hurwitz' Theorem that

$$x_{j+1,n} = x_{jn} - \frac{1 + o(1)}{\tilde{K}_n\left(x_{jn}, x_{jn}\right)}.$$

(d) Suppose the conclusion is wrong. Then for some infinite sequence $\mathcal{S}$ of positive integers n and corresponding $j = j(n)$, either for $k = j$ or $k = j+1$, $y_{jn} - x_{kn} = o\left(\frac{1}{n}\right)$. Let us assume this is true for infinitely many $k = j$ and all $n \in \mathcal{S}$, so that

$$y_{jn} = x_{jn} - \frac{\varepsilon_n}{\tilde{K}_n\left(x_{jn}, x_{jn}\right)}$$

where the $\{\varepsilon_n\}$ have limit 0. (The other case is similar). The uniform universality limit in (a) gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{K_n\left(x_{jn}, y_{jn}\right)}{K_n\left(x_{jn}, x_{jn}\right)} e^{\frac{n}{\tilde{K}_n\left(x_{jn},x_{jn}\right)} Q'(x_{jn})\varepsilon_n} = \mathbb{S}(0) = 1.$$

Also then as $\frac{n}{\tilde{K}_n(x_{jn},x_{jn})} Q'\left(x_{jn}\right)$ is bounded by (b), the Christoffel-Darboux formula gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\gamma_{n-1}}{\gamma_n} \frac{\left|p_n\left(y_{jn}\right) p_{n-1}\left(x_{jn}\right)\right|}{\varepsilon_n} = 1.$$

Let

$$z_{jn} = \frac{1}{2}\left(x_{j+1,n} + x_{jn}\right) = x_{jn} - \frac{1 + o(1)}{2\tilde{K}_n\left(x_{jn}, x_{jn}\right)},$$

in view of (c). Also, for large enough n, $z_{jn} \in \left(x_{j+1,n}, x_{jn}\right)$ again by (c). Exactly as above, the uniform universality limit gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\gamma_{n-1}}{\gamma_n} \frac{\left|p_n\left(z_{jn}\right) p_{n-1}\left(x_{jn}\right)\right|}{(1/2)} = \mathbb{S}\left(\frac{1}{2}\right).$$

Taking the ratio of this limit and its analogue for y_{jn} gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \left|\frac{p_n\left(y_{jn}\right)}{p_n\left(z_{jn}\right)}\right| \frac{2}{\varepsilon_n} = \mathbb{S}\left(\frac{1}{2}\right)^{-1}.$$

Then as y_{jn} gives a maximum of $\left|p_n\right| e^{-nQ}$ in $\left(x_{j+1,n}, x_{jn}\right)$,

$$1 \le \left|\frac{p_n\left(y_{jn}\right) e^{-nQ(y_{jn})}}{p_n\left(z_{jn}\right) e^{-nQ(z_{jn})}}\right| = \varepsilon_n \frac{1}{2} \mathbb{S}\left(\frac{1}{2}\right)^{-1} e^{n[Q(z_{jn}) - Q(y_{jn})]} \left(1 + o(1)\right).$$

Here as Q' is bounded, while $z_j - y_{jn} = O\left(\frac{1}{n}\right)$, this last inequality gives a contradiction as $n \to \infty, n \in \mathcal{S}$. Then the result follows. ■

Proof of Theorem 2.1
(a) We apply Theorem 2.3 with

$$\tau_n = \frac{n}{\tilde{K}_n(\xi_n, \xi_n)} \text{ and } \Psi_n = -\frac{1}{\tilde{K}_n(\xi_n, \xi_n)} \frac{p_n'(\xi_n)}{p_n(\xi_n)} = -\frac{\tau_n}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn}}.$$

Here $\{\tau_n\}$ are bounded above and below for $\xi_n \in J$ by Lemma 4.2(b). Our choice of Ψ_n gives the first condition in (2.9). Next, our hypothesis (2.2) and the spacing in Lemma 4.2(c), as well as the bounds on the reproducing kernel show that for some $\eta > 0$,

$$\frac{1}{n^2} \sum_{x_{kn} \in (y_{jn} - \eta, y_{jn} + \eta)} \frac{1}{(\xi_n - x_{kn})^2} \leq C. \tag{4.8}$$

Indeed, we assumed that

$$\min_k |\xi_n - x_{kn}| \geq \frac{c}{n}.$$

Moreover, if $x_{k_0 n}$ is the closest zero to ξ_n, then from Lemma 4.2(c),

$$|\xi_n - x_{kn}| \geq C |k - k_0| / n$$

for $x_{kn} \in (\xi_n - \eta, \xi_n + \eta)$ and some $\eta > 0$ independent of j. These last two estimates easily yield (4.8). The remaining part of the sum is trivially bounded:

$$\frac{1}{n^2} \sum_{x_{kn} \notin (y_{jn} - \eta, y_{jn} + \eta)} \frac{1}{(\xi_n - x_{kn})^2} \leq \frac{1}{n\eta^2}.$$

We have shown (2.9) holds, so from Theorem 2.3 for appropriate subsequences $\mathcal{S}$,

$$\lim_{n \to \infty, n \in \mathcal{S}} \frac{p_n\left(\mu_n, \xi_n + \frac{z}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{p_n(\mu_n, \xi_n)} e^{-\frac{1}{\tilde{K}_n(\xi_n, \xi_n)} \frac{p_n'(\xi_n)}{p_n(\xi_n)} z} = \cos \pi z + \alpha \sin \pi z,$$

where by choice of Ψ_n,

$$\alpha = \lim_{n \to \infty, n \in \mathcal{S}} \left[\frac{\tau_n}{n} \frac{p_n'(\mu_n, \xi_n)}{p_n(\mu_n, \xi_n)} + \Psi_n \right] = 0.$$

As the limit is independent of the subsequence, it holds as $n \to \infty$.
(b) Here as $\xi_n = y_{jn}$ is a local max of $|p_n(\mu_n, \cdot)| e^{-nQ(\cdot)}$, we have

$$\frac{p_n'}{p_n}(y_{jn}) - nQ_n'(y_{jn}) = 0$$

so choosing Ψ_n as in (a), we have

$$\Psi_n = -\frac{1}{\tilde{K}_n(y_{jn}, y_{jn})} \frac{p_n'(y_{jn})}{p_n(y_{jn})} = -\frac{nQ_n'(y_{jn})}{\tilde{K}_n(y_{jn}, y_{jn})}.$$

Also by Lemma 4.2(d), we have the necessary lower bound (2.2) for the distance between y_{jn} and $\{x_{kn}\}_{k=1}^{n}$. Thus the result follows from (a). ■

Next, we turn to what is needed to prove Theorem 2.2:

Lemma 4.3

Assume the hypotheses of Theorem 2.2. Let J be a compact subinterval of $(-1,1)$).

(a) Uniformly for $\xi \in J$ and u, v in compact subsets of the plane,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{-\frac{n}{\tilde{K}_n(\xi\ \xi)} Q_n'(\xi)(u+v)} = \mathbb{S}(u-v).$$

(b) Uniformly for $\xi \in J$

$$\lim_{n\to\infty} \left(\frac{1}{n} \tilde{K}_n(\xi,\xi) - \sigma_{Q_n}(\xi) \right) = 0.$$

Here, there exists $\hat{C} > 0$ such that uniformly for $t \in (-1,1)$ and $n \geq 1$,

$$\hat{C}^{-1} \leq \sigma_{Q_n}(t) / \sqrt{1-t^2} \leq \hat{C}.$$

(c) Uniformly for $x_{jn}, x_{j+1,n} \in J$,

$$\tilde{K}_n(x_{jn}, x_{jn})(x_{jn} - x_{j+1,n}) = 1 + o(1).$$

(d) If $y_{jn} \in (x_{j+1,n}, x_{jn})$ is a local maximum of $\left|p_n(\mu_n, \cdot) e^{-nQ_n}\right|$, then for $k = j, j+1$,

$$|x_{kn} - y_{jn}| \geq C/n.$$

Here C is independent of j, depending only on J.

Proof

(a) In [12, Theorem 15.1, p. 155], it is proven that uniformly for ξ in a compact subinterval of $(-1,1)$and u, v in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \mathbb{S}(u-v),$$

that is

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{n\left[2Q_n(\xi) - Q_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}\right) - Q_n\left(\xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)\right]} = \mathbb{S}(u-v).$$

Now using the uniform Lipschitz condition of order α on $\{Q_n'\}$, we see that

$$\begin{aligned} & 2Q_n(\xi) - Q_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}\right) - Q_n\left(\xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right) \\ = \quad & -\frac{Q_n'(\xi)}{\tilde{K}_n(\xi,\xi)}(u+v) + O\left(\frac{|u|^{\alpha+1} + |v|^{\alpha+1}}{\tilde{K}_n(\xi,\xi)^{\alpha+1}}\right) \\ = \quad & -\frac{Q_n'(\xi)}{\tilde{K}_n(\xi,\xi)}(u+v) + o\left(\frac{1}{n}\right), \end{aligned}$$

since $\tilde{K}_n(\xi,\xi) \geq Cn$, see [12, Theorem 2.1(b), p. 9]. It follows that we have the limit

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{-\frac{nQ_n'(\xi)}{\tilde{K}_n(\xi,\xi)}(u+v)} = \mathbb{S}(u-v). \tag{4.9}$$

for real u, v. To extend it to complex u, v, we use Theorem 1.2 and its extension (1.13) in [10, p. 748, p. 749]. Let us verify the hypotheses of Theorem 1.2 there. Firstly, the measures have the correct form. Secondly the equilibrium measures were shown to satisfy a Lipschitz condition of positive order in [12, Theorem 3.1, p. 15], which is much more than the equicontinuity required in [10]. Next, the $\{Q_n'\}$ satisfy a Lipschitz condition and are uniformly bounded in compact sets, which is more than the requirements in [10]. The requisite upper and lower bounds for the Christoffel functions appear in [12, Theorem 2.1, p. 9]. Finally, the asymptotics for Christoffel functions in [12, Theorem 2.2(c), p. 11] give, uniformly for a in compact subsets of the real line,

$$\frac{\lambda_n\left(\mu_n, \xi + \frac{a}{n}\right)}{\lambda_n(\mu_n,\xi)} \frac{e^{-2nQ_n(\xi)}}{e^{-2nQ_n\left(\xi+\frac{a}{n}\right)}} = \frac{\sigma_{Q_n}(\xi) + o(1)}{\sigma_{Q_n}\left(\xi + \frac{a}{n}\right) + o(1)} = 1 + o(1)$$

so all hypotheses of Theorem 1.2 in [10] are satisfied, so we obtain (4.9) uniformly for u, v in compact subsets of the real line, and hence also from (1.13), for u, v in compact subsets of the plane.
(b) The asymptotics for the Christoffel functions were established in [12, Theorem 2.2(c), p. 11]. The estimate for σ_{Q_n} appears in [12, Theorem 3.1(a), p. 15].
(c) The asymptotics for the spacing of zeros were established in [12, Theorem 2.2(d), p. 11].
(d) The proof is exactly the same as in Lemma 4.2. ■

Proof of Theorem 2.2
This follows from Theorem 2.3 and Lemma 4.3 in exactly the same that Theorem 2.1 followed from Theorem 2.3 and Lemma 4.2. ■

Proof of Corollary 2.4
We assume the hypotheses of Theorem 2.2. The proof under the hypotheses of Theorem 2.1 is very similar. Observe first that

$$\begin{aligned} & \frac{\mu_n'\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{\mu_n'(y_{jn})} \\ = {} & \exp\left(n\left[Q_n(y_{jn}) - Q_n\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)\right]\right) \\ = {} & \exp\left(-n\frac{Q_n'(\xi)\,x}{\tilde{K}_n(y_{jn},y_{jn})}\right), \end{aligned} \tag{4.10}$$

for some ξ between $y_{jn}, y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}$. The assumed uniform Lipschitz condition on $\{Q_n'\}$ and the fact that $\tilde{K}_n(y_{jn}, y_{jn}) \geq Cn$ allow us to continue this as

(4.11)
$$\frac{\mu_n'\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{\mu_n'(y_{jn})} = \exp\left(-n\frac{Q_n'(y_{jn})\,x}{\tilde{K}_n(y_{jn},y_{jn})} + o(1)\right) = \exp(\Psi_n x)(1+o(1)),$$

where as above,

$$\Psi_n = -\frac{nQ_n'(y_{jn})}{\tilde{K}_n(y_{jn},y_{jn})}. \tag{4.12}$$

Then Theorem 2.2(b) gives, uniformly for x in a compact subset of $\mathbb{R}$,

$$\frac{(p_n\mu_n')\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{(p_n\mu_n')(y_{jn})} = \cos\pi x + o(1). \tag{4.13}$$

Next, the differentiated form of the limit (2.4) with Q_n instead of Q, gives, also locally uniformly in x,

$$\begin{aligned} &\frac{p_n'\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{p_n(y_{jn})}\frac{1}{\tilde{K}_n(y_{jn},y_{jn})}e^{\Psi_n x} + \frac{p_n\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{p_n(y_{jn})}e^{\Psi_n x}\Psi_n \\ = \ & -\pi\sin\pi x + o(1). \end{aligned}$$

Recall that we may differentiate (2.4) because it holds uniformly in compact subsets of $\mathbb{C}$. In view of (4.10) to (4.12) above and our Lipschitz condition on Q_n, we can recast this as

$$\begin{aligned} &\frac{1}{\tilde{K}_n(y_{jn},y_{jn})}\left[\frac{(p_n'\mu_n')\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{p_n\mu_n'(y_{jn})} + \frac{p_n\mu_n''\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{p_n\mu_n'(y_{jn})}\right] \\ = \ & -\pi\sin\pi x + o(1). \end{aligned}$$

so that

$$\frac{1}{\pi\tilde{K}_n(y_{jn},y_{jn})}\frac{(p_n\mu_n')'\left(y_{jn} + \frac{x}{\tilde{K}_n(y_{jn},y_{jn})}\right)}{(p_n\mu_n')(y_{jn})} = -\sin\pi x + o(1).$$

This and (4.13) give the result. ■

5. Proof of Theorem 3.1

We begin with

Lemma 5.1
Assume the hypotheses of Theorem 3.1, and in particular,

$$\lim_{n\to\infty}\frac{K_n(1+\rho_n u, 1+\rho_n v)}{K_n(1,1)}e^{-\Phi_n(u+v)} = \frac{\mathbb{A}i(u,v)}{\mathbb{A}i(0,0)}, \tag{5.1}$$

uniformly for u, v in compact subsets of $\mathbb{C}$. Assume also that through the subsequence $\mathcal{S}$, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z} = f(z). \tag{5.2}$$

Then

$$f(z) = \frac{Ai'(z)}{Ai'(0)} + c_0 \left\{ Ai(z)\, Ai'(0) - Ai'(z)\, Ai(0) \right\}, \tag{5.3}$$

where

$$c_0 = \frac{1}{Ai'(0)^2} \lim_{n\to\infty, n\in\mathcal{S}} \left\{ \rho_n \frac{p_n'(1)}{p_n(1)} + \Phi_n \right\}. \tag{5.4}$$

Proof

As in Lemma 4.1, we have for complex u, z, w,

$$\frac{K_n(z,w)}{p_n(z)\, p_n(w)} (w-z) = \frac{K_n(u,z)}{p_n(z)\, p_n(u)} (u-z) + \frac{K_n(w,u)}{p_n(u)\, p_n(w)} (w-u).$$

Replace z, w, u respectively by $1+\rho_n z$, $1+\rho_n w, 1+\rho_n u$. Divide each denominator by $p_n(1)^2$ and each numerator by $K_n(1,1)$ as well as ρ_n. Take limits through the subsequence $\mathcal{S}$. Observe that the first term on the left becomes

$$\frac{K_n(1+\rho_n z, 1+\rho_n w)\, e^{\Phi_n(z+w)}}{K_n(1,1)} \frac{w-z}{\left(\frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z}\right)\left(\frac{p_n(1+\rho_n w)}{p_n(1)} e^{\Phi_n w}\right)}$$

and that this has the subsequential limit $\frac{\mathbb{A}i(z,w)}{\mathbb{A}i(0,0)} \frac{w-z}{f(z)f(w)}$. Similar considerations hold for the two terms on the right, so we obtain, if $f(u)\, f(w) f(z) \neq 0$, that

$$\frac{\mathbb{A}i(z,w)}{\mathbb{A}i(0,0)} \frac{w-z}{f(z)\, f(w)} = \frac{\mathbb{A}i(u,z)}{\mathbb{A}i(0,0)} \frac{u-z}{f(u)\, f(z)} + \frac{\mathbb{A}i(w,u)}{\mathbb{A}i(0,0)} \frac{w-u}{f(u)\, f(w)}.$$

Hence analytic continuation shows that for all u, z, w

$$f(u)\, \mathbb{A}i(z,w)(w-z) = f(w)\, \mathbb{A}i(u,z)(u-z) + f(z)\, \mathbb{A}i(w,u)(w-u).$$

Next, the definition (3.2) of the Airy kernel easily gives

$$Ai(u)\, \mathbb{A}i(z,w)(w-z) = Ai(w)\, \mathbb{A}i(u,z)(u-z) + Ai(z)\, \mathbb{A}i(w,u)(w-u).$$

Multiplying the last identity by c and subtracting gives

$$\begin{aligned} & [f(u) - cAi(u)]\, \mathbb{A}i(z,w)(w-z) \\ = \; & [f(w) - cAi(w)]\, \mathbb{A}i(u,z)(u-z) + [f(z) - cAi(z)]\, \mathbb{A}i(w,u)(w-u). \end{aligned}$$

Since (as follows from (5.2)) $f(0) = 1$, setting $u = 0$ and $c = Ai(0)^{-1}$ gives

$$0 = -[f(w) - cAi(w)]\, \mathbb{A}i(0,z)\, z + [f(z) - cAi(z)]\, \mathbb{A}i(w,0)\, w$$

and hence as long as the denominators are non-0, we have for all complex w, z,

$$\frac{f(w) - cAi(w)}{\mathbb{A}i(w,0)\,w} = \frac{f(z) - cAi(z)}{\mathbb{A}i(z,0)\,z}.$$

Then both sides must be constant, so for some number d,

$$\begin{aligned} f(z) &= \frac{Ai(z)}{Ai(0)} + dz\mathbb{A}i(z,0) \\ &= \frac{Ai(z)}{Ai(0)} + d\left\{Ai(z)\,Ai'(0) - Ai'(z)\,Ai(0)\right\} \end{aligned}$$

by (3.2). Differentiating and setting $z = 0$, gives

$$\begin{aligned} f'(0) &= \frac{Ai'(0)}{Ai(0)} + d\left\{Ai'(0)^2 - Ai''(0)\,Ai(0)\right\} \\ &= \frac{Ai'(0)}{Ai(0)} + dAi'(0)^2, \end{aligned}$$

as $Ai''(0) = 0$, see (3.4). Moreover, differentiating the asymptotic relation (5.2) as we can, gives

$$f'(z) = \lim_{n\to\infty, n\in\mathcal{S}} \left\{\rho_n \frac{p_n'(1+\rho_n z)}{p_n(1)} e^{\Phi_n z} + \frac{p_n(1+\rho_n z)}{p_n(1)} e^{\Phi_n z}\Phi_n\right\},$$

so

$$f'(0) = \lim_{n\to\infty, n\in\mathcal{S}} \left\{\rho_n \frac{p_n'(1)}{p_n(1)} + \Phi_n\right\}$$

and then

$$\begin{aligned} d &= \frac{1}{Ai'(0)^2}\left[\lim_{\mathcal{S}}\left\{\rho_n \frac{p_n'(1)}{p_n(1)} + \Phi_n\right\} - \frac{Ai'(0)}{Ai(0)}\right]. \\ &= c_0 - \frac{1}{Ai(0)\,Ai'(0)}, \end{aligned}$$

where c_0 is given by (5.4). Then

$$\begin{aligned} f(z) &= \frac{Ai(z)}{Ai(0)} + c_0\left\{Ai(z)\,Ai'(0) - Ai'(z)\,Ai(0)\right\} \\ &\quad - \frac{1}{Ai(0)\,Ai'(0)}\left\{Ai(z)\,Ai'(0) - Ai'(z)\,Ai(0)\right\}, \end{aligned}$$

which gives (5.3). ■

We turn to the

Proof of Theorem 3.1

(I)⇒(II)

$$\begin{aligned}
&\log\left|\frac{p_n(1+\rho_n z)}{p_n(1)}e^{\Phi_n z}\right| \\
= &\sum_{j=1}^{n}\log\left|\left(1+\frac{\rho_n z}{1-x_{jn}}\right)\right|+\Phi_n \operatorname{Re} z \\
= &\frac{1}{2}\sum_{j=1}^{n}\log\left(1+\frac{2\rho_n \operatorname{Re} z}{1-x_{jn}}+\frac{(\rho_n|z|)^2}{(1-x_{jn})^2}\right)+\Phi_n \operatorname{Re} z \\
\leq &\operatorname{Re} z\left[\rho_n\sum_{j=1}^{n}\frac{1}{1-x_{jn}}+\Phi_n\right]+\frac{(\rho_n|z|)^2}{2}\sum_{j=1}^{n}\frac{1}{(1-x_{jn})^2}.
\end{aligned}$$

Then our hypotheses (3.9) give the uniform boundedness.

(II)⇒(I)

Suppose we have the uniform boundedness (3.10). Then by normality from every subsequence, we can choose another subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty,n\in\mathcal{S}}\frac{p_n(1+\rho_n z)}{p_n(1)}e^{\Phi_n z}=f(z),$$

where f is an entire function. Then also from (3.10), with $R=1$,

$$\sup_{|z|\leq 1}|f(z)|\leq C_1.$$

Because of the uniform convergence for z in compact subsets of $\mathbb{C}$, the differentiated sequence also converges, so

$$\lim_{n\to\infty,n\in\mathcal{S}}\left|\rho_n\frac{p_n'(1)}{p_n(1)}+\Phi_n\right|=|f'(0)|.$$

By Cauchy's inequalities for derivatives, $|f'(0)|$ is bounded above independently of the subsequence $\mathcal{S}$, so

$$\sup_{n\in\mathcal{T}}\left|\rho_n\sum_{j=1}^{n}\frac{1}{1-x_{jn}}+\Phi_n\right|<\infty.$$

This gives the first relation in (3.9). Next, setting $z=iy$, we have for real y,

$$C_1\geq\log\left|\frac{p_n(1+i\rho_n y)}{p_n(1)}e^{\Phi_n iy}\right|=\frac{1}{2}\sum_{j=1}^{n}\log\left(1+\frac{\rho_n^2y^2}{(1-x_{jn})^2}\right).$$

Let $y = 1$. Then also for each j,

$$\begin{aligned} C_1 &\geq \frac{1}{2}\log\left(1+\frac{\rho_n^2}{(1-x_{jn})^2}\right) \\ &\Rightarrow e^{2C_1} \geq 1+\frac{\rho_n^2}{(1-x_{jn})^2} \\ &\Rightarrow C_2 := e^{2C_1}-1 \geq \frac{\rho_n^2}{(1-x_{jn})^2}. \end{aligned}$$

Now there exists C_3 depending only on C_2 such that

$$\log(1+t) \geq C_3 t \text{ for } t \in [0, C_2].$$

Then

$$\begin{aligned} C_1 &\geq \log\left|\frac{p_n(1+i\rho_n)}{p_n(1)}e^{\Phi_n i}\right| \\ &= \frac{1}{2}\sum_{j=1}^{n}\log\left(1+\frac{\rho_n^2}{(1-x_{jn})^2}\right) \\ &\geq \frac{C_3}{2}\sum_{j=1}^{n}\frac{\rho_n^2}{(1-x_{jn})^2}. \end{aligned}$$

So we also have the second relation in (3.9).

(II)⇒(III)

Because of the uniform boundedness, we can extract a subsequence $\mathcal{S}$ of $\mathcal{T}$ such that

$$\lim_{n\in\mathcal{S}}\frac{p_n(1+\rho_n z)}{p_n(1)}e^{\Phi_n i} = f(z)$$

uniformly for z in compact subsets of $\mathbb{C}$. Then Lemma 5.1 shows that f has the form (5.3-5.4) and hence also (3.11-12).

(III)⇒(II)

Since d is bounded independently of the subsequence, we obtain the uniform boundedness in (3.10). ■

References

[1] G. Akemann, J. Baik, P. DiFrancesco (eds.), *The Oxford Handbook of Random Matrix Theory*, Oxford University Press, Oxford, 2011.

[2] P. Bleher and A. Its, *Random matrix models and their applications,* Cambridge University Press, Cambridge, 2001.

[3] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[4] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory,* Communications in Pure and Applied Mathematics, 52(1999), 1335-1425.

[5] P. J. Forrester, *Log-gases and Random matrices*, London Mathematical Society Mongraphs, Vol. 34, Princeton University Press, Princeton, 2010.

[6] A.B. Kuijlaars, *Universality*, (in) The Oxford Handbook of Random Matrix Theory, (eds. G. Akemann et al), Oxford University Press, Oxford, 2011, pp. 103-134.

[7] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Maths. Research Notices, 30(2002), 1575-1600.

[8] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations at the Origin of the Spectrum,* Commun. Math. Phys., 243(2003), 163-191.

[9] Eli Levin and D.S. Lubinsky, *Applications of Universality Limits to Zeros and Reproducing Kernels of Orthogonal Polynomials*, Journal of Approximation Theory, 150(2008), 69-95.

[10] Eli Levin and D.S. Lubinsky, *Universality Limits in the Bulk for Varying Measures,* Advances in Mathematics, 219(2008), 743-779.

[11] Eli Levin and D.S. Lubinsky, *Universality Limits at the Soft Edge of the Spectrum via Classical Complex Analysis,* International Maths Research Notices (2010) doi: 10.1093/imrn/rnq185 , 13(2011), 3006-3070.

[12] Eli Levin and D.S. Lubinsky, *Bounds and Asymptotics for Orthogonal Polynomials for Varying Weights*, Springer Briefs in Mathematics, Springer, Cham, Switzerland, 2018.

[13] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915-939.

[14] D̊.S. Lubinsky, *Pointwise Asymptotics for Orthonormal Polynomials at the Endpoints of the Interval via Universality*, International Maths Research Notices, https://doi.org/10.1093/imrn/rny042

[15] D.S.Lubinsky, *Local Asymptotics for Orthonormal Polynomials in the Interior of the Support via Universality*, Proc. of the Amer. Math. Soc., 147(2019), 3887-3886.

[16] D.S. Lubinsky, *Local Asymptotics for Orthonormal Polynomials on the Unit Circle Via Universality*, to appear in Journal d'Analyse Math.

[17] K. T.-R. McLaughlin, *Asymptotic analysis of random matrices with external source and a family of algebraic curves*, Nonlinearity 20 (2007), 1547–1571.

[18] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights,* Intern. Math. Res. Papers, 2006, Article ID 48673, pages 1-78.

[19] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights,*" Intern. Math. Res. Notices 2008, Article ID rnn075, pages 1-66, 2008.

[20] L. Pastur and M. Schcherbina, *Universality of the local eigenvalue statistics for a class of Unitary Invariant Random Matrix Ensembles*, J. Statistical Physics, 86(1997), 109-147.

[21] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer, New York, 1997.

[22] B. Simon, *Orthogonal Polynomials on the Unit Circle, Parts 1 and 2,* American Mathematical Society, Providence 2005.

[23] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, Journal d'Analyse de Mathematique, 105 (2008), 345-362.

[24] B. Simon, *Szegö's theorem and its Descendants: Spectral Theory for L_2 Perturbations of Orthogonal Polynomials*, Princeton University Press, Princeton, 2011.

[25] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[26] T. Tao, *Topics in Random Matrix Theory,* Graduate Studies in Mathematics, vol. 132, American Mathematical Society, Providence, 2012.

[27] V. Totik, *Asymptotics for Christoffel functions with Varying Weights,* Advances in Applied Maths., 25(2000), 322-351.

[28] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361-391.

[29] V. Totik, *Universality under Szegő's Condition,* Canad. Math. Bull., 59(2016), 211-224.

[30] V. Totik, *Oscillatory Behavior of Orthogonal Polynomials*, manuscript.

[31] C.A. Tracy and H. Widom, *Level-Spacing Distributions and the Airy Kernel,* Comm. Math. Phys., 159(1994), 151-174.

[32] O. Vallee and M. Soares, *Airy Functions and Applications to Physics*, World Scientific, Singapore, 2004.

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