

ASYMPTOTIC BEHAVIOR OF NIKOLSKII CONSTANTS FOR POLYNOMIALS ON THE UNIT CIRCLE

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ABSTRACT. Let $q > p > 0$, and consider the Nikolskii constants

$$\Lambda_{n,p,q} = \inf_{\deg(P) \leq n-1} \frac{\|P\|_p}{\|P\|_q},$$

where the norm is with respect to normalized Lebesgue measure on the unit circle. We prove that

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{p,q} \leq \mathcal{E}_{p,q},$$

where

$$\mathcal{E}_{p,q} = \inf \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_q(\mathbb{R})}},$$

and the inf is taken over all entire functions f of exponential type at most π . We conjecture that the lim sup can be replaced by a limit.

Nikolskii Inequalities, Paley-Wiener Spaces 42C05

1. INTRODUCTION¹

Define the normalized L_p norms of polynomials P on the unit circle:

$$\|P\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^p d\theta \right)^{1/p}, \text{ if } p < \infty$$

and

$$\|P\|_{\infty} = \sup_{|z|=1} |P(z)|.$$

Classic Nikolskii inequalities assert that given $q > p > 0$, there exists C depending on p, q , such that for $n \geq 1$ and polynomials P of degree $\leq n$,

$$(1.1) \quad \frac{\|P\|_p}{\|P\|_q} \geq C n^{\frac{1}{q} - \frac{1}{p}}.$$

These inequalities are useful in studying convergence of orthonormal expansions and Lagrange interpolation, and in analyzing quadrature and discretization of integrals. A proof for trigonometric polynomials, which includes this case, appears in [1, Theorem 2.6, page 102]. The converse sharp inequality, namely

$$\frac{\|P\|_p}{\|P\|_q} \leq 1$$

follows from Hölder's inequality. It is a longstanding problem to determine the sharp constants in (1.1). In a recent paper dealing with L_p Christoffel functions,

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we obtained the asymptotically sharp ($n \rightarrow \infty$) form for $q = \infty$, and $p > 0$ [4]. In this paper, we obtain asymptotic upper bounds for $q > p > 0$. We emphasize that p and q are not necessarily dual/conjugate L_p exponents.

The asymptotics involve the Paley-Wiener space L_π^p , $0 < p \leq \infty$. This is the set of all entire functions f satisfying $\|f\|_{L_p(\mathbb{R})} < \infty$, and for some $C > 0$,

$$|f(z)| \leq Ce^{\pi|z|}, \quad z \in \mathbb{C}.$$

Note that $L_\pi^p \subset L_\pi^q$ for $q > p$. We define

$$(1.2) \quad \mathcal{E}_{p,q} = \inf \left\{ \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_q(\mathbb{R})}} : f \in L_\pi^p \right\}.$$

Also define for $n \geq 1$, the n th Nikolskii constant,

$$(1.3) \quad \Lambda_{n,p,q} = \inf_{\deg(P) \leq n-1} \frac{\|P\|_p}{\|P\|_q}.$$

We prove:

Theorem 1

Let $q > p > 0$. Then

$$(1.4) \quad \limsup_{n \rightarrow \infty} \Lambda_{n,p,q} n^{\frac{1}{p} - \frac{1}{q}} \leq \mathcal{E}_{p,q}.$$

In [4], we showed that

$$\lim_{n \rightarrow \infty} \Lambda_{n,p,\infty} n^{\frac{1}{p}} = \mathcal{E}_{p,\infty},$$

so for $q = \infty$, the lim sup can be replaced by a limit. We offer:

Conjecture

$$(1.5) \quad \lim_{n \rightarrow \infty} \Lambda_{n,p,q} n^{\frac{1}{p} - \frac{1}{q}} = \mathcal{E}_{p,q}.$$

In the sequel, C, C_1, C_2, \dots , denote positive constants independent of n, x, t , and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. For real x , we use $[x]$ to denote the greatest integer $\leq x$, and $[x]_+ = \max\{0, x\}$. We prove the upper bound in Theorem 1 in Section 2, and discuss some of the difficulties of proving the Conjecture in Section 3.

Acknowledgement

In an earlier version of this paper, we proved a weak asymptotic lower bound, as some evidence towards the conjecture. A closer look at this bound showed, however, that it is zero and therefore useless. We thank Vili Totik for this observation.

2. PROOF OF THEOREM 1

We shall use Lagrange interpolation at the roots of unity. Let $n \geq 2$, and for $|j| \leq [n/2]$, we let

$$z_{jn} = e^{2\pi i j/n},$$

and define the corresponding fundamental polynomial

$$(2.1) \quad \ell_{jn}(z) = \frac{1}{n} \frac{z^n - 1}{z z_{jn} - 1}.$$

Throughout, we also use the sinc kernel

$$S(t) = \frac{\sin \pi t}{\pi t}.$$

We start with:

Lemma 2.1

Assume that $C > 1$ and $k = k(n)$ is such that

$$C^{-1} \leq \frac{k}{n} \leq C, \quad n \geq 1.$$

Then as $n \rightarrow \infty$,

$$(2.2) \quad \ell_{jk} \left(e^{2\pi it/n} \right) = (-1)^j e^{i\pi tk/n} S \left(\frac{tk}{n} - j \right) + o(1),$$

uniformly for j and t with

$$(2.3) \quad \frac{|j|}{n} = o(1); \quad \frac{t}{n} = o(1).$$

Proof

We see that

$$\begin{aligned} \ell_{jk} \left(e^{2\pi it/n} \right) &= \frac{1}{k} \frac{e^{i\pi tk/n} \sin \left(\frac{\pi tk}{n} \right)}{e^{i\pi \left(\frac{t}{n} - \frac{j}{k} \right)} \sin \left(\pi \frac{t}{n} - \pi \frac{j}{k} \right)} \\ &= \frac{1}{k} \frac{e^{i\pi tk/n} (-1)^j \sin \left(\pi \left(\frac{tk}{n} - j \right) \right)}{e^{i\pi \left(\frac{t}{n} - \frac{j}{k} \right)} \sin \left(\frac{\pi}{k} \left(\frac{tk}{n} - j \right) \right)} \\ &= \frac{e^{i\pi tk/n} (-1)^j S \left(\frac{tk}{n} - j \right)}{e^{i\pi \left(\frac{t}{n} - \frac{j}{k} \right)} S \left(\frac{1}{k} \left(\frac{tk}{n} - j \right) \right)}. \end{aligned}$$

Here $e^{i\pi \left(\frac{t}{n} - \frac{j}{k} \right)} = 1 + o(1)$ uniformly for j and t satisfying (2.3). Moreover, by continuity of S at 0, $S \left(\frac{1}{k} \left(\frac{tk}{n} - j \right) \right) = S \left(\frac{t}{n} - \frac{j}{k} \right) = 1 + o(1)$ uniformly for the same range of j and t . ■

Now for each $f \in L^p_\pi$, and any $p > 0$, a result of Plancherel and Polya [2, p. 506], [5] asserts that

$$(2.4) \quad \sum_{n=-\infty}^{\infty} |f(n)|^p \leq C \int_{-\infty}^{\infty} |f(t)|^p dt,$$

where C is independent of f . The converse inequality, with appropriate C , holds only for $p > 1$. Thus for $p > 1$, and some C_1, C_2 independent of f , [3, p. 152]

$$(2.5) \quad C_1 \sum_{n=-\infty}^{\infty} |f(n)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p dt \leq C_2 \sum_{n=-\infty}^{\infty} |f(n)|^p.$$

As a consequence, any such function f admits an expansion

$$(2.6) \quad f(z) = \sum_{j=-\infty}^{\infty} f(j) S(z-j),$$

that converges locally uniformly in the plane. Indeed, for $p > 1$, this follows from the Plancherel-Polya theorem [3, p. 152] that we have just quoted. For $p \leq 1$, (2.4), (2.5) also imply that $f \in L_\pi^2$, so yet again (2.6) holds. Note too that $L_\pi^p \subset L_\pi^q$ for $q > p$. In particular, if $f \in L_\pi^p$ for some $p > 0$, then $\|f\|_{L_\infty(\mathbb{R})} < \infty$.

Lemma 2.2

Let $q > p > 1$. Then

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \leq \mathcal{E}_{p,q}.$$

Proof

Let $f \in L_\pi^p$, not the zero function. Fix $m \geq 1$ so large that at least one of $f(j)$, $|j| \leq m$, is not 0. Let

$$(2.7) \quad S_n(z) = \sum_{|j| \leq m} f(j) (-1)^j \ell_{jn}(z).$$

We have

$$\Lambda_{n,p,q} \leq \frac{\|S_n\|_p}{\|S_n\|_q}.$$

Let $r > 1$ and $s > 0$. Lemma 2.1 gives

$$\lim_{n \rightarrow \infty} \frac{n}{2\pi} \int_{-2\pi r/n}^{2\pi r/n} |S_n(e^{i\theta})|^s d\theta = \lim_{n \rightarrow \infty} \int_{-r}^r |S_n(e^{2\pi i t/n})|^s dt = \int_{-r}^r \left| \sum_{|j| \leq m} f(j) S(t-j) \right|^s dt.$$

(2.8)

Next, we estimate the rest of the integral. Let $z = e^{i\theta}$, $\theta \in [0, \pi]$. If $0 \leq j \leq [n/2]$,

(2.9)

$$|\ell_{jn}(z)| \leq \min \left\{ 1, \frac{2}{n|z - z_{jn}|} \right\} \leq \min \left\{ 1, \frac{1}{n \left| \sin \left(\frac{\theta - 2j\pi/n}{2} \right) \right|} \right\} \leq \min \left\{ 1, \frac{\pi}{|n\theta - 2j\pi|} \right\},$$

by the inequality $|\sin t| \geq \frac{2}{\pi} |t|$, $|t| \leq \frac{\pi}{2}$. For $0 > j \geq -[n/2]$, we have instead

$$|\ell_{jn}(z)| \leq |\ell_{-jn}(z)| \leq \min \left\{ 1, \frac{\pi}{|n\theta - 2|j|\pi|} \right\}.$$

Hence if $r \geq 2m$, and $\pi \geq \theta \geq 2\pi r/n$

$$|S_n(z)| \leq \left(\sum_{|j| \leq m} |f(j)| \right) \frac{2\pi}{n|\theta|}.$$

The same estimate holds for $-\pi \leq \theta \leq -2\pi r/n$. Then for some C independent of n, f, r ,

$$\begin{aligned}
 & \frac{n}{2\pi} \int_{2\pi r/n \leq |\theta| \leq \pi} |S_n(z)|^p d\theta \\
 & \leq C \left(\sum_{|j| \leq m} |f(j)| \right)^p n \int_{2\pi r/n \leq |\theta| \leq \pi} \frac{d\theta}{|n\theta|^p} \\
 (2.10) \quad & \leq C \left(\sum_{|j| \leq m} |f(j)| \right)^p r^{1-p},
 \end{aligned}$$

where again, C is independent of n and r . Combined with (2.8), for $s = q, p$, this gives

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} & \leq \limsup_{n \rightarrow \infty} \frac{\left(\frac{n}{2\pi} \int_{-\pi}^{\pi} |S_n(z)|^p d\theta \right)^{1/p}}{\left(\frac{n}{2\pi} \int_{-\pi}^{\pi} |S_n(z)|^q d\theta \right)^{1/q}} \\
 & \leq \frac{\left(\int_{-r}^r \left| \sum_{|j| \leq m} f(j) S(t-j) \right|^p dt + C \left(\sum_{|j| \leq m} |f(j)| \right)^p r^{1-p} \right)^{1/p}}{\left(\int_{-r}^r \left| \sum_{|j| \leq m} f(j) S(t-j) \right|^q dt \right)^{1/q}}.
 \end{aligned}$$

Recall that m is fixed. Letting $r \rightarrow \infty$ gives

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \leq \frac{\left(\int_{-\infty}^{\infty} \left| \sum_{|j| \leq m} f(j) S(t-j) \right|^p dt \right)^{1/p}}{\left(\int_{-\infty}^{\infty} \left| \sum_{|j| \leq m} f(j) S(t-j) \right|^q dt \right)^{1/q}}.$$

Now the triangle inequality and the Polya-Plancherel equivalence (2.5) allow us to let $m \rightarrow \infty$, giving

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \leq \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_q(\mathbb{R})}}.$$

As we may choose any $f \in L_{\pi}^p$, we obtain the result. \blacksquare

Next, we handle the more difficult case $p \leq 1$. We let

$$U_k(z) = \frac{1}{k} \sum_{j=0}^{k-1} z^j = \frac{1}{k} \frac{1-z^k}{1-z} = \ell_{0,k}(z).$$

Observe that from Lemma 2.1, uniformly for t in compact sets, as $k \rightarrow \infty$ subject to the restrictions $C^{-1} \leq k/n \leq C$,

$$(2.11) \quad U_k \left(e^{2\pi i t/n} \right) = e^{i\pi t k/n} S \left(\frac{tk}{n} \right) + o(1).$$

Lemma 2.3

Let $0 < p \leq 1$ and $q > p$. Then

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \leq \mathcal{E}_{p,q}.$$

Proof

Let $f \in L^p_\pi$. Let $\varepsilon \in (0, \frac{1}{2})$. Choose a positive integer k such that $kp \geq 2$ and let

$$S_n(z) = \left(\sum_{|j| \leq [\log n]} f(j) (-1)^j \ell_{j, n - [\varepsilon n]}(z) \right) U_{[\frac{\varepsilon}{k}n]}(z)^k,$$

a polynomial of degree $\leq n - 1$. Fix $r > 0$, $s > 0$. As $|U_{[\frac{\varepsilon}{k}n]}(z)| \leq 1$ for $|z| \leq 1$, we have from Lemma 2.1 and (2.11),

$$\begin{aligned} & \frac{n}{2\pi} \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^s d\theta \\ &= \int_{-r}^r \left| \left(\sum_{|j| \leq [\log n]} f(j) (-1)^j \ell_{j, n - [\varepsilon n]}(e^{2\pi i t/n}) \right) U_{[\frac{\varepsilon}{k}n]}^k(e^{2\pi i t/n}) \right|^s dt \\ &= \int_{-r}^r \left| \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) + o\left(\sum_{|j| \leq [\log n]} |f(j)| \right) \right|^s \left(\left| S\left(\frac{\varepsilon}{k}t\right) \right| + o(1) \right)^{ks} dt \\ &= \int_{-r}^r \left| \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) \right|^s \left| S\left(\frac{\varepsilon}{k}t\right) \right|^{ks} dt + o(1). \end{aligned}$$

Here we are using the fact that

$$D = \sum_{j=-\infty}^{\infty} |f(j)| \leq \|f\|_{L^\infty(\mathbb{R})}^{1-p} \sum_{j=-\infty}^{\infty} |f(j)|^p < \infty,$$

recall (2.4), and that each $f \in L^p_\pi$ is bounded on the real line. Next, uniformly for $t \in [-r, r]$,

$$\begin{aligned} & \left| f(t(1-\varepsilon)) - \sum_{|j| \leq [\log n]} f(j) S(t(1-\varepsilon) - j) \right| \\ & \leq \sum_{|j| > [\log n]} |f(j)| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{aligned}$$

It follows that

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{n}{2\pi} \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^s d\theta \\ (2.12) \quad &= \int_{-r}^r |f(t(1-\varepsilon))|^s \left| S\left(\frac{\varepsilon}{k}t\right) \right|^{ks} dt. \end{aligned}$$

Next, for all $|z| \leq 1$, $|\ell_{jn}(z)| \leq 1$, so with $z = e^{i\theta}$, $\theta \in [-\pi, \pi]$,

$$\begin{aligned} |S_n(z)| &\leq D \left| U_{[\frac{\varepsilon}{k}n]}(z) \right|^k \\ &\leq D \left(\frac{2}{\left[\frac{\varepsilon}{k}n\right] |1-z|} \right)^k \leq D \left(\frac{\pi}{\left[\frac{\varepsilon}{k}n\right] |\theta|} \right)^k. \end{aligned}$$

Hence,

$$\begin{aligned}
 & \frac{n}{2\pi} \int_{2\pi r/n \leq |\theta| \leq \pi} |S_n(z)|^p d\theta \\
 & \leq CD^p n \int_{2\pi r/n \leq |\theta| \leq \pi} \left(\frac{1}{\lfloor \frac{\varepsilon}{k} n \rfloor |\theta|} \right)^{kp} d\theta \\
 & \leq CD^p \int_{|t| \geq 2\pi r} |t|^{-kp} dt \leq CD^p r^{-kp+1}.
 \end{aligned}$$

Here C is independent of r, n , but depends on ε, k . Combining this with (2.12) gives

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \\
 & \leq \limsup_{n \rightarrow \infty} \frac{\left(\frac{n}{2\pi} \int_{-\pi}^{\pi} |S_n(z)|^p d\theta \right)^{1/p}}{\left(\frac{n}{2\pi} \int_{-\pi}^{\pi} |S_n(z)|^q d\theta \right)^{1/q}} \\
 & \leq \limsup_{n \rightarrow \infty} \frac{\left(\frac{n}{2\pi} \int_{-\pi}^{\pi} |S_n(z)|^p d\theta \right)^{1/p}}{\left(\frac{n}{2\pi} \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^q d\theta \right)^{1/q}} \\
 & \leq \frac{\left(\int_{-r}^r |f(t(1-\varepsilon))|^p |S(\frac{\varepsilon}{k}t)|^{kp} dt + CD^p r^{-kp+1} \right)^{1/p}}{\left(\int_{-r}^r |f(t(1-\varepsilon))|^q |S(\frac{\varepsilon}{k}t)|^{kq} dt \right)^{1/q}}.
 \end{aligned}$$

Since the left-hand side is independent of r , we can let $r \rightarrow \infty$ to obtain

$$\begin{aligned}
 \limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} & \leq \frac{\left(\int_{-\infty}^{\infty} |f(t(1-\varepsilon))|^p |S(\frac{\varepsilon}{k}t)|^{kp} dt \right)^{1/p}}{\left(\int_{-\infty}^{\infty} |f(t(1-\varepsilon))|^q |S(\frac{\varepsilon}{k}t)|^{kq} dt \right)^{1/q}} \\
 & = \left(\frac{1}{1-\varepsilon} \right)^{1/p - 1/q} \frac{\left(\int_{-\infty}^{\infty} |f(t)|^p |S(\frac{\varepsilon}{k(1-\varepsilon)}t)|^{kp} dt \right)^{1/p}}{\left(\int_{-\infty}^{\infty} |f(t)|^q |S(\frac{\varepsilon}{k(1-\varepsilon)}t)|^{kq} dt \right)^{1/q}}.
 \end{aligned}$$

Now we can let $\varepsilon \rightarrow 0+$, and use dominated convergence, noting that $|S(t)| \leq 1$ for all t and $S(0) = 1$. We obtain

$$\limsup_{n \rightarrow \infty} n^{\frac{1}{p} - \frac{1}{q}} \Lambda_{n,p,q} \leq \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_q(\mathbb{R})}}.$$

and taking the inf's over all f gives the result. ■

3. REMARKS ON PROVING THE CONJECTURE

One needs to prove

$$\liminf_{n \rightarrow \infty} \Lambda_{n,p,q} n^{\frac{1}{p} - \frac{1}{q}} \geq \mathcal{E}_{p,q}.$$

This was achieved for $q = \infty$ in [4], but is much easier in that case. The reason is that in considering

$$\frac{\|P\|_p}{\|P\|_\infty},$$

one can assume $\|P\|_\infty = P(0) = 1$, and then only has to deal with integrals in the numerator. For $q < \infty$, one has to consider the fact that integrals over several different subarcs may make substantial contributions to $\|P\|_q$. It is very likely that in an extremal polynomial P attaining the inf $\Lambda_{n,p,q}$, the polynomial is "concentrated" around the point, where its maximum modulus on the circle is attained. That is, the absolute value of the polynomial decays away from this point, which can be assumed to be 1. Once one has a suitable form of concentration, one can estimate tail integrals much as in Section 2. Unfortunately, all our attempts to prove this "concentration" or "decay" failed.

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