Orthogonal Polynomials for Exponential Weights $x^{2\rho}e^{-2Q(x)}$ on $[0,d)$, II

Eli Levin

Mathematics Department, The Open University of Israel, P.O. Box 808, Raanana 43107, Israel

Doron Lubinsky

The School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 USA

Abstract

Let $I = [0,d)$, where d is finite or infinite. Let $W_\rho(x) = x^\rho \exp(-Q(x))$, where $\rho > -\frac{1}{2}$ and Q is continuous and increasing on I, with limit ∞ at d. We obtain further bounds on the orthonormal polynomials associated with the weight W_ρ^2, finer spacing on their zeros, and estimates of their associated fundamental polynomials of Lagrange interpolation. In addition, we obtain weighted Markov and Bernstein inequalities.

In Honor of Barry Simon's 60th Birthday

1 Introduction and Results [1]

Let

$$I = [0,d), \tag{1.1}$$

where $0 < d \leq \infty$. Let $Q : I \to [0,\infty)$ be continuous, and

$$W = \exp(-Q) \tag{1.2}$$

be such that all moments $\int_I x^n W(x)\,dx, n \geq 0$, exist. We call W an exponential weight on I. For $\rho > -\frac{1}{2}$, we set

$$W_\rho(x) := x^\rho W(x), \qquad x \in I.$$

Email address: lubinsky@math.gatech.edu (Doron Lubinsky).

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The orthonormal polynomial of degree n for W^2 is denoted by $p_n(W^2, x)$ or just $p_n(x)$. That for W_ρ^2 is denoted by $p_n\left(W_\rho^2, x\right)$ or just $p_{n,\rho}(x)$. Thus

$$\int_I p_{n,\rho}(x)\, p_{m,\rho}(x)\, x^{2\rho} W^2(x)\, dx = \delta_{mn}$$

and

$$p_{n,\rho}(x) = \gamma_{n,\rho} x^n + \cdots,$$

where $\gamma_{n,\rho} = \gamma_n\left(W_\rho^2\right) > 0$.

In the predecessor to this paper [3], we established bounds for $p_{n,\rho}$, estimates of the associated Christoffel functions, spacing of the zeros of the orthonormal polynomials, and restricted range inequalities. In this paper, we shall establish further bounds on the orthonormal polynomials, more precise spacing of their zeros, estimates for their fundamental polynomials, and Markov-Bernstein inequalities. We denote the zeros of $p_{n,\rho}$ by

$$x_{nn} < x_{n-1,n} < \cdots < x_{2n} < x_{1n}.$$

As in [3], we use results from [2] by defining an even weight W^* corresponding to the one-sided weight W. Given I and W as in (1.1) and (1.2), let

$$I^* := \left(-\sqrt{d}, \sqrt{d}\right)$$

and for $x \in I^*$,

$$Q^*(x) := Q\left(x^2\right);$$
$$W^*(x) := \exp\left(-Q^*(x)\right).$$

We say that $f : I \to (0, \infty)$ is *quasi-increasing* if there exists $C > 0$ such that

$$f(x) \le C f(y), \qquad 0 < x < y < d.$$

Of course, any increasing function is quasi-increasing. The notation

$$f(x) \sim g(x)$$

means that there are positive constants C_1, C_2 such that for the relevant range of x,

$$C_1 \le f(x)/g(x) \le C_2.$$

Similar notation is used for sequences and sequences of functions.

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda)$, $C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. We denote the polynomials of degree $\leq n$ by $\mathcal{P}_n$.

Following is our class of weights:

Definition 1.1 Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$satisfies the following properties:
(a) $\sqrt{x}Q'(x)$ *is continuous in* I, *with limit* 0 *at* 0 *and* $Q(0) = 0$;
(b) Q'' exists in $(0, d)$, while $Q^{*\prime\prime}$ is positive in $\left(0, \sqrt{d}\right)$;
(c)

$$\lim_{x \to d-} Q(x) = \infty.$$

(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \qquad x \in (0, d) \tag{1.3}$$

is quasi-increasing in $(0, d)$, with

$$T(x) \geq \Lambda > \frac{1}{2}, \qquad x \in (0, d). \tag{1.4}$$

(e) There exists $C_1 > 0$ such that

$$\frac{|Q''(x)|}{Q'(x)} \leq C_1 \frac{Q'(x)}{Q(x)}, \qquad \text{a.e. } x \in (0, d). \tag{1.5}$$

Then we write $W \in \mathcal{L}(C^2)$. If also there exists a compact subinterval J of I^*, and $C_2 > 0$ such that

$$\frac{Q^{*\prime\prime}(x)}{|Q^{*\prime}(x)|} \geq C_2 \frac{|Q^{*\prime}(x)|}{Q^*(x)}, \qquad \text{a.e. } x \in I^* \backslash J, \tag{1.6}$$

then we write $W \in \mathcal{L}(C^2+)$.

Remarks See [3] for further orientation on this class of weights and this topic. Here are some examples of Q that satisfy the above conditions:
(I)

$$Q(x) = x^\alpha, \qquad x \in [0, \infty)$$

where $\alpha > \frac{1}{2}$.
(II)

$$Q(x) = \exp_k(x^\alpha) - \exp_k(0), \qquad x \in [0, \infty)$$

where $\alpha > \frac{1}{2}$ and $k \geq 0$. Here we set

$$\exp_0(x) := x$$

and for $k \geq 1$,

$$\exp_k(x) = \underbrace{\exp(\exp(\exp \cdots \exp(x)) \cdots)}_{k \text{ times}}$$

is the kth iterated exponential.
(III) An example on the finite interval $I=[0,1)$ is

$$Q(x)=\exp_k((1-x)^{-\alpha})-\exp_k(1),\qquad x\in[0,1),$$

where $\alpha>0$ and $k\geq 0$.

One of the important descriptive quantities we need is the Mhaskar-Rakhmanov-Saff number a_t, [2], [4], [5] defined for $t>0$ as the positive root of the equation

$$t=\frac{1}{\pi}\int_0^1\frac{a_tuQ'\left(a_tu\right)}{\sqrt{u\left(1-u\right)}}du. \tag{1.7}$$

One of our main results is:

Theorem 1.2 *Let $\rho>-\frac{1}{2}$, $0<\beta<1$, and let $W\in\mathcal{L}\left(C^2+\right)$. Let $p_{n,\rho}\left(x\right)$ be the nth orthonormal polynomial for the weight W_ρ^2. Then uniformly for $n\geq 1$,*

$$\sup_{x\in I}|p_{n,\rho}(x)|W(x)\left(x+\frac{a_n}{n^2}\right)^\rho\sim\left(\frac{n}{a_n}\right)^{1/2} \tag{1.8}$$

and

$$\sup_{x\in[a_{\beta n},d)}|p_{n,\rho}(x)|W(x)\left(x+\frac{a_n}{n^2}\right)^\rho\sim a_n^{-1/2}\left(nT\left(a_n\right)\right)^{1/6}. \tag{1.9}$$

If $W\in\mathcal{L}\left(C^2\right)$, these estimates hold with $\sim$ replaced by $\leq C$.

Remark In [3], we proved the estimate

$$\sup_{x\in I}|p_{n,\rho}(x)|W(x)\left(x+\frac{a_n}{n^2}\right)^\rho|\left(x+a_nn^{-2}\right)\left(a_n-x\right)|^{1/4}\sim 1, \tag{1.10}$$

assuming that $W\in\mathcal{L}\left(C^2\right)$.

Next, we turn to pointwise bounds on orthogonal polynomials and their derivatives. Let

$$\eta_t=\left(tT\left(a_t\right)\right)^{-2/3},\qquad t>0, \tag{1.11}$$

and

$$\varphi_t(x):=\begin{cases}\frac{\sqrt{x+a_tt^{-2}}(a_{2t}-x)}{t\sqrt{a_t-x+a_t\eta_t}}, & x\in[0,a_t];\\ \varphi_t(a_t), & x>a_t;\\ \varphi_t\left(0\right), & x<0.\end{cases} \tag{1.12}$$

Recall that the fundamental polynomials at the zeros of $p_{n,\rho}$ are polynomials $\ell_{jn}\in\mathcal{P}_{n-1}$ satisfying

$$\ell_{jn}(x_{kn})=\delta_{kj}.$$

Theorem 1.3 *Let $W\in\mathcal{L}\left(C^2+\right)$ and $\rho>-\frac{1}{2}$. There exists n_0 such that uniformly for $n\geq n_0, 1\leq j\leq n$,*

(a)

$$|p'_{n,\rho}W_\rho|(x_{jn}) \sim \varphi_n(x_{jn})^{-1}\left[x_{jn}\left(a_n - x_{jn}\right)\right]^{-1/4}; \tag{1.13}$$

(b)

$$|p_{n-1,\rho}W_\rho|(x_{jn}) \sim a_n^{-1}\left[x_{jn}\left(a_n - x_{jn}\right)\right]^{1/4}; \tag{1.14}$$

(c)

$$\max_{x\in I}\left|\ell_{jn}(x)W(x)\left(x+\frac{a_n}{n^2}\right)^\rho\right|W_\rho^{-1}(x_{jn}) \sim 1; \tag{1.15}$$

(d) For $j \le n-1$ and $x \in [x_{j+1,n}, x_{jn}]$,

$$\begin{aligned}|p_{n,\rho}W_\rho|(x) &\sim \min\{|x-x_{jn}|, |x-x_{j+1,n}|\}\\ &\quad\times \varphi_n(x_{jn})^{-1}\left[x_{jn}\left(a_n - x_{jn}\right)\right]^{-1/4}.\end{aligned} \tag{1.16}$$

If we assume instead that $W \in \mathcal{L}(C^2)$, then (a) holds with $\sim$ replaced by $\le C$ and (b) holds with $\sim$ replaced by $\ge C$.

Concerning the spacing of the zeros, we prove

Theorem 1.4 *Let $W \in \mathcal{L}(C^2+)$ and $\rho > -\frac{1}{2}$. Uniformly for $n \ge 1$ and $1 \le j < n$,*

$$x_{jn} - x_{j+1,n} \sim \varphi_n(x_{jn}). \tag{1.17}$$

In [3], we proved the upper bound implicit in (1.17), assuming that $W \in \mathcal{L}(C^2)$.

Finally, we turn to Markov-Bernstein Inequalities. For these, we need a modification of φ_t, namely

$$\varphi_t^{\#}(x) = \sqrt{\frac{x}{x+a_t t^{-2}}}\varphi_t(x) = \frac{\sqrt{x}\left(a_{2t}-x\right)}{t\sqrt{a_t - x + a_t\eta_t}}, \qquad x \in [0, a_t], \tag{1.18}$$

and

$$\varphi_t^{\#}(x) = \varphi_t^{\#}(a_t), \qquad x > a_t. \tag{1.19}$$

Theorem 1.5 (Bernstein Inequality) *Let $W \in \mathcal{L}(C^2)$. Let $0 < p \le \infty$ and let $\beta > -\frac{1}{p}$ if $p < \infty$and $\beta \ge 0$ if $p = \infty$. Then for $n \ge 1$ and $P \in \mathcal{P}_n$, and for some $C \ne C(n,P)$,*

$$\|(PW)'(x)\,\varphi_n^{\#}(x)\,x^\beta\|_{L_p(I)} \le C\|(PW)(x)\,x^\beta\|_{L_p(I)}. \tag{1.20}$$

Theorem 1.6 (Markov Inequality) *Let $W \in \mathcal{L}(C^2)$. Let $0 < p \le \infty$, $0 < \gamma < 1$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \ge 0$ if $p = \infty$. Then for $n \ge 1$ and $P \in \mathcal{P}_n$,*

$$\|(P'W)(x)\,x^\beta\|_{L_p(I)} \le C\frac{n^2}{a_n}\|(PW)(x)\,x^\beta\|_{L_p(I)} \tag{1.21}$$

and

$$\| (P'W)(x)\, x^{\beta} \|_{L_p[a_{\gamma n}, d)} \leq C \frac{n}{a_n} \sqrt{T(a_n)} \| (PW)(x)\, x^{\beta} \|_{L_p(I)}. \tag{1.22}$$

Since $T(a_n) << n^2$, we see from the last two inequalities, the special role played by 0: the rate of growth of $P'W$ can be far larger near 0 than near a_n. We shall show that (1.21) is sharp as regards the rate of growth in n, at least in L_2 and for $\beta = 0$. More precisely, in Section 7, we show that

$$\|p_n'' W\|_{L_2(I)} \sim \frac{n^2}{a_n} \|p_n' W\|_{L_2(I)}, \qquad n \geq 1. \tag{1.23}$$

We note that all the above results are valid under weaker conditions on W. All we need is that W^* satisfies the conditions for the corresponding result in [2]. However, for simplicity, we use just one class of weights in this paper. We note too that for the case where Q is of polynomial growth on $I = [0, \infty)$, Theorems 1.3 and 1.4 follow from results of Kasuga and Sakai [1].

This paper is organised as follows. In the next section, we list technical estimates. In Section 3, we prove the Markov-Bernstein inequalities of Theorems 1.5 and 1.6. In Section 4 we estimate a certain function $A^{\#}_{n,\rho}(x)$. In Section 5, we prove Theorem 1.2 and Theorem 1.3(a), (b). In Section 6, we prove Theorem 1.3(c), (d) and Theorem 1.4. Finally in Section 7, we prove (1.23).

Finally, we illustrate some of the results above on specific weights. Throughout p, ρ, β are as in Theorem 1.6.

Example 1 Let $I = [0, \infty)$, $\alpha > \frac{1}{2}$ and

$$Q(x) = x^{\alpha}, \qquad x \in [0, \infty).$$

In this special case

$$a_t = \left(\sqrt{\pi} \frac{\Gamma(\alpha)}{\Gamma\left(\alpha + \frac{1}{2}\right)} \right)^{1/\alpha} t^{1/\alpha}$$

and

$$\eta_t = (\alpha t)^{-2/3}, \qquad t > 0.$$

(I) The Markov inequality takes the following form: for $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\| (P'W)(x)\, x^{\beta} \|_{L_p(0,\infty)} \leq C n^{2 - \frac{1}{\alpha}} \| (PW)(x)\, x^{\beta} \|_{L_p(0,\infty)}.$$

Moreover, given $0 < \gamma < 1$,

$$\| (P'W)(x)\, x^{\beta} \|_{L_p(a_{\gamma n}, \infty)} \leq C n^{1 - \frac{1}{\alpha}} \| (PW)(x)\, x^{\beta} \|_{L_p(0,\infty)}.$$

(II) The sup norm bound on the orthonormal polynomials takes the form

$$\|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\|_{L_\infty(0,\infty)}\sim n^{\frac{1}{2}\left[1-\frac{1}{\alpha}\right]}.$$

Moreover,

$$\|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\|_{L_\infty(a_{\gamma n},\infty)}\sim n^{-\frac{1}{2\alpha}+\frac{1}{6}}.$$

Example 2 Let $I=[0,\infty)$, $k\geq 1$ and $\alpha>\frac{1}{2}$. Let

$$Q(x)=\exp_k(x^\alpha)-\exp_k(0),\qquad x\in[0,\infty).$$

We also need the jth iterated logarithm: let $\log_0(x):=x$ and for $j\geq 1$,

$$\log_j(x)=\underbrace{\log(\log(\log\cdots\log(x)))}_{j\text{ times}},\qquad x>\exp_{j-1}(0).$$

Here as $n\to\infty$,

$$\begin{aligned}a_n&=(\log_k n)^{1/\alpha}(1+o(1));\\ T(a_n)&\sim\prod_{j=1}^{k}\log_j n;\\ \eta_n&\sim\left(n\prod_{j=1}^{k}\log_j n\right)^{-2/3}.\end{aligned}$$

(I) The Markov inequality takes the following form: for $n\geq\exp_k(1)$ and $P\in\mathcal{P}_n$,

$$\|(P'W)(x)\,x^\beta\|_{L_p(0,\infty)}\leq C\frac{n^2}{(\log_k n)^{1/\alpha}}\|(PW)(x)\,x^\beta\|_{L_p(0,\infty)}.$$

Moreover, for $n\geq\exp_k(1)$,

$$\|(P'W)(x)\,x^\beta\|_{L_p(a_{\gamma n},\infty)}\leq C\frac{n}{(\log_k n)^{1/\alpha}}\left(\prod_{j=1}^{k}\log_j n\right)^{1/2}\|(PW)(x)\,x^\beta\|_{L_p(0,\infty)}.$$

(II) The sup norm bound on the orthonormal polynomials takes the form

$$\|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\|_{L_\infty(0,\infty)}\sim n^{1/2}(\log_k n)^{-1/(2\alpha)}.$$

Moreover, for $n\geq\exp_k(1)$,

$$\|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\|_{L_\infty(a_{\gamma n},\infty)}\sim(\log_k n)^{-1/(2\alpha)}\left(n\prod_{j=1}^{k}\log_j n\right)^{1/6}.$$

Example 3 Let $I = [0, 1)$, $\alpha > 0$, and

$$Q(x) = (1 - x)^{-\alpha} - 1, \qquad x \in [0, 1).$$

Here

$$\begin{aligned} 1 - a_n &\sim n^{-\left(\frac{1}{\alpha + \frac{1}{2}}\right)}; \\ T(a_n) &\sim n^{\frac{1}{\alpha + \frac{1}{2}}}; \\ \eta_n &\sim n^{-\frac{2}{3}\left(\frac{2\alpha+3}{2\alpha+1}\right)}. \end{aligned}$$

(I) The Markov inequality takes the following form: for $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\| (P'W)(x) x^{\beta} \|_{L_p[0,1]} \leq C n^2 \| (PW)(x) x^{\beta} \|_{L_p[0,1]}.$$

Moreover, given $0 < \gamma < 1$,

$$\| (P'W)(x) x^{\beta} \|_{L_p[a_{\gamma n},1]} \leq C n^{\frac{2\alpha+2}{2\alpha+1}} \| (PW)(x) x^{\beta} \|_{L_p[0,1]}.$$

(II) The sup norm bound on the orthonormal polynomials takes the form

$$\| p_{n,\rho}(x) W(x) \left(x + \frac{a_n}{n^2}\right)^{\rho} \|_{L_\infty[0,1]} \sim n^{\frac{1}{2}}.$$

Moreover,

$$\| p_{n,\rho}(x) W(x) \left(x + \frac{a_n}{n^2}\right)^{\rho} \|_{L_\infty[a_{\gamma n},1]} \sim n^{\frac{2\alpha+3}{2\alpha+1}\frac{1}{6}}.$$

Example 4 Let $I = [0, 1)$, $k \geq 1$ and $\alpha > 0$. Let

$$Q(x) = \exp_k((1 - x)^{-\alpha}) - \exp_k(1), \qquad x \in [0, 1).$$

Here as $n \to \infty$,

$$\begin{aligned} 1 - a_n &= (\log_k n)^{-1/\alpha} (1 + o(1)); \\ T(a_n) &\sim (\log_k n)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n; \\ \eta_n &\sim \left(n (\log_k n)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{-2/3}. \end{aligned}$$

(I) The Markov inequality takes the following form: assume that p, β are as above. Then for $n \geq 1$ and $P \in \mathcal{P}_n$,

$$\| (P'W)(x) x^{\beta} \|_{L_p[0,1]} \leq C n^2 \| (PW)(x) x^{\beta} \|_{L_p[0,1]}.$$

Moreover,

$$\| (P'W)(x)\, x^{\beta} \|_{L_p[a_{\gamma n},1]} \leq Cn \left((\log_k n)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{1/2} \| (PW)(x)\, x^{\beta} \|_{L_p[0,1]}.$$

(II) The sup norm bound on the orthonormal polynomials takes the form

$$\|p_{n,\rho}(x)\, W(x) \left(x + \frac{a_n}{n^2} \right)^{\rho} \|_{L_\infty[0,1]} \sim n^{1/2}.$$

Moreover,

$$\|p_{n,\rho}(x)\, W(x) \left(x + \frac{a_n}{n^2} \right)^{\rho} \|_{L_\infty[a_{\gamma n},1]} \sim \left(n\, (\log_k n)^{1+1/\alpha} \prod_{j=1}^{k-1} \log_j n \right)^{1/6}.$$

2 Technical Estimates

The classes $\mathcal{L}(C^2)$ and $\mathcal{L}(C^2+)$ are defined in such a way that W^* becomes part of the corresponding classes $\mathcal{F}(C^2)$ and $\mathcal{F}(C^2+)$ in [2, p. 7]. In [3, Lemma 2.2] we proved that

$$W \in \mathcal{L}\left(C^2\right) \Leftrightarrow W^* \in \mathcal{F}\left(C^2\right)$$

and

$$W \in \mathcal{L}\left(C^2+\right) \Leftrightarrow W^* \in \mathcal{F}\left(C^2+\right).$$

Thus we can apply results from [2] to W^*. We denote the (positive) Mhaskar-Rakhmanov-Saff number for W^* by a_t^*. In [3, eqn. (2.6)] we showed that

$$a_{t/2} = a_t^{*2}. \tag{2.1}$$

We shall also use the quantity $\eta_t = (tT(a_t))^{-2/3}$, and its analogue for Q^*

$$\eta_t^* = \{tT^*(a_t^*)\}^{-2/3},$$

where

$$T^*(x) = x\frac{Q^{*\prime}(x)}{Q^*(x)} = 2T\left(x^2\right).$$

We note that [3, (2.9)]

$$\eta_{2t}^* = 4^{-2/3}\eta_t. \tag{2.2}$$

Lemma 2.1 *Let $W \in \mathcal{L}(C^2)$.*
(a) Uniformly for $t > 0$, we have

$$Q'(a_t) \sim \frac{t}{a_t}\sqrt{T(a_t)}. \tag{2.3}$$

(b) For fixed $L > 1$ and uniformly for $t > 0$,

$$a_{Lt} \sim a_t. \tag{2.4}$$

(c) Fix $L > 0$. Then uniformly for $t > 0$,

$$Q'(a_{Lt}) \sim Q'(a_t); \quad T(a_{Lt}) \sim T(a_t) \quad \text{and} \quad \eta_{Lt} \sim \eta_t. \tag{2.5}$$

(d) For some $\varepsilon > 0$, and for large enough t,

$$T(a_t) \leq Ct^{2-\varepsilon} \tag{2.6}$$

and

$$\eta_t T(a_t) \leq Ct^{-\varepsilon} = o(1). \tag{2.7}$$

Proof. See Lemma 3.1 in [3]. □

Some further estimates involving a_t:

Lemma 2.2 *Let $W \in \mathcal{L}(C^2)$.*
(a) We have for $t > 0$,

$$|1 - \frac{a_t}{a_s}| \sim \frac{1}{T(a_t)}|1 - \frac{t}{s}|, \qquad \frac{1}{2} \leq \frac{s}{t} \leq 2\,. \tag{2.8}$$

(b) Given fixed $L > 1$, we have uniformly for $t > 0$,

$$|1 - \frac{a_{Lt}}{a_t}| \sim \frac{1}{T(a_t)}. \tag{2.9}$$

(c) For $x \in [0, a_t)$,

$$Q'(x) \leq \frac{Ct}{\sqrt{x\,(a_t - x)}}. \tag{2.10}$$

(d) Assume also $W \in \mathcal{L}(C^2+)$ and let $L > 1$. There exist C and t_0 such that for $t \geq t_0$,

$$\frac{a_{Lt}Q'(a_{Lt})}{a_t Q'(a_t)} \geq 1 + C. \tag{2.11}$$

Proof.
(a), (b) See Lemma 3.2 in [3].

(c) See Lemma 3.3 in [3].
(d) Note that (cf. (2.1))

$$\frac{\sqrt{a_{Lt}}Q'(a_{Lt})}{\sqrt{a_t}Q'(a_t)} = \frac{Q^{*\prime}(a^*_{2Lt})}{Q^{*\prime}(a^*_{2t})} \geq 1 + C$$

by Proposition 13.1 in [3, pp. 359–360]. Since $a_{Lt} \geq a_t$, we then obtain (2.11). □

Next, a lemma on the functions φ_m and $\varphi^{\#}_m$. We shall also sometimes need the corresponding function for W^*, which we denote by φ^*_m. This is defined in $[-a^*_m, a^*_m]$ [2, p. 19] by

$$\varphi^*_m(x) = \frac{|x^2 - a^{2*}_{2m}|}{m\sqrt{|x + a^*_m| + a^*_m \eta^*_m}\sqrt{|x - a^*_m| + a^*_m \eta^*_m}} \tag{2.12}$$

and to be constant in $(-\infty, -a^*_m]$ and $[a^*_m, \infty)$.

Lemma 2.3 *Let* $W \in \mathcal{L}(C^2)$.
(a) For $x \in [0, a_m]$,

$$\varphi^*_{2m}\left(\sqrt{x}\right) \sim \frac{\varphi_m(x)}{\sqrt{x + a_m m^{-2}}} = \frac{\varphi^{\#}_m(x)}{\sqrt{x}}. \tag{2.13}$$

(b) Let $C > 0$. *Uniformly for* m *and* n *such that*

$$\left|1 - \frac{m}{n}\right| \leq CT(a_n)\eta_n, \tag{2.14}$$

we have uniformly for $x \in I$

$$\varphi_n(x) \sim \varphi_m(x). \tag{2.15}$$

Moreover, uniformly for $x \in I^*$,

$$\varphi^*_n(x) \sim \varphi^*_m(x). \tag{2.16}$$

(c) For $n \geq 1$ *and* $x \in [a_n n^{-2}, d)$,

$$\varphi^{\#}_n(x) \leq Cx. \tag{2.17}$$

(d) Let $L > 0$, $0 < \beta < 1$. *Then uniformly for* $n \geq 1$ *and* $x \in [a_n/n^2, a_{\beta n}]$,

$$\varphi_n(x)\left[x\left(a_n\left[1 + L\eta_n\right] - x\right)\right]^{1/2} \sim \frac{x(a_n - x)}{n}. \tag{2.18}$$

(e) Let $\varepsilon, L > 0$. *Then uniformly for* $n \geq 1$ *and* $x \in [\varepsilon a_n, a_n(1 + L\eta_n)]$,

$$\varphi_n(x)\left[x\left(a_n\left[1 + L\eta_n\right] - x\right)\right]^{1/2} \sim \frac{a_n^2}{nT(a_n)}. \tag{2.19}$$

(f) Uniformly for $n \geq 1$, $1 \leq j \leq n-1$, and $x \in [x_{j+1,n}, x_{jn}]$,

$$\varphi_n(x_{jn}) \sim \varphi_n(x). \tag{2.20}$$

Proof.
(a) Since $a_{2m}^{*2} = a_m$, we see that in $[0, a_m]$,

$$\begin{aligned}\varphi_{2m}^*\left(\sqrt{x}\right) &= \frac{|x - a_{2m}|}{m\sqrt{\left|\sqrt{x}+\sqrt{a_m}\right| + \sqrt{a_m}\eta_{2m}^*}\sqrt{\left|\sqrt{x}-\sqrt{a_m}\right| + \sqrt{a_m}\eta_{2m}^*}} \\ &\sim \frac{a_{2m} - x}{m\sqrt{a_m - x + a_m\eta_m}} \sim \frac{\varphi_m(x)}{\sqrt{x + a_m m^{-2}}} = \frac{\varphi_m^{\#}(x)}{\sqrt{x}},\end{aligned}$$

by (1.12) and (1.18).
(b) Firstly Lemma 9.7 in [2, p. 264] gives (2.16). Using (a), we obtain in $[0, a_n]$,

$$\frac{\varphi_m(x)}{\sqrt{x + a_m m^{-2}}} \sim \frac{\varphi_n(x)}{\sqrt{x + a_n n^{-2}}}.$$

Here (2.14) and (2.7) show that $1 - \frac{m}{n} \to 0, n \to \infty$. Hence $m \sim n$, so $a_m \sim a_n$. Then (2.15) follows in $[0, a_n]$. Since we may assume $m \geq n$ and φ_n and φ_m are constant outside $[0, a_n]$ and $[0, a_m]$ respectively, we obtain (2.15) in I.
(c) Now in $[0, a_n]$, we see from (1.18) and then (2.9), (2.7) that

$$\begin{aligned}\frac{\varphi_n^{\#}(x)}{x} &\sim \frac{a_{2n} - x}{n\sqrt{x}\sqrt{a_n - x + a_n\eta_n}} \\ &\sim \begin{cases}\frac{\sqrt{a_n - x}}{n\sqrt{x}}, & x \in \left[0, a_{n/2}\right], \\ \frac{\sqrt{a_n}}{nT(a_n)\sqrt{a_n - x + a_n\eta_n}}, & x \in \left[a_{n/2}, a_n\right]\end{cases} \\ &\leq C\begin{cases}\frac{\sqrt{a_n}}{n\sqrt{x}}, & x \in \left[0, a_{n/2}\right], \\ \frac{1}{nT(a_n)\sqrt{\eta_n}}, & x \in \left[a_{n/2}, a_n\right]\end{cases} \\ &\leq C, x \in \left[a_n n^{-2}, a_n\right],\end{aligned} \tag{2.21}$$

recall that $\eta_n = (nT(a_n))^{-2/3} = o(1)$. Since $\varphi_n^{\#}(x) = \varphi_n^{\#}(a_n)$, $x \geq a_n$, this inequality persists in $[a_n, d)$.
(d) For this range of x,

$$|x - a_{2n}| \sim |x - a_n| \quad \text{and} \quad a_n - x + a_n\eta_n \sim a_n - x,$$

while $x + a_n/n^2 \sim x$, so (2.18) follows easily from (1.12).
(e) For this range of x,

$$|x - a_{2n}| \sim \frac{a_n}{T(a_n)} \quad \text{and} \quad x \sim a_n$$

and so at least in $[0, a_n]$,

$$\varphi_n(x) \sim \frac{a_n^{3/2}}{nT(a_n)} \frac{1}{\sqrt{a_n - x + a_n n^{-2}}}$$

whence

$$\varphi_n(x) \left[x \left(a_n \left[1 + 2L\eta_n\right] - x\right)\right]^{1/2} \sim \frac{a_n^2}{nT(a_n)}.$$

This persists in $[a_n, a_n(1 + L\eta_n)]$, since φ_n is constant there.
(f) This follows from (7.14) (and its preceding lines) in [3] and Lemma 4.3 there. □

Next, we restate some restricted range inequalities from [3]. For $t \geq 0$, we denote by $\mathbb{P}_t$ the set of all functions of the form

$$P(z) = c \exp\left(\int \log|z - \xi| d\nu(\xi)\right),$$

where $\nu \geq 0$, $\nu(\mathbb{C}) \leq t$, $c \geq 0$, and the support of ν is compact. These are the exponentials of potentials of mass $\leq t$. In particular if $t \geq n$, then $P \in \mathcal{P}_n \Rightarrow |P| \in \mathbb{P}_t$.

Lemma 2.4 *Let $W \in \mathcal{L}(C^2)$. Let $0 < p \leq \infty$, $\sigma \in \mathbb{R}$, and $L, \lambda \geq 0$. Let $\beta > -\frac{1}{p}$ if $p < \infty$ and $\beta \geq 0$ if $p = \infty$.*
(a) There exist C_1, t_0 such that for $t \geq t_0$ and $P \in \mathbb{P}_t$,

$$\|(PW)(x)\,x^{\beta}\|_{L_p(I)} \leq C_1 \|(PW)(x)\,x^{\beta}\|_{L_p[La_t t^{-2}, a_t(1-\lambda\eta_t)]}. \tag{2.22}$$

(b) Given $r > 1$, we have for some C, t_0, $\alpha > 0$and $t \geq t_0$, and $P \in \mathbb{P}_t$,

$$\begin{aligned}&\|(PW)(x)\,x^{\beta}\|_{L_p(a_{rt}, d)} \\ &\quad\leq \exp(-Ct^{\alpha})\,\|(PW)(x)\,x^{\beta}\|_{L_p(0, a_t)}.\end{aligned} \tag{2.23}$$

(c) There exist C_1, $t_0 > 0$ such that for $t \geq t_0$ and $P \in \mathbb{P}_t$,

$$\|(PW)(x)\left(x + \frac{a_t}{t^2}\right)^{\sigma}\|_{L_p(I)} \leq C_1 \|(PW)(x)\left(x + \frac{a_t}{t^2}\right)^{\sigma}\|_{L_p[La_t t^{-2}, a_t(1-\lambda\eta_t)]}. \tag{2.24}$$

Proof.
(a), (b) See Theorem 5.2 in [3].
(c) In Lemma 8.7 in [3], we proved that

$$\begin{aligned}&\|(PW)(x)\left(x + \frac{a_t}{t^2}\right)^{\sigma}\|_{L_p(I)} \\ &\quad\leq C_1 \|(PW)(x)\left(x + \frac{a_t}{t^2}\right)^{\sigma}\|_{L_p[La_t t^{-2}, a_{2t}(1-\lambda\eta_{2t})]}.\end{aligned}$$

So it suffices to estimate $\| (PW)(x) \left(x+\frac{a_t}{t^2}\right)^\sigma \|_{L_p[a_t(1-\lambda\eta_t),a_{2t}(1-\lambda\eta_{2t})]}$. We see that it is bounded by a constant times $\| (PW)(x)\, x^\sigma \|_{L_p[a_t(1-\lambda\eta_t),a_{2t}(1-\lambda\eta_{2t})]}$ and if $\sigma \geq 0$, we can apply (a) of this Lemma to deduce (2.24). If instead $\sigma < 0$, we use

$$
\begin{aligned}
&\| (PW)(x)\, x^\sigma \|_{L_p[a_t(1-\lambda\eta_t),a_{2t}(1-\lambda\eta_{2t})]} \\
&\quad \leq C a_t^\sigma \| (PW)(x) \|_{L_p[a_t(1-\lambda\eta_t),a_{2t}(1-\lambda\eta_{2t})]} \\
&\quad \leq C a_t^\sigma \| (PW)(x) \|_{L_p[La_t t^{-2},a_t(1-\lambda\eta_t)]} \\
&\quad \leq C \| (PW)(x) \left(x+\frac{a_t}{t^2}\right)^\sigma \|_{L_p[La_t t^{-2},a_t(1-\lambda\eta_t)]},
\end{aligned}
$$

by first (a) of this lemma and then as $\sigma < 0$. □

Finally, we need polynomials that behave like x^ρ :

Lemma 2.5 *Let $\rho \in \mathbb{R}$ and $L \in (0,1)$. For $n \geq 1$, there exist polynomials R_n of degree $\leq n$ such that*

$$R_n(x) \sim \left(x+\frac{a_n}{n^2}\right)^\rho, \qquad x \in [0, a_{2n}]; \tag{2.25}$$

$$|R_n'(x)| \leq C x^{\rho-1}, \qquad x \in \left[La_n n^{-2}, a_{2n}\right]. \tag{2.26}$$

Proof. See Lemma 6.3 in [3]. □

3 Markov-Bernstein Inequalities

We begin by proving:

Lemma 3.1 *Let k be a non-negative integer and $0 < p \leq \infty$. Let $W \in \mathcal{L}(C^2)$. Then for $n \geq 1$ and $P \in \mathcal{P}_n$,*

$$\| (PW)'(y)\, \varphi_n^{\#}(y)\, y^{\frac{k}{2}-\frac{1}{2p}} \|_{L_p(I)} \leq C \| (PW)(y)\, y^{\frac{k}{2}-\frac{1}{2p}} \|_{L_p(I)}. \tag{3.1}$$

Proof. Let us suppose that $p < \infty$, ($p = \infty$ is simpler) and let

$$R(x) := x^k P\left(x^2\right).$$

By Theorem 10.1(a) in [2, p. 293],

$$\| (RW^*)'\, \varphi_{2n+k}^* \|_{L_p(I^*)} \leq C \|RW^*\|_{L_p(I^*)}.$$

Since

$$(RW^*)'(x) = 2x^{k+1} (PW)'\left(x^2\right) + kx^{k-1} (PW)\left(x^2\right),$$

we obtain after a substitution $x = \sqrt{y}$,

$$\int_I \left|2y^{\frac{k+1}{2}} (PW)'(y) + ky^{\frac{k-1}{2}} (PW)(y)\right|^p \varphi^*_{2n+k}(\sqrt{y})^p \frac{dy}{\sqrt{y}} \leq C \int_I |PW|^p (y)\, y^{\frac{kp}{2}} \frac{dy}{\sqrt{y}}. \tag{3.2}$$

Here by Lemma 2.3(b),

$$\varphi^*_{2n+k}(\sqrt{y}) \sim \varphi^*_{2n}(\sqrt{y}),$$

while by Lemma 2.3(a),

$$\varphi^*_{2n}(\sqrt{y}) \sim \frac{\varphi_n(y)}{\sqrt{\min\{y, a_n\} + a_n n^{-2}}} \sim \frac{\varphi^\#_n(y)}{\sqrt{\min\{y, a_n\}}}, \tag{3.3}$$

uniformly for $n \geq 1$ and $y \in I$. (For $y \geq a_n$, this follows by constancy of φ_n and so on.) Then we obtain from (3.2) and (3.3) that

$$\begin{aligned}
&\int_I \left|(PW)' \varphi^\#_n(y)\right|^p y^{\frac{kp-1}{2}} dy \\
&\leq C \int_I \left|y^{\frac{k+1}{2}} (PW)'(y)\, \varphi^*_{2n+k}(\sqrt{y})\right|^p \frac{dy}{\sqrt{y}} \\
&\leq C \int_I |PW|^p (y)\, y^{\frac{kp-1}{2}} dy + Ck^p \int_I |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\varphi^\#_n(y)}{\sqrt{y}\sqrt{\min\{y, a_n\}}}\right)^p dy \\
&\leq C \int_I |PW|^p (y)\, y^{\frac{kp-1}{2}} dy + Ck^p \int_0^{a_n n^{-2}} |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\varphi^\#_n(y)}{y}\right)^p dy
\end{aligned} \tag{3.4}$$

by Lemma 2.3(c), and since $\varphi^\#_n(y) = \varphi^\#_n(a_n)\,, y \geq a_n$. Of course if $k = 0$, the second term vanishes. We now assume that $k \geq 1$. From (2.21) in the proof of Lemma 2.3(c), we see that

$$\begin{aligned}
\int_0^{a_n n^{-2}} |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\varphi^\#_n(y)}{y}\right)^p dy &\leq C \int_0^{a_n n^{-2}} |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\sqrt{a_n}}{n\sqrt{y}}\right)^p dy \\
&\leq C \int_{a_n n^{-2}}^{a_n} |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\sqrt{a_n}}{n\sqrt{y}}\right)^p dy,
\end{aligned}$$

by our restricted range inequality Lemma 2.4(a). This is applicable since $k \geq 1$, so that

$$\frac{kp-1}{2} - \frac{p}{2} \geq -\frac{1}{2} > -1.$$

We continue this as

$$\int_0^{a_n n^{-2}} |PW|^p (y)\, y^{\frac{kp-1}{2}} \left(\frac{\varphi^\#_n(y)}{y}\right)^p dy \leq C \int_{a_n n^{-2}}^{a_n} |PW|^p (y)\, y^{\frac{kp-1}{2}} dy.$$

This and (3.4) give the result. □

Now we can prove a preliminary form of Theorem 1.5. There are no restrictions on the power x^β here, so the result has some independent interest.

Theorem 3.2 *Let $0 < p \leq \infty$, $L > 0$ and $\beta \in \mathbb{R}$. Let $W \in \mathcal{L}(C^2)$. Then for $n \geq 1$ and $P \in \mathcal{P}_n$, and for some $C \neq C(n, P)$,*

$$\|(PW)'(x)\,\varphi_n^{\#}(x)\,x^\beta\|_{L_p(La_n n^{-2},d)} \leq C\|\,(PW)(x)\,x^\beta\|_{L_p(a_n n^{-2},a_n)}. \tag{3.5}$$

Proof. Assume $p < \infty$. The case $p = \infty$ is easier. We split

$$\begin{aligned}
&\int_{La_n n^{-2}}^{d} \left|(PW)'\,\varphi_n^{\#}\right|^p(y)\,y^{\beta p}dy \\
&\quad = \left(\int_{La_n n^{-2}}^{\frac{1}{2}a_n} + \int_{\frac{1}{2}a_n}^{d}\right)\left|(PW)'\,\varphi_n^{\#}\right|^p(y)\,y^{\beta p}dy \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Choose $\rho \in \mathbb{R}$ such that

$$\beta p = -\frac{1}{2} + \rho p$$

and let $\{R_n\}$ be the polynomials from Lemma 2.5, satisfying (2.25) and (2.26). Note that PR_n has degree at most $2n$. We see that

$$\begin{aligned}
I_1 &= \int_{La_n n^{-2}}^{\frac{1}{2}a_n}\left|(PW)'\,\varphi_n^{\#}\right|^p(y)\,y^{\beta p}dy \\
&\leq C\int_{La_n n^{-2}}^{\frac{1}{2}a_n}\left|(PW)'\,\varphi_n^{\#}\right|^p(y)\,R_n^p(y)\,y^{-\frac{1}{2}}dy \\
&= C\int_{La_n n^{-2}}^{\frac{1}{2}a_n}\left|\left[(PR_nW)' - PWR_n'\right]\varphi_n^{\#}\right|^p(y)\,y^{-\frac{1}{2}}dy \\
&\leq C\int_{La_n n^{-2}}^{\frac{1}{2}a_n}\left|(PR_nW)'\,\varphi_{2n}^{\#}\right|^p(y)\,y^{-\frac{1}{2}}dy \\
&\quad + C\int_{La_n n^{-2}}^{\frac{1}{2}a_n}\left|PWR_n'\varphi_n^{\#}\right|^p(y)\,y^{-\frac{1}{2}}dy \\
&\leq C\int_I |PR_nW|^p(y)\,y^{-1/2}dy \\
&\quad + C\int_{La_n n^{-2}}^{\frac{1}{2}a_n}|PW|^p(y)\left(\frac{\varphi_n^{\#}(y)}{y}\right)^p y^{\beta p}dy,
\end{aligned}$$

by Lemma 3.1 with $k = 0$, since $\varphi_{2n}^{\#} \sim \varphi_n^{\#}$ in $\left[a_n n^{-2}, \frac{1}{2}a_n\right]$ and by (2.26). Using our restricted range inequality Lemma 2.4(a) and using (2.25) and (2.17), we can continue this as

$$I_1 \leq C\int_{La_n n^{-2}}^{a_{2n}}|PW|^p(y)\,y^{\beta p}dy.$$

Lemma 2.4(c) allows us to continue this as

$$I_1 \leq C \int_{a_n n^{-2}}^{a_n} |PW|^p (y)\, y^{\beta p} dy. \tag{3.6}$$

Next, to handle I_2, we choose a positive integer k so large that

$$\lambda := \beta p - \frac{kp-1}{2} < 0.$$

Then

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}a_n}^{d} \left|(PW)' \varphi_n^{\#}\right|^p (y)\, y^{\beta p} dy \\ &\leq \left(\frac{1}{2}a_n\right)^{\lambda} \int_{\frac{1}{2}a_n}^{d} \left|(PW)' \varphi_n^{\#}\right|^p (y)\, y^{\frac{kp-1}{2}} dy \\ &\leq C \left(\frac{1}{2}a_n\right)^{\lambda} \int_I |PW|^p (y)\, y^{\frac{kp-1}{2}} dy, \end{aligned}$$

by Lemma 3.1. Using our restricted range inequality Lemma 2.4(a), and $\lambda < 0$, we continue this as

$$\begin{aligned} I_2 &\leq C \left(\frac{1}{2}a_n\right)^{\lambda} \int_{a_n n^{-2}}^{a_n} |PW|^p (y)\, y^{\frac{kp-1}{2}} dy \\ &\leq C \int_{a_n n^{-2}}^{a_n} |PW|^p (y)\, y^{\beta p} dy. \end{aligned}$$

Together this and (3.6) give (3.5). □

Proof of Theorem 1.6 Let $P \in \mathcal{P}_n$. We shall use Theorem 3.2 and treat P as a polynomial of degree $\leq 2n$. First write

$$\begin{aligned} &|(P'W)(x)|\, x^{\beta} \\ &\leq \left|(PW)'(x)\, \varphi_{2n}^{\#}(x)\, x^{\beta}\right| \varphi_{2n}^{\#}(x)^{-1} + \left|(PW)(x)\, x^{\beta}\right| Q'(x). \end{aligned} \tag{3.7}$$

Here in $[a_n n^{-2}, a_n]$,

$$\varphi_{2n}^{\#}(x) = \frac{\sqrt{x}}{2n} \frac{a_{4n} - x}{\sqrt{a_{2n} - x + a_{2n}\eta_{2n}}} \geq C \varphi_{2n}^{\#}\left(a_n n^{-2}\right) \sim \frac{a_n}{n^2},$$

note that by (2.9) and (2.6),

$$a_{4n} - x \sim a_{2n} - x + a_{2n}\eta_{2n} \geq a_{2n} - a_n \sim \frac{a_n}{T(a_n)} \gg \frac{a_n}{n^2}.$$

Also by (2.10), in $[a_n n^{-2}, a_n]$,

$$Q'(x) \leq \frac{Cn}{\sqrt{x(a_{2n} - x)}} \leq C \max\left\{\frac{n^2}{a_n}, \frac{n\sqrt{T(a_n)}}{a_n}\right\} \sim \frac{n^2}{a_n},$$

by (2.6). Our restricted range inequality Lemma 2.4(a), followed by (3.7), give

$$\begin{aligned}
&\| (P'W)(x)\, x^{\beta} \|_{L_p(I)} \le C \| (P'W)(x)\, x^{\beta} \|_{L_p[a_n n^{-2}, a_n]} \\
&\le C\frac{n^2}{a_n} \left(\| (PW)'(x)\, \varphi_{2n}^{\#}(x)\, x^{\beta} \|_{L_p[a_n n^{-2}, a_n]} + \| (PW)(x)\, x^{\beta} \|_{L_p[a_n n^{-2}, a_n]} \right) \\
&\le C\frac{n^2}{a_n} \| (PW)(x)\, x^{\beta} \|_{L_p(I)},
\end{aligned}$$

by Theorem 3.2. Thus we have (1.21). The proof of (1.22) is similar: in $[a_{\gamma n}, d)$,

$$\varphi_{2n}^{\#}(x) \sim \varphi_{2n}(x) \ge C\varphi_{2n}(a_{\gamma n}) \sim \frac{a_n}{n\sqrt{T(a_n)}}$$

while in $[a_{\gamma n}, a_n]$ (cf. (2.3)),

$$Q'(x) \sim Q'(a_n) \sim \frac{n}{a_n}\sqrt{T(a_n)}. \qquad \square$$

Finally we turn to the

Proof of Theorem 1.5 In view of Theorem 3.2, we need only estimate the norm over $(0, a_n n^{-2})$. Now

$$\begin{aligned}
&\| (PW)'(x)\, \varphi_n^{\#}(x)\, x^{\beta} \|_{L_p(0, a_n n^{-2})} \\
&\le C\Big[\| (P'W)(x)\, \varphi_n^{\#}(x)\, x^{\beta} \|_{L_p(0, a_n n^{-2})} \\
&\quad + \| (PW)(x)\, Q'(x)\, \varphi_n^{\#}(x)\, x^{\beta} \|_{L_p(0, a_n n^{-2})} \Big] \\
&= C\left[I_1 + I_2 \right].
\end{aligned}$$

Note that

$$\varphi_n^{\#}(x) \sim \frac{\sqrt{a_n x}}{n} \le C\frac{a_n}{n^2}, \qquad x \in \left[0, \frac{a_n}{n^2}\right]. \tag{3.8}$$

Then (1.21) gives

$$I_1 \le C\frac{a_n}{n^2} \| (P'W)(x)\, x^{\beta} \|_{L_p(I)} \le C \| (PW)(x)\, x^{\beta} \|_{L_p(I)}.$$

Next, by (3.8) and then (2.10), for $x \in [0, a_n n^{-2}]$,

$$Q'(x)\, \varphi_n^{\#}(x) \le C\frac{n}{\sqrt{a_n x}} \frac{\sqrt{a_n x}}{n} = C.$$

So

$$I_2 \le C \| (PW)(x)\, x^{\beta} \|_{L_p(I)}.$$

Thus we have shown that

$$\| (PW)'(x)\, \varphi_n^{\#}(x)\, x^{\beta} \|_{L_p(0, a_n n^{-2})} \le C \| (PW)(x)\, x^{\beta} \|_{L_p(I)}.$$

This and Theorem 3.2 give the result. □

4 Estimation of $A^{\#}_{n,\rho}(x)$

We now estimate the function

$$A^{\#}_{n,\rho}(x) = \frac{2}{x}\int_I (p_{n,\rho}W_\rho)^2(t)\overline{Q(x,t)}dt$$

where

$$\overline{Q(x,t)} = \frac{xQ'(x) - tQ'(t)}{x-t}.$$

It plays a key role in estimation of $p_{n,\rho}(x)$. Using our bound (1.10) for $p_{n,\rho}$ we shall prove:

Theorem 4.1 *Assume that $W \in \mathcal{L}(C^2+)$, that $\rho > -\frac{1}{2}$ and let $L > 1$. Then $\exists C, n_0 > 0$ such that for $n \geq n_0$ and $x \in [a_n n^{-2}/L, a_n(1 + L\eta_n)]$,*

$$A^{\#}_{n,\rho}(x) \sim \varphi_n(x)^{-1}\left[x\left(a_n(1+2L\eta_n) - x\right)\right]^{-1/2}. \qquad (4.1)$$

If we assume instead that $W \in \mathcal{L}(C^2)$, this holds with $\sim$ replaced by $\leq C$.

Proof of the upper bound in (4.1) We fix $M > 1, \varepsilon \in \left(0, \frac{1}{2}\right)$ and set

$$\mathcal{J}_n := \left[\frac{a_n}{Mn^2}, \varepsilon a_n\right].$$

We assume, as in [3, eqn. (8.18)] that M is large enough so that

$$x_{nn,\rho} > \frac{a_n}{Mn^2}.$$

Let, as in [3, eqns. (8.20)–(8.21)],

$$\Psi_n(x) := (p_{n,\rho}W)^2(x)\left(x + \frac{a_n}{n^2}\right)^{2\rho} \left|\left(x + \frac{a_n}{n^2}\right)(a_n - x)\right|^{1/2}$$

and

$$\Theta_n(x) := A^{\#}_{n,\rho}(x)\varphi_n(x)|x(a_n - x)|^{1/2}.$$

We distinguish two ranges of x:

(I) Upper bounds for $x \in [a_n n^{-2}/M, \varepsilon a_n]$

We note that for this range of x,

$$x\left(a_n(1+2L\eta_n) - x\right) \sim x\left(a_n - x\right)$$

with constants in $\sim$ independent of n and x. Then

$$A^{\#}_{n,\rho}(x)\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2}\sim\Theta_n(x).$$

By Lemma 8.6 in [3], for some $\varepsilon>0$,

$$\|\Theta_n\|_{L_\infty(\mathcal{J}_n)}\leq C+\|\Psi_n\|_{L_\infty(I)}\leq C_1,$$

by (1.10). So choosing $M\geq L$, we have the upper bound implicit in (4.1) for this range of x.

(II) Upper Bounds for $x\in[\varepsilon a_n,a_n(1+L\eta_n)]$

Write

$$a_n(1+2L\eta_n)=a_m$$

so that

$$1-\frac{a_n}{a_m}\sim\eta_n.$$

We choose m in this way to ensure that for some small enough α,β and large enough n,

$$\left[\beta a_m m^{-2},a_m\left(1-\alpha\eta_m\right)\right]\supseteq\left[\varepsilon a_n,a_n\left(1+L\eta_n\right)\right]. \tag{4.2}$$

By (2.8) and (2.7),

$$1-\frac{n}{m}\sim T\left(a_n\right)\left(1-\frac{a_n}{a_m}\right)\sim T\left(a_n\right)\eta_n\to 0, \tag{4.3}$$

as $n\to\infty$. Moreover, for $x\in[0,a_n\left(1-\eta_n\right)]$, we have

$$|a_m-x|\sim|a_n-x|$$

so (1.10) gives the bound

$$\left|p_{n,\rho}\left(x\right)W\left(x\right)\right|\left(x+\frac{a_n}{n^2}\right)^{\rho}\left|x\left(a_m-x\right)\right|^{1/4}\leq C. \tag{4.4}$$

By our restricted range inequality Lemma 2.4(a), this then holds throughout I. We split

$$\begin{aligned}A^{\#}_{n,\rho}\left(x\right)&=\frac{2}{x}\left[\int_0^{a_n n^{-2}}+\int_{a_n n^{-2}}^{a_m}+\int_{a_m}^{a_{2n}}+\int_{a_{2n}}^{d}\right]\left(p_{n,\rho}W_\rho\right)^2\left(t\right)\overline{Q\left(x,t\right)}dt\\&=:I_1+I_2+I_3+I_4.\end{aligned}$$

Estimation of I_1

For $t\in[0,a_n n^{-2}]$, the monotonicity of $uQ'\left(u\right)$ gives

$$\overline{Q(x,t)}\leq CQ'\left(x\right)$$

so our bound (1.10) on $p_{n,\rho}$ gives

$$\begin{aligned} I_1 &\leq C\frac{Q'(x)}{x}\int_0^{a_n n^{-2}} \frac{1}{\sqrt{(t+a_n n^{-2})(a_n-t)}}\left(\frac{t}{t+a_n n^{-2}}\right)^{2\rho} dt \\ &\leq C\frac{Q'(x)}{xn}\int_0^1 \frac{1}{\sqrt{s+1}}\left(\frac{s}{s+1}\right)^{2\rho} ds, \end{aligned}$$

where we made the substitution $t = a_n n^{-2} s$. Then from Lemma 2.3(e) and using monotonicity of $uQ'(u)$ and the fact that $m \sim n$ gives

$$\begin{aligned} &I_1\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2} \\ &\quad\leq C\frac{Q'(a_m)}{a_n n}\frac{a_n^2}{nT(a_n)} \leq \frac{C}{n\sqrt{T(a_n)}} \leq C. \end{aligned}$$

Estimation of I_2
Our bound (4.4) on $p_{n,\rho}$ gives

$$I_2 \leq \frac{C}{a_n}\int_0^{a_m} \frac{\overline{Q(x,t)}}{\sqrt{t(a_m-t)}}dt \leq \frac{C}{a_n}\sqrt{\frac{x}{a_m-x}}\sigma_m(x),$$

where

$$\sigma_m(x) = \frac{1}{\pi^2}\sqrt{\frac{a_m-x}{x}}\int_0^{a_m}\frac{\overline{Q(x,t)}}{\sqrt{t(a_m-t)}}dt$$

is the density of the equilibrium measure of total mass m for the field Q. It is shown in [3, (4.10)] that

$$\sigma_m(x) \sim \varphi_m^{-1}(x) \qquad \text{in } \left[\beta a_m m^{-2}, a_m(1-\alpha\eta_m)\right]$$

for any fixed $\beta, \alpha > 0$, so

$$I_2 \leq \frac{C}{\sqrt{a_m(a_m-x)}}\varphi_m^{-1}(x).$$

Here $\eta_m \sim \eta_n$. Now for small enough α, β, we saw at (4.2) that this last range contains $[\varepsilon a_n, a_n(1+L\eta_n)]$, while Lemma 2.3(b) and (4.3) show that $\varphi_m \sim \varphi_n$. Then

$$\begin{aligned} &I_2\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2} \\ &\quad\leq CI_2\varphi_m(x)\left[x(a_m-x)\right]^{1/2} \leq C. \end{aligned}$$

Estimation of I_3
For $t \in [a_n, a_{2n}]$ there exists $s \in [\varepsilon a_n, a_{2n}]$ such that

$$\overline{Q(x,t)} = (uQ'(u))'_{|u=s} = sQ''(s) + Q'(s).$$

Here by (1.5),

$$sQ''(s) \leq C\frac{sQ'(s)^2}{Q(s)} = CT(s)\,Q'(s)$$

and as T is bounded below, while $sQ'(s)$ is increasing, we see that

$$\overline{Q(x,t)} \leq CT(s)\,Q'(s) \leq CT(a_{2n})\,Q'(a_{2n}) \leq C\frac{n}{a_n}T(a_n)^{3/2},$$

recall (2.3). Then

$$\begin{aligned} I_3 &\leq C\frac{n}{a_n^2}T(a_n)^{3/2}\int_{a_m}^{a_{2n}}\frac{dt}{\sqrt{t(t-a_n)}} \\ &\leq C\frac{n}{a_n^{5/2}}T(a_n)^{3/2}(a_{2n}-a_n)^{1/2} \leq C\frac{nT(a_n)}{a_n^2}. \end{aligned}$$

Then (2.19) gives

$$I_3\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2} \leq C.$$

Estimation of I_4
Next,

$$\begin{aligned} I_4 &\leq \frac{C}{a_n}\int_{a_{2n}}^{d}(p_{n,\rho}W_\rho)^2(t)\frac{tQ'(t)}{t-x}dt \\ &\leq \frac{C}{a_n\left(a_{2n}-a_n\left(1+L\eta_n\right)\right)}\int_{a_{2n}}^{d}(p_{n,\rho}W_\rho)^2(t)tQ'(t)\,dt \\ &\leq \frac{CT(a_n)}{a_n^2}\int_{a_{2n}}^{d}(p_{n,\rho}W_\rho)^2(t)tQ'(t)\,dt \\ &\leq \frac{CnT(a_n)}{a_n^2}, \end{aligned}$$

by (2.8) and the identity

$$\int_I tQ'(t)\,(p_{n,\rho}W_\rho)^2(t)\,dt = n+\rho+\frac{1}{2}. \tag{4.5}$$

The latter follows by integrating by parts and using orthogonality. Using (2.19) again, we obtain

$$I_4\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2} \leq C.$$

Finally combining the estimates for I_1, I_2, I_3, I_4 gives

$$A_{n,\rho}^{\#}(x)\varphi_n(x)\left[x\left(a_n(1+2L\eta_n)-x\right)\right]^{1/2} \leq C. \qquad \square$$

In proving the lower bounds for $A_{n,\rho}^{\#}$, we need an estimate related to the identity (4.5):

Theorem 4.2 *Let $W \in \mathcal{L}(C^2)$. There exists $\alpha > 1$ such that uniformly for $r \in [0, 2n]$*

$$\int_{[0,a_{2n}]\setminus[a_{r/\alpha},a_{\alpha r}]} (p_{n,\rho}W_\rho)^2(t)\, tQ'(t)\, dt \sim n. \tag{4.6}$$

Proof. We use (4.5) and show that the integrals over $[a_{2n}, d)$ and $\left[a_{r/\alpha}, a_{\alpha r}\right]$ are small. Write

$$\rho = \beta + j$$

where $\beta \leq 0$ and j is a non-negative integer. Then

$$\int_{a_{2n}}^{d} (p_{n,\rho}W_\rho)^2(t)\, tQ'(t)\, dt \leq \frac{1}{2}a_{2n}^{2\beta} \int_{a_{2n}}^{d} P(t) \frac{d}{dt}\left(-W^2(t)\right) dt,$$

where

$$P(t) = p_{n,\rho}^2(t)\, t^{1+2j}.$$

Integrating by parts gives

$$\int_{a_{2n}}^{d} P(t) \frac{d}{dt}\left(-W^2(t)\right) dt = \left(PW^2\right)(a_{2n}) + \int_{a_{2n}}^{d} P'(t)\, W^2(t)\, dt.$$

Because of our bounds on $p_{n,\rho}$, we know that in $[0, a_n(1-\eta_n)]$, $|P|\, W^2$ is bounded by a power of n (recall a_n is of polynomial growth). Our restricted range inequality then shows that it is bounded in I by a power of n and moreover Lemma 2.4(b) gives that for some $C > 0$,

$$\left(PW^2\right)(a_{2n}) = O(e^{-n^C}).$$

Our Markov-Bernstein inequality Theorem 1.6 then shows that $P'W^2$ is bounded by a power of n in I. The same is then true of the L_1 norm of $P'W^2$ over $[0, a_{2n}]$. Another application of our restricted range inequalities to the weight W^2 (rather than W) shows that (at least for large enough n),

$$\int_{a_{2n}}^{d} P'(t)\, W^2(t)\, dt = O\left(e^{-n^C}\right).$$

So

$$\int_{a_{2n}}^{d} (p_{n,\rho}W_\rho)^2(t)\, tQ'(t)\, dt = O\left(e^{-n^C}\right).$$

In view of (4.5), it now suffices to show that given $\eta > 0$, there exists $\alpha \in (1, \frac{3}{2}]$ such that uniformly for $r \in [0, 2n]$,

$$I = \int_{a_{r/\alpha}}^{a_{\alpha r}} (p_{n,\rho}W_\rho)^2(t)\, tQ'(t)\, dt \leq \eta n. \tag{4.7}$$

We note first that it is an easy consequence of (2.8) and (2.5) that

$$a_{\alpha r} - a_{r/\alpha} \leq C\frac{a_r}{T(a_r)}\left(\alpha - \frac{1}{\alpha}\right) \tag{4.8}$$

with C independent of r, α, since $\alpha \in [1, \frac{3}{2}]$. Moreover, by our bound (1.10) on $p_{n,\rho}$, and by (2.5),

$$I \leq C \int_{a_{r/\alpha}}^{a_{\alpha r}} \frac{tQ'(t)}{\sqrt{t\,|a_n - t|}} dt \leq C a_r^{\frac{1}{2}} Q'(a_r) \int_{a_{r/\alpha}}^{a_{\alpha r}} \frac{dt}{\sqrt{|a_n - t|}}.$$

If $a_{\alpha r} \leq a_n$, we continue this as

$$\begin{aligned} I &\leq C a_r^{\frac{1}{2}} Q'(a_r) \int_{a_{r/\alpha}}^{a_{\alpha r}} \frac{dt}{\sqrt{a_{\alpha r} - t}} \leq C a_r^{\frac{1}{2}} Q'(a_r) \sqrt{a_{\alpha r} - a_{r/\alpha}} \\ &\leq C r \left(\alpha - \frac{1}{\alpha}\right)^{1/2} \leq C n \left(\alpha - 1\right)^{1/2}, \end{aligned}$$

by (4.8) and (2.3). If $a_{\alpha/r} > a_n$, we similarly continue this as

$$I \leq C a_r^{\frac{1}{2}} Q'(a_r) \int_{a_{r/\alpha}}^{a_{\alpha r}} \frac{dt}{\sqrt{t - a_{r/\alpha}}} \leq C n \left(\alpha - 1\right)^{1/2}.$$

If $a_{\alpha r} > a_n > a_{\alpha/r}$, we continue this as

$$I \leq C a_r^{\frac{1}{2}} Q'(a_r) \sqrt{a_{\alpha r} - a_{r/\alpha}} \leq C n \left(\alpha - 1\right)^{1/2}.$$

In summary, in all cases, if α is close enough to 1, we obtain (4.7). □

Proof of the Lower Bounds for $A_{n,\rho}^{\#}(x)$ Let us write $x = a_r$ for $x \in [a_n n^{-2}/L, a_n (1 + L\eta_n)]$. By Lemma 2.2(d), for $t \in [0, a_{2n}] \backslash [a_{r/\alpha}, a_{r\alpha}]$,

$$\overline{Q(x,t)} \geq \frac{C}{|x - t|} \max\left\{tQ'(t), xQ'(x)\right\} \geq \frac{CtQ'(t)}{a_{2n} - x}.$$

Note that while (2.11) holds for t in a neighbourhood omitting 0, the last inequality is valid for $n \geq n_0$ for all $x \in [0, a_n(1 + L\eta_n)]$ (that is, even for x, t near 0). Then

$$\begin{aligned} &\frac{1}{x} \int_0^d (p_{n,\rho} W_\rho)^2(t) \overline{Q(x,t)} dt \\ &\quad \geq \frac{C}{x\left(a_{2n} - x\right)} \int_{[0,a_{2n}] \backslash [a_{r/\alpha}, a_{\alpha r}]} \left(p_{n,\rho} W_\rho\right)^2(t)\, tQ'(t)\, dt \\ &\quad \geq C \frac{n}{x(a_{2n} - x)} \end{aligned}$$

by (4.6). So

$$A_{n,\rho}^{\#}(x) \geq C \frac{n}{x} \frac{1}{a_{2n} - x}. \tag{4.9}$$

Firstly if $x \in [a_n n^{-2}, a_{n/2}]$, Lemma 2.3(d) gives as $a_{2n} - x \sim a_n - x$,

$$A^{\#}_{n,\rho}(x)\varphi_n(x)[x\,(a_n\,(1+2L\eta_n)-x)]^{1/2} \geq C. \tag{4.10}$$

Thus we obtain a lower bound to match the upper bound for $A^{\#}_{n,\rho}$ that we have already proven. Next, if $x \in [a_{n/2}, a_n(1+L\eta_n)]$, we obtain from (4.9) that at least for large n,

$$A^{\#}_{n,\rho}(x) \geq C\frac{n}{(a_{2n}-a_{n/2})a_n} \sim \frac{nT(a_n)}{a_n^2}$$

so Lemma 2.3(e) gives (4.10) again. □

5 The Proof of Theorems 1.2 and 1.3(a),(b)

In the sequel, we shall need Christoffel functions,

$$\lambda_n\left(W_\rho^2, x\right) = \inf_{\deg(P)\leq n-1} \frac{\int_I (PW_\rho)^2}{P^2(x)},$$

the Christoffel numbers

$$\lambda_{jn} = \lambda_n\left(W_\rho^2, x_{jn}\right)$$

and the reproducing kernel

$$K_{n,\rho}(x,t) = \sum_{j=0}^{n-1} p_{j,\rho}(x)\, p_{j,\rho}(t)\,.$$

Lemma 5.1 *Let $W \in \mathcal{L}(C^2)$.*
(a)

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p'_{n,\rho}(x_{jn})p_{n-1,\rho}(x_{jn}); \tag{5.1}$$

$$p'_{n,\rho}(x_{jn}) = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} A^{\#}_{n,\rho}(x_{jn})p_{n-1,\rho}(x_{jn}). \tag{5.2}$$

(b)

$$\lambda_n\left(W_\rho^2, x\right) \sim \varphi_n(x)\, W^2(x)\left(x+\frac{a_n}{n^2}\right)^{2\rho} \quad \text{in } [0, a_n(1+L\eta_n)] \tag{5.3}$$

and

$$\lambda_n\left(W_\rho^2, x\right) \geq C\varphi_n(x)\, W^2(x)\left(x+\frac{a_n}{n^2}\right)^{2\rho} \quad \text{in } I. \tag{5.4}$$

(c)

$$1-\frac{x_{1n}}{a_n} \sim \eta_n \quad \text{and} \quad x_{nn} \sim \frac{a_n}{n^2} \tag{5.5}$$

and uniformly in j, n,

$$x_{jn} - x_{j+1,n} \leq C\varphi_n(x_{jn}). \tag{5.6}$$

Proof.
(a) See [3, (8.5) and Lemma 8.3].
(b) See Theorem 1.3 in [3].
(c) See Theorem 1.4 in [3]. □

Next, we prove the upper bound implicit in (1.8).

Lemma 5.2 *Let $W \in \mathcal{L}(C^2)$.*
(a) For $n \geq 1$,

$$\sup_{x \in I} (p_{n,\rho} W)(x) \left(x + \frac{a_n}{n^2}\right)^{\rho} \leq C\sqrt{\frac{n}{a_n}}; \tag{5.7}$$

(b) For $n \geq 1$,

$$\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} \sim a_n. \tag{5.8}$$

Proof.
(a) Now for $x \in [a_n n^{-2}, a_n(1-\eta_n)]$, and for large enough n, (1.10) gives

$$\begin{aligned} |p_{n,\rho}(x) W(x)| x^{\rho} &\leq \frac{C}{[x(a_n - x)]^{1/4}} \\ &\leq C \max\left\{\frac{1}{a_n^2 n^{-2}}, \frac{1}{a_n^2 \eta_n}\right\}^{1/4} \\ &= C a_n^{-1/2} \max\left\{n^{1/2}, (nT(a_n))^{1/6}\right\} \leq C a_n^{-1/2} n^{1/2}, \end{aligned}$$

by (2.6). Then the restricted range inequality Lemma 2.4(a) gives (5.7).
(b) The ideas are standard, but we provide the details. Firstly, from our restricted range inequality and Cauchy-Schwarz,

$$\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} = \int_I x p_{n,\rho}(x) p_{n-1,\rho}(x) W_\rho^2(x)\, dx \leq C a_n.$$

We proceed to prove a corresponding lower bound. From the definition (1.12) of φ_n,

$$\varphi_n(x) \sim \frac{a_n}{n}, \qquad x \in \left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]. \tag{5.9}$$

Then for $x_{jn}, x_{j-1,n} \in \left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]$, Lemma 5.1(c) ensures that

$$x_{j-1,n} - x_{jn} \leq C\frac{a_n}{n}.$$

Here C is independent of j and n. It then follows that for large enough n, the number of zeros of $p_{n,\rho}$ lying in this interval is at least $\geq C_1 n$. (We only have to show that $\left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]$ contains at least one zero. If not, we easily obtain a contradiction using Lemma 5.1(c)). Recall from Lemma 5.1(a), the identities

$$\begin{aligned}\lambda_{jn}^{-1} &= \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}\left(x_{jn}\right) p_{n,\rho}'\left(x_{jn}\right) \\ &= \left(\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n-1,\rho}\left(x_{jn}\right)\right)^2 A_{n,\rho}^{\#}\left(x_{jn}\right).\end{aligned}$$

Applying our bound of Theorem 4.1 on $A_{n,\rho}^{\#}$ and also (5.9) gives for $x_{jn} \in \left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]$,

$$\left(\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}\right)^{-2} \leq C\lambda_{jn} p_{n-1,\rho}^2\left(x_{jn}\right) \frac{n}{a_n^2}.$$

Adding over the $\geq C_1 n$ zeros $x_{jn} \in \left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]$ gives

$$\begin{aligned}C_1 n \left(\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}}\right)^{-2} &\leq C\frac{n}{a_n^2} \sum_{x_{jn} \in \left[\frac{1}{4}a_n, \frac{1}{2}a_n\right]} \lambda_{jn} p_{n-1,\rho}^2\left(x_{jn}\right) \\ &\leq C\frac{n}{a_n^2} \int_I p_{n-1,\rho}^2 W_\rho = C\frac{n}{a_n^2}.\end{aligned}$$

Here we have used the Gauss quadrature formula. Hence

$$\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} \geq Ca_n.$$

□

Proof of Theorem 1.3(a),(b) We use (5.1) and (5.2) in the form

$$\lambda_{jn}^{-1} = \frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} p_{n,\rho}'(x_{jn}) p_{n-1,\rho}(x_{jn}) = p_{n,\rho}'(x_{jn})^2 / A_{n,\rho}^{\#}(x_{jn}) \tag{5.10}$$

so that

$$|p_{n,\rho}' W_\rho|(x_{jn}) = [\lambda_{jn}^{-1} W_\rho^2(x_{jn}) A_{n,\rho}^{\#}(x_{jn})]^{1/2}.$$

Substituting the upper bounds for $A_{n,\rho}^{\#}(x_{jn})$ from Theorem 4.1 and the lower bounds for λ_{jn} from Lemma 5.1(b) in this last expression gives the upper bounds for $|p_{n,\rho}' W_\rho|(x_{jn})$ that are implicit in (1.13). We also use (5.5). When $W \in \mathcal{L}(C^2+)$ we also have matching lower bounds for $A_{n,\rho}^{\#}$, so we obtain $\sim$ relations for $|p_{n,\rho}' W_\rho|(x_{jn})$. The identity (5.10) above, in the form

$$|p_{n-1,\rho} W_\rho|(x_{jn}) = \lambda_{jn}^{-1} W_\rho^2(x_{jn}) / [\frac{\gamma_{n-1,\rho}}{\gamma_{n,\rho}} |p_{n,\rho}' W_\rho|(x_{jn})]$$

and the previous lemma then gives the required estimates for $|p_{n-1,\rho} W_\rho|(x_{jn})$.□

Proof of Theorem 1.2 We already proved the upper bound implicit in (1.8) in Lemma 5.2. For the corresponding lower bound, we use our Markov-Bernstein inequality Theorem 3.2 in the form

$$\begin{aligned}|(p'_{n,\rho}W_\rho)(x_{nn})\varphi_n^{\#}(x_{nn})| &= |(p_{n,\rho}W)'(x_{nn})\varphi_n^{\#}(x_{nn})x_{nn}^{\rho}| \\ &\leq C \max_{x\in[a_n n^{-2},a_n]} |\left(p_{n,\rho}W\right)(x)\, x^{\rho}|.\end{aligned}$$

Recall that $x_{nn} \sim a_n n^{-2}$, so Theorem 3.2 is applicable. Substituting the bound (1.13) for $|p'_{n,\rho}W_\rho|(x_{nn})$ proved above and using $\varphi_n^{\#}(x_{nn}) \sim \varphi_n(x_{nn})$ gives

$$\max_{x\in[a_n n^{-2},a_n]} |p_{n,\rho}W_\rho|(x) \geq C\sqrt{\frac{n}{a_n}}.$$

Thus we have (1.8). For (1.9), observe from (1.10) that

$$|xp_{n,\rho}(x)\,W_\rho(x)| \leq Cx^{3/4}\,(a_n - x)^{-1/4}, \qquad x \in \left[a_n n^{-2}, a_n\,(1-\eta_n)\right].$$

Maximizing the right-hand side over this interval gives

$$\begin{aligned}&\left|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho+1}\right| \\ &\quad\leq 2\left|xp_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right| \leq Ca_n^{1/2}\,(nT\,(a_n))^{1/6},\end{aligned}$$

and then our restricted range inequality Lemma 2.4(a) shows that

$$\sup_{x\in I}\left|p_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho+1}\right| \leq Ca_n^{1/2}\,(nT\,(a_n))^{1/6}. \tag{5.11}$$

Moreover, the bound (1.10) gives

$$\begin{aligned}\max_{x\in[0,a_{\beta n}]}\left|xp_{n,\rho}(x)\,W(x)\left(x+\frac{a_n}{n^2}\right)^{\rho}\right| &\leq Ca_n^{3/4}\,(a_n - a_{\beta n})^{-1/4} \\ &\leq Ca_n^{1/2}T\,(a_n)^{1/4} \\ &= o\left(a_n^{1/2}\,(nT\,(a_n))^{1/6}\right),\end{aligned} \tag{5.12}$$

by (2.6). Next, our Markov-Bernstein inequality Theorem 3.2 gives

$$\begin{aligned}|x_{1n}(p'_{n,\rho}W_\rho)(x_{1n})\varphi_n(x_{1n})| &= |(p_{n,\rho}W)'(x_{1n})\varphi_n(x_{1n})x_{1n}^{1+\rho}| \\ &\leq C \max_{x\in[a_n n^{-2},a_n]} |\left(p_{n,\rho}W\right)(x)\, x^{1+\rho}|.\end{aligned}$$

Substituting in the bounds for $|p'_{n,\rho}W_\rho|(x_{1n})$ from (1.13), and using

$$a_n - x_{1n} \sim a_n\eta_n$$

gives

$$\max_{x\in[a_n n^{-2},a_n]} |xp_{n,\rho}W_\rho|(x) \geq Ca_n^{1/2}\left(nT\left(a_n\right)\right)^{1/6}.$$

Combining this, (5.11) and (5.12) gives (1.9). □

6 Lagrange Interpolation Polynomials

In this section, we prove Theorem 1.3(c), (d) and Theorem 1.4. The most difficult part is the upper bound implicit in (1.15), namely,

$$\Delta_{jn}(x) := |\ell_{jn}W|(x)\left(x+\frac{a_n}{n^2}\right)^\rho W_\rho^{-1}(x_{jn}) \leq C \tag{6.1}$$

with C independent of j, n, x. Since

$$\Delta_{jn}(x_{jn}) = \left(\frac{x_{jn}+\frac{a_n}{n^2}}{x_{jn}}\right)^\rho \sim 1,$$

(1.15) follows from (6.1). We begin with two independent bounds for Δ_{jn}. We shall use the notation

$$\pi_n(x) := x(a_n(1+\eta_n)-x). \tag{6.2}$$

Lemma 6.1 *Assume that $W \in \mathcal{L}(C^2)$ and $\rho > -\frac{1}{2}$. Uniformly in j, n and $x \in I$,*
(a)

$$\Delta_{jn}(x) \leq C\left(\frac{\varphi_n(x_{jn})}{\varphi_n(x)}\right)^{1/2}; \tag{6.3}$$

(b)

$$\Delta_{jn}(x) \leq C\frac{\varphi_n(x_{jn})}{|x-x_{jn}|}\left|\frac{\pi_n(x_{jn})}{\pi_n(x)}\right|^{1/4}. \tag{6.4}$$

Proof.
(a) We use the Cauchy-Schwarz inequality on the identity

$$\ell_{jn}(x) = K_{n,\rho}(x,x_{jn})/K_{n,\rho}(x_{jn},x_{jn})$$

to deduce

$$\begin{aligned}&|\ell_{jn}W|(x)W_\rho^{-1}(x_{jn})\\ &\quad\leq \left(\frac{K_{n,\rho}(x,x)W^2(x)}{K_{n,\rho}(x_{jn},x_{jn})W_\rho^2(x_{jn})}\right)^{1/2} = \left(\frac{\lambda_n^{-1}(W_\rho,x)W^2(x)}{\lambda_n^{-1}(W_\rho,x_{jn})W_\rho^2(x_{jn})}\right)^{1/2}.\end{aligned}$$

Applying the Christoffel function bounds (5.3) and (5.4) to $\Delta_{jn}(x)$ gives the result.
(b) By our bounds for p_n from (1.10),

$$|p_{n,\rho}W|(x)\left(x+\frac{a_n}{n^2}\right)^{\rho} \le C|\pi_n(x)|^{-1/4}. \tag{6.5}$$

Substituting this and the bounds for $|p'_{n,\rho}W_\rho|(x_{jn})$ from (1.13) into

$$|\ell_{jn}W|(x)\left(x+\frac{a_n}{n^2}\right)^{\rho} W_\rho^{-1}(x_{jn}) = \frac{|p_{n,\rho}W|(x)}{|x-x_{jn}||p'_{n,\rho}W_\rho|(x_{jn})}\left(x+\frac{a_n}{n^2}\right)^{\rho} \tag{6.6}$$

gives the result. □

We shall also find the following simple observation useful:

Lemma 6.2 *There exists n_0 such that for $n \ge n_0$ and $s,t \in [0,a_n]$,*

$$|\pi_n(s)| \ge 2|\pi_n(t)| \Rightarrow |s-t| \ge \frac{|\pi_n(s)|}{4a_n}. \tag{6.7}$$

Proof. Now for $x \in [0,a_n]$,

$$|\pi'_n(x)| = |a_n(1+\eta_n) - 2x| \le 2a_n$$

for n large enough, so if s,t are as above, then for some ξ between s,t

$$\frac{1}{2}|\pi_n(s)| \le |\pi_n(s)-\pi_n(t)| = |\pi'_n(\xi)||s-t| \le 2a_n|s-t|.$$ □

We break down the proof of (6.1) into 2 lemmas, considering various ranges of x_{jn}.

Lemma 6.3 *For $x_{jn} \in [0, a_{n/2}]$ and $x \in I$,*

$$\Delta_{jn}(x) \le C. \tag{6.8}$$

Proof. We prove the upper bound for $\Delta_{jn}(x)$ separately for two ranges of $x \in [a_n n^{-2}, a_n(1-\eta_n)]$. (Then the result follows for all x from the restricted range inequality Lemma 2.4(c).) From Lemma 2.3(d), uniformly in j and n,

$$\varphi_n(x_{jn}) \sim \pi_n(x_{jn})^{1/2}/n. \tag{6.9}$$

(Recall that $x_{nn} \ge Ca_n n^{-2}$.) We shall substitute this and relevant estimates for $\varphi_n(x)$ in (6.3) and (6.4).
(I) $x \in [a_n n^{-2}, a_{3n/4}]$
From Lemma 2.3(d),

$$\varphi_n(x) \sim \pi_n(x)^{1/2}/n.$$

Then our bound (6.3) becomes

$$\Delta_{jn}(x) \leq C\left(\frac{\pi_n(x_{jn})}{\pi_n(x)}\right)^{1/4}.$$

If $\pi_n(x_{jn}) \leq 2\pi_n(x)$, we obtain the desired bound. In the contrary case, where $\pi_n(x_{jn}) > 2\pi_n(x)$, the previous lemma gives

$$|x - x_{jn}| \geq \frac{|\pi_n(x_{jn})|}{4a_n} \tag{6.10}$$

so (6.4) becomes, with the aid of (6.9),

$$\Delta_{jn}(x) \leq C\frac{a_n}{n}[\pi_n(x)\pi_n(x_{jn})]^{-1/4} \leq C\frac{a_n}{n}\pi_n(x)^{-1/2} \leq C,$$

as π_n attains its minimum over $[a_n n^{-2}, a_{3n/4}]$ at $a_n n^{-2}$, and that minimum $\sim a_n^2 n^{-2}$.

(II) $x \in [a_{3n/4}, a_n(1-\eta_n)]$

For this range of x, Lemma 2.3(e) gives

$$\varphi_n(x) \sim \pi_n(x)^{-1/2}\frac{a_n^2}{nT(a_n)}.$$

Moreover,

$$\pi_n(x) \sim a_n(a_n(1+\eta_n) - x) \leq a_n(a_n(1+\eta_n) - a_{3n/4})$$

so

$$\pi_n(x) \leq C_1\frac{a_n^2}{T(a_n)}.$$

(Recall (2.9) and that $\eta_n = o(1/T(a_n))$.) Then if

$$\pi_n(x_{jn}) \leq 2C_1\frac{a_n^2}{T(a_n)}$$

we obtain from (6.3) and (6.9) that

$$\Delta_{jn}(x) \leq C\left(\pi_n(x_{jn})^{1/2}\pi_n(x)^{1/2}\frac{T(a_n)}{a_n^2}\right)^{1/2} \leq C_2.$$

In the contrary case, where

$$\pi_n(x_{jn}) > 2C_1\frac{a_n^2}{T(a_n)} \geq 2\pi_n(x),$$

the previous lemma gives (6.10) again, and hence as above, (6.4) gives

$$\Delta_{jn}(x) \leq C\frac{a_n}{n}[\pi_n(x)\pi_n(x_{jn})]^{-1/4}.$$

Now for the current range of x, we have as usual,

$$\pi_n(x) \geq \min\left\{\pi_n\left(a_{3n/4}\right), \pi_n\left(a_n\left(1-\eta_n\right)\right)\right\} \sim a_n^2\eta_n,$$

so

$$\Delta_{jn}(x) \leq C\frac{a_n}{n}\left[a_n^2\eta_n \cdot \frac{a_n^2}{T(a_n)}\right]^{-1/4} = C\left[\frac{T(a_n)}{n^2}\right]^{5/12} \leq C,$$

by (2.6). □

Lemma 6.4 *For $x_{jn} \in [a_{n/2}, d)$ and $x \in I$,*

$$\Delta_{jn}(x) \leq C. \tag{6.11}$$

Proof. Recall that for some $M > 0$ and large enough n, $x_{1n} \leq a_n(1 - M\eta_n)$. Then for $x_{jn} \geq a_{n/2}$, Lemma 2.3(e) shows that

$$\varphi_n(x_{jn}) \sim \pi_n(x_{jn})^{-1/2}\frac{a_n^2}{nT(a_n)}. \tag{6.12}$$

We shall substitute this and relevant estimates for $\varphi_n(x)$ in (6.3) and (6.4), for two different ranges of x.
(I) $x \in [a_n n^{-2}, a_{n/4}]$
From Lemma 2.3(d),

$$\varphi_n(x) \sim \pi_n(x)^{1/2}/n.$$

Moreover,

$$|x - x_{jn}| \geq a_{n/2} - a_{n/4} \geq \frac{Ca_n}{T(a_n)}.$$

From (6.4),

$$\Delta_{jn}(x) \leq C\frac{a_n}{n}[\pi_n(x)\pi_n(x_{jn})]^{-1/4}.$$

Here

$$\pi_n(x_{jn}) \geq Ca_n^2\eta_n;$$
$$\pi_n(x) \geq C\frac{a_n^2}{n^2}.$$

Then we obtain

$$\Delta_{jn}(x) \leq C\left\{\frac{T(a_n)}{n^2}\right\}^{1/6} \leq C$$

by (2.6).
(II) $x \in [a_{n/4}, a_n(1-\eta_n)]$
For this range of x, Lemma 2.3(d) gives

$$\varphi_n(x) \sim \pi_n(x)^{-1/2}\frac{a_n^2}{nT(a_n)}.$$

Here (6.3) becomes

$$\Delta_{jn}(x) \leq C\left(\frac{\pi_n(x)}{\pi_n(x_{jn})}\right)^{1/4}.$$

If $\pi_n(x) \leq 2\pi_n(x_{jn})$, the result follows. In the contrary case, Lemma 6.2 gives

$$|x - x_{jn}| \geq \frac{\pi_n(x)}{4a_n}$$

so (6.4) becomes

$$\Delta_{jn}(x) \leq \frac{C}{\pi_n(x)^{5/4}\pi_n(x_{jn})^{1/4}} \frac{a_n^3}{nT(a_n)} \leq \frac{C}{(a_n^2\eta_n)^{3/2}} \frac{a_n^3}{nT(a_n)} \leq C,$$

by definition of η_n. □

With the proof of (6.1), and hence Theorem 1.3(c) complete, we turn to an auxiliary result for Theorem 1.3(d):

Lemma 6.5 *Let $W \in \mathcal{L}(C^2)$. Uniformly in j, n and for $x \in [x_{j+1,n}, x_{jn}]$,*

$$(\ell_{jn}W_\rho)(x)W_\rho^{-1}(x_{jn}) + (\ell_{j+1,n}W_\rho)(x)W_\rho^{-1}(x_{j+1,n}) \sim 1. \qquad (6.13)$$

Proof. From Lemma 7.5 in [3], for $x \in [x_{j+1,n}, x_{jn}]$

$$(\ell_{jn}W)(x)W^{-1}(x_{jn}) + (\ell_{j+1,n}W)(x)W^{-1}(x_{j+1,n}) \geq 1.$$

Note that uniformly in j and n,

$$x_{j+1,n} \sim x_{jn}. \qquad (6.14)$$

Indeed from (5.6) and then (2.20), (1.18) and (2.17),

$$0 \leq \frac{x_{jn} - x_{j+1,n}}{x_{j+1,n}} \leq C\frac{\varphi_n(x_{jn})}{x_{j+1,n}} \sim \frac{\varphi_n(x_{j+1,n})}{x_{j+1,n}} \sim \frac{\varphi_n^{\#}(x_{j+1,n})}{x_{j+1,n}} \leq C.$$

So we have (6.14) and hence

$$(\ell_{jn}W_\rho)(x)W_\rho^{-1}(x_{jn}) + (\ell_{j+1,n}W_\rho)(x)W_\rho^{-1}(x_{j+1,n}) \geq C.$$

The corresponding upper bound follows from Theorem 1.3(c). □

Proof of Theorem 1.4 We already know from Lemma 5.1 that uniformly in j, n

$$x_{jn} - x_{j+1,n} \leq C\varphi_n(x_{j+1,n})$$

and must prove the corresponding lower bound. First note from our Markov-Bernstein inequality Theorem 3.2 and since $\varphi_n \sim \varphi_n^{\#}$ in $[x_{nn}, d)$ that

$$\begin{aligned}&\|(\ell_{jn}W)'(x)\,\varphi_n(x)\,x^{\rho}\|_{L_\infty[x_{nn},d)}W_\rho^{-1}(x_{jn})\\&\quad\le C\|\,(\ell_{jn}W)\,(x)\,x^{\rho}\|_{L_\infty[a_n n^{-2},a_n]}W_\rho^{-1}(x_{jn})\le C_1\end{aligned}$$

with C_1 independent of j, n. Then for some ξ between x_{jn} and $x_{j+1,n}$,

$$\begin{aligned}1 &= (\ell_{jn}W)(x_{jn})W^{-1}(x_{jn}) - (\ell_{jn}W)(x_{j+1,n})W^{-1}(x_{jn})\\&= (\ell_{jn}W)'(\xi)W^{-1}(x_{jn})(x_{jn} - x_{j+1,n})\\&\le C\varphi_n(\xi)^{-1}\xi^{-\rho}x_{jn}^{\rho}(x_{jn} - x_{j+1,n}).\end{aligned}$$

Since Lemma 2.3(f) shows that

$$\varphi_n(x_{jn}) \sim \varphi_n(\xi) \sim \varphi_n(x_{j+1,n}) \text{ and } x_{jn} \sim \xi$$

(cf. (6.14)) we obtain the required lower bound. □

Proof of Theorem 1.3(d) From Lemma 6.5, for $x \in [x_{j+1,n}, x_{jn}]$, (and recall, if necessary, the expression (6.6) for ℓ_{jn})

$$\begin{aligned}|p_{n,\rho}W_\rho|(x)&\left\{\frac{1}{|x-x_{jn}||p'_{n,\rho}W_\rho|(x_{jn})}\right.\\&\quad\left.+\frac{1}{|x-x_{j+1,n}||p'_{n,\rho}W_\rho|(x_{j+1,n})}\right\}\sim 1.\end{aligned}\tag{6.15}$$

Now we know from Lemma 2.3 that

$$\varphi_n(x_{jn}) \sim \varphi_n(x_{j+1,n}).$$

We also claim that uniformly in j and n,

$$a_n - x_{jn} \sim a_n - x_{j+1,n}.\tag{6.16}$$

Once we have this claim, Theorem 1.3(a) and (6.14) give

$$|p'_{n,\rho}W_\rho|(x_{jn}) \sim |p'_{n,\rho}W_\rho|(x_{j+1,n})$$

so we obtain

$$\begin{aligned}|p_{n,\rho}W_\rho|(x) &\sim |p'_{n,\rho}W_\rho|(x_{jn})\left\{\frac{1}{|x-x_{jn}|}+\frac{1}{|x-x_{j+1,n}|}\right\}^{-1}\\&\sim |p'_{n,\rho}W_\rho|(x_{jn})\min\{|x-x_{jn}|, |x-x_{j+1,n}|\}.\end{aligned}$$

Substituting in (1.13), gives (1.16). We turn to the proof of (6.16): from (5.6),

$$0 \leq 1 - \frac{a_n - x_{jn}}{a_n - x_{j+1,n}} = \frac{x_{jn} - x_{j+1,n}}{a_n - x_{j+1,n}} \leq C \frac{\varphi_n\left(x_{j+1,n}\right)}{a_n - x_{j+1,n}}.$$

If $x_{j+1,n} \leq a_{n/2}$, we continue this using (2.18) as

$$\leq C \frac{\sqrt{x_{j+1,n}}}{n}\left(a_n - x_{j+1,n}\right)^{-1/2} \leq C \frac{\sqrt{x_{j+1,n}}}{n} \sqrt{\frac{T\left(a_n\right)}{a_n}} \leq C.$$

If $x_{j+1,n} > a_{n/2}$, we continue this using (2.19) as

$$\leq \frac{C}{\left(a_n - x_{j+1,n}\right)^{3/2}} \frac{a_n^{3/2}}{nT\left(a_n\right)} \leq \frac{C}{\left(a_n \eta_n\right)^{3/2}} \frac{a_n^{3/2}}{nT\left(a_n\right)} = C. \qquad \square$$

7 Sharpness of the Markov Inequality

We prove the sharpness of the Markov inequality (1.21) in L_2 and with $\beta = 0$:

Theorem 7.1 *Let $\{p_n\}$ denote the orthonormal polynomials for the weight W^2, where $W \in \mathcal{L}\left(C^2+\right)$. Then for $n \geq 1$,*

$$\|p_n'' W\|_{L_2(I)} \sim \frac{n^2}{a_n} \|p_n' W\|_{L_2(I)}. \tag{7.1}$$

Proof. By the Gauss quadrature formula, and then (1.13) and (5.3), followed by (1.12),

$$\begin{aligned}
\|p_n' W\|_{L_2(I)}^2 &= \sum_{j=1}^{n} \lambda_{jn} p_n'\left(x_{jn}\right)^2 \\
&\sim \sum_{j=1}^{n} \varphi_n^{-1}\left(x_{jn}\right)\left[x_{jn}\left(a_n - x_{jn}\right)\right]^{-1/2} \\
&\sim n \sum_{j=1}^{n} \frac{1}{x_{jn}\left(a_{2n} - x_{jn}\right)} \\
&= \frac{n}{a_{2n}} \sum_{j=1}^{n}\left(\frac{1}{x_{jn}} + \frac{1}{a_{2n} - x_{jn}}\right) \\
&=: \Sigma_1 + \Sigma_2.
\end{aligned} \tag{7.2}$$

Here

$$\Sigma_1 = -\frac{n}{a_{2n}} \frac{p_n'(0)}{p_n(0)}.$$

By our Markov inequality (1.21) and our bound (1.8) on p_n,

$$|p_n'(0)| \leq C\frac{n^2}{a_n}\|p_n W\|_{L_\infty(I)} \leq C\frac{n^{5/2}}{a_n^{3/2}},$$

while as $|p_n|$ is convex in $(-\infty, x_{nn})$,

$$|p_n(0)| \geq |p_n'(x_{nn})|\, x_{nn} \sim \sqrt{\frac{n}{a_n}},$$

by (1.13) and (5.5). Then

$$|\Sigma_1| \leq C\frac{n^3}{a_n^2}.$$

In the other direction, we can use just a single term in Σ_1 :

$$\Sigma_1 \geq \frac{n}{a_{2n}}\frac{1}{x_{nn}} \sim \frac{n^3}{a_n^2}.$$

Thus

$$\Sigma_1 \sim \frac{n^3}{a_n^2}. \tag{7.3}$$

Next, we estimate Σ_2. Let

$$x_{0n} = x_{1n} + a_n\eta_n.$$

By our spacing of zeros, namely (1.17),

$$\begin{aligned}\Sigma_2 &\sim \frac{n}{a_n}\sum_{j=1}^{n}(x_{j-1,n} - x_{jn})\frac{\varphi_n^{-1}(x_{jn})}{a_{2n} - x_{jn}} \\ &\leq C\frac{n}{a_n}\int_0^{x_{0n}}\frac{dx}{\varphi_n(x)(a_{2n} - x)},\end{aligned}$$

recall (2.20). We continue thus using (1.12) as

$$\begin{aligned}\Sigma_2 &\sim \frac{n^2}{a_n}\int_0^{x_{0n}}\frac{\sqrt{a_{2n} - x + a_n\eta_n}}{(a_{2n} - x)^2}\frac{dx}{\sqrt{x + a_n n^{-2}}} \\ &\leq C\frac{n^2}{a_n}\left[\begin{array}{c} a_n^{-3/2}\int_0^{\frac{1}{2}a_n}\frac{dx}{\sqrt{x+a_n n^{-2}}} + a_n^{-1/2}\int_{\frac{1}{2}a_n}^{a_n/2}\frac{dx}{(a_n - x)^{3/2}} \\ +a_n^{-1/2}\int_{a_{n/2}}^{x_{0n}}\frac{\sqrt{a_{2n}-a_{n/2}}}{(a_{2n}-x)^2}dx \end{array}\right] \\ &\leq C\frac{n^2}{a_n}\left[a_n^{-1} + a_n^{-1/2}\left(a_n - a_{n/2}\right)^{-1/2} + a_n^{-1/2}\frac{\sqrt{a_{2n} - a_{n/2}}}{a_{2n} - a_n}\right] \\ &\leq C\frac{n^2}{a_n^2}T(a_n)^{1/2} = o\left(\frac{n^3}{a_n^2}\right).\end{aligned}$$

Here we have used (2.9) and (2.6). This last relation, (7.2) and (7.3) give

$$\|p_n'W\|_{L_2(I)}^2 \sim \frac{n^3}{a_n^2}. \tag{7.4}$$

Now we obtain a lower bound for the norm of $p_n''W$. As p_n' has opposite sign at x_{nn} and $x_{n-1,n}$, and W^{-1} is bounded near 0,

$$\begin{aligned}\left|p_n'\left(x_{nn}\right)\right| < \left|p_n'\left(x_{nn}\right) - p_n'\left(x_{n-1,n}\right)\right| &= \left|\int_{x_{nn}}^{x_{n-1,n}} p_n''\right| \\ &\leq C\left(x_{n-1,n} - x_{nn}\right)^{1/2}\left(\int_{x_{nn}}^{x_{n-1,n}} \left(p_n''W\right)^2\right)^{1/2} \\ &< C\left(x_{n-1,n} - x_{nn}\right)^{1/2}\|p_n''W\|_{L_2(I)}\end{aligned}$$

so

$$\begin{aligned}\|p_n''W\|_{L_2(I)}^2 &> C\left|p_n'\left(x_{nn}\right)\right|^2\left(x_{n-1,n} - x_{nn}\right)^{-1} \\ &\geq C\varphi_n\left(x_{nn}\right)^{-3}\left(x_{nn}a_n\right)^{-1/2} \sim n^7 a_n^{-4},\end{aligned}$$

by (1.13), (1.17) and (1.12). This and (7.4) give

$$\|p_n''W\|_{L_2(I)}^2 / \|p_n'W\|_{L_2(I)}^2 \geq Cn^4 a_n^{-2},$$

that is,

$$\|p_n''W\|_{L_2(I)} \geq C\frac{n^2}{a_n}\|p_n'W\|_{L_2(I)}^2.$$

The converse direction is an immediate consequence of the Markov inequality (1.21). □

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