BOUNDS ON ORTHOGONAL POLYNOMIALS AND SEPARATION OF THEIR ZEROS

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Abstract. Let $\{p_n\}$ denote the orthonormal polynomials associated with a measure μ with compact support on the real line. Let μ be regular in the sense of Stahl, Totik, and Ullmann, and I be a subinterval of the support in which μ is absolutely continuous, while μ' is positive and continuous there. We show that boundedness of the $\{p_n\}$ in that subinterval is closely related to the spacing of zeros of p_n and p_{n-1} in that interval. One ingredient is proving that "local limits" imply universality limits.

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1. Results

Let μ be a finite positive Borel measure with compact support, which we denote by supp$[\mu]$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

The zeros of p_n are real and simple. We list them in decreasing order:

$$x_{1n} > x_{2n} > ... > x_{n-1,n} > x_{nn}.$$

They interlace the zeros y_{jn} of p_n' :

$$p_n'(y_{jn}) = 0 \text{ and } y_{jn} \in (x_{j+1,n}, x_{jn}), \, 1 \leq j \leq n-1.$$

It is a classic result that the zeros of p_n and p_{n-1} also interlace. The three term recurrence relation has the form

$$(x - b_n) p_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x),$$

where for $n \geq 1$,

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} = \int x p_{n-1}(x) p_n(x) d\mu(x); \; b_n = \int x p_n^2(x) d\mu(x).$$

Uniform boundedness of orthonormal polynomials is a long studied topic. For example, given an interval I, one asks whether

$$\sup_{n\geq 1} \|p_n\|_{L_\infty(I)} < \infty.$$

There is an extensive literature on this fundamental question - see for example [1], [2], [3], [4], [12]. In this paper, we establish a connection to the distance between zeros of p_n and p_{n-1}.

The results require more terminology: we let $dist\,(a, \mathbb{Z})$ denote the distance from a real number a to the integers. We say that μ is *regular* (in the sense of Stahl, Totik, and Ullmann) if for every sequence of non-zero polynomials $\{P_n\}$ with degree P_n at most n,

$$\limsup_{n\to\infty} \left(\frac{|P_n(x)|}{\left(\int |P_n|^2 \, d\mu\right)^{1/2}} \right)^{1/n} \leq 1$$

for quasi-every $x \in$supp$[\mu]$ (that is except in a set of logarithmic capacity 0). If the support consists of finitely many intervals, and $\mu' > 0$ a.e. in each subinterval, then μ is regular, though much less is required [15]. An equivalent formulation involves the leading coefficients $\{\gamma_n\}$ of the orthonormal polynomials for μ :

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{cap\,(\text{supp}\,[\mu])},$$

where cap denotes logarithmic capacity.

Recall that the equilibrium measure for the compact set supp$[\mu]$ is the probability measure that minimizes the energy integral

$$\int\int \log \frac{1}{|x-y|} d\nu\,(x)\, d\nu\,(y)$$

amongst all probability measures ν supported on supp$[\mu]$. If I is an interval contained in supp$[\mu]$, then the equilibrium measure is absolutely continuous in I, and moreover its density, which we denote throughout by ω, is positive and continuous in the interior I^o of I [13, p.216, Thm. IV.2.5]. Given sequences $\{x_n\}, \{y_n\}$ of non-0 real numbers, we write

$$x_n \sim y_n$$

if there exists $C > 1$ such that for $n \geq 1$,

$$C^{-1} \leq x_n/y_n < C.$$

Similar notation is used for functions and sequences of functions.

Our main result is

Theorem 1.1
Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $A > 0$. The following are equivalent:
(a) There exists $C > 0$ such that for $n \geq 1$ and $x_{jn} \in I$,

$$dist\left(n\omega\left(x_{jn}\right)\left(x_{jn} - x_{j,n-1}\right), \mathbb{Z}\right) \geq C. \tag{1.1}$$

(b) There exists $C > 0$ such that for $n \geq 1$ and $y_{jn} \in I$,

$$dist\left(n\omega\left(y_{jn}\right)\left(y_{jn} - y_{j,n-1}\right), \mathbb{Z}\right) \geq C. \tag{1.2}$$

(c) Uniformly for $n \geq 1$ and $x \in I$,

$$\left\|p_{n-1}\right\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]}\left\|p_n\right\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \sim 1. \tag{1.3}$$

(d) There exists $C > 0$ such that for $n \geq 1$ and $x \in I$,

$$\left\|p_{n-1}\right\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]}\left\|p_n\right\|_{L_\infty[x-\frac{A}{n},x+\frac{A}{n}]} \leq C. \tag{1.4}$$

Moreover, under any of (a), (b), (c), (d), we have

$$\sup_{n\geq 1}\sup_{x\in I}\left|\left|x-b_n\right|^{1/2} p_n\left(x\right)\right| < \infty. \tag{1.5}$$

Remarks
(a) The main idea behind the proof is that universality limits and "local" limits give

$$\left|p_{n-1}\left(y_{j,n-1}\right)p_n\left(y_{jn}\right)\right|\left|\sin\left[\pi n\omega\left(y_{jn}\right)\left(y_{jn}-y_{j,n-1}\right)\right]+o\left(1\right)\right| \sim 1,$$

uniformly in j, n, while p_n has a local extremum at y_{jn}.
(b) We could replace $x_{j,n-1} - x_{jn}$ in (1.1) by $x_{j,n-1} - x_{j,n+k}$, for any fixed integer k (see Lemma 4.1).
(b) Under additional assumptions, involving the spacing of zeros of p_n and p_{n-2}, we can remove the factor $\left|x-b_n\right|^{1/2}$ in (1.5):

Theorem 1.2
Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Let ω be the density of the equilibrium measure for the support of μ. Let $A > 0$. Assume that (1.1) holds in I. The following are equivalent:
(a) There exist $C_1 > 0$ such that for $n \geq 1$ and $x_{jn} \in I$,

$$\left|n\left(x_{jn} - x_{j-1,n-2}\right)\right| \geq C_1\left|x_{jn} - b_{n-1}\right|. \tag{1.6}$$

(b) Uniformly for $x \in I$ and $n \geq 1$,

$$\|p_n\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \sim 1. \tag{1.7}$$

(c)

$$\sup_{n\geq 1} \|p_n\|_{L_\infty(I)} < \infty. \tag{1.8}$$

Remark

We note that because of the interlacing, both x_{jn} and $x_{j-1,n-2}$ belong to the interval $(x_{j,n-1}, x_{j-1,n-1})$.

Two important ingredients in our proofs are universality and local limits. The so-called universality limit involves the reproducing kernel

$$K_n(x,y) = \sum_{k=0}^{n-1} p_k(x)\, p_k(y) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(y) - p_{n-1}(x)\, p_n(y)}{x-y}.$$

(1.9)

For x in the interior of supp$[\mu]$ (the "bulk" of the support), at least when $\mu'(x)$ is finite and positive, the universality limit typically takes the form [6], [8], [14], [17]

$$\lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{\mu'(x)K_n(x,x)}, x+\frac{b}{\mu'(x)K_n(x,x)}\right)}{K_n(x,x)} = \mathbb{S}(a-b), \tag{1.10}$$

uniformly for a, b in compact subsets of $\mathbb{C}$. Here $\mathbb{S}$ is the sinc kernel,

$$\mathbb{S}(a) = \frac{\sin \pi a}{\pi a}.$$

Universality limits holds far more generally than pointwise asymptotics for orthonormal polynomials, that at one stage were used to prove them. In a series of recent papers [7], [9], [10], [11], it was shown that one can go in the other direction, namely from universality limits, to "local ratio limits" for orthogonal polynomials.

Under fairly general conditions on μ, the *Christoffel function* $K_n(x,x)$ admits the asymptotic [16]

$$\lim_{n\to\infty} \frac{1}{n} K_n(x,x)\, \mu'(x) = \omega(x)$$

for x in the interior of the support of μ. This allows us to reformulate the universality limit (1.10) as

$$\lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{n\omega(x)}, x+\frac{b}{n\omega(x)}\right)\mu'(x)}{n\omega(x)} = \mathbb{S}(a-b), \tag{1.11}$$

uniformly for a, b in compact subsets of $\mathbb{C}$.

Using this universality limit, we proved in [10]:

Theorem A

Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Then

$$\lim_{n\to\infty} \frac{p_n\left(y_{jn} + \frac{z}{n\omega(y_{jn})}\right)}{p_n\left(y_{jn}\right)} = \cos \pi z \tag{1.12}$$

uniformly for $y_{jn} \in J$ and z in compact subsets of $\mathbb{C}$.

A secondary goal of this paper is to prove a converse of Theorem A, namely to show that local limits such as (1.12) imply a universality limit like (1.11). For measures on the unit circle this was undertaken in [11] - however the results necessarily take a quite different form.

Theorem 1.3

Let μ be a measure with compact support. Assume that we are given a bounded sequence of real numbers $\{\xi_n\}$ such that

$$\sup_{n\geq 1} n \left|\xi_n - \xi_{n-1}\right| < \infty, \tag{1.13}$$

and a sequence $\{\tau_n\}$ of positive numbers with $\tau_n \sim 1$ such that

$$\lim_{n\to\infty} \frac{\tau_n}{\tau_{n-1}} = 1 \tag{1.14}$$

and uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{p_n\left(\xi_n + \frac{\tau_n}{n} z\right)}{p_n\left(\xi_n\right)} = \cos \pi z. \tag{1.15}$$

Let $A > 0$. Then uniformly for a, b in compact subsets of $\mathbb{C}$, and x_n such that

$$|x_n - \xi_n| \leq \frac{A}{n} \tag{1.16}$$

we have

$$\frac{K_n\left(x_n + \frac{\tau_n}{n}a, x_n + \frac{\tau_n}{n}b\right)}{K_n\left(x_n, x_n\right)} = \mathbb{S}\left(a-b\right) + o\left(\frac{\frac{\gamma_{n-1}}{\gamma_n} n \left|p_{n-1}\left(\xi_{n-1}\right) p_n\left(\xi_n\right)\right|}{K_n\left(x_n, x_n\right)}\right). \tag{1.17}$$

Moreover,

$$K_n\left(x_n+\tfrac{\tau_n}{n}a, x_n+\tfrac{\tau_n}{n}b\right) / K_n\left(x_n,x_n\right) = \mathbb{S}\left(a-b\right)+o\left(1\right), \tag{1.18}$$

provided either

$$\liminf_{n\to\infty}\ \text{dist}\left(\frac{n}{\tau_n}\left(\xi_n-\xi_{n-1}\right),\mathbb{Z}\right)>0 \tag{1.19}$$

or

$$\sup_{n\geq 1}\frac{\frac{\gamma_{n-1}}{\gamma_n}n\left|p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\right|}{K_n\left(x_n,x_n\right)}<\infty. \tag{1.20}$$

We prove Theorem 1.3 in the next section and Theorem 1.1 in Section 3. Theorem 1.2 is proved in Section 4. In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, θ. The same symbol does not necessarily denote the same constant in different occurences.

2. Proof of Theorem 1.3

Throughout this section, we assume the hypotheses of Theorem 1.3. Write for $n\geq 1$ and $m=n-1,n$,

$$x_n=\xi_m+\Delta_{n,m}\frac{\tau_m}{m} \tag{2.1}$$

and

$$\chi_n=\left(\frac{\tau_n}{n}\right)/\left(\frac{\tau_{n-1}}{n-1}\right). \tag{2.2}$$

Recall from (1.14) that $\chi_n\to 1$ as $n\to\infty$. Note too that in view of (1.13), (1.14), (1.16), $\{\Delta_{n,n}\}$ and $\{\Delta_{n,n-1}\}$ are bounded sequences. We start with:

Lemma 2.1
(a) Uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{\tau_n}{n}\frac{p_n'\left(\xi_n+\frac{\tau_n}{n}z\right)}{p_n\left(\xi_n\right)}=-\pi\sin\pi z. \tag{2.3}$$

(b) Uniformly for a,b in compact subsets of $\mathbb{C}$,

$$\begin{aligned}&\left(p_n\left(x_n+\frac{\tau_n}{n}a\right)-p_n\left(x_n+\frac{\tau_n}{n}b\right)\right)/p_n\left(\xi_n\right)\\ = &\ -\pi\int_b^a\sin\pi\left(\Delta_{n,n}+t\right)\ dt+o\left(\left|a-b\right|\right).\end{aligned}$$

(c) Moreover,

$$\begin{aligned} & \left(p_{n-1}\left(x_n+\frac{\tau_n}{n}a\right)-p_{n-1}\left(x_n+\frac{\tau_n}{n}b\right)\right)/p_{n-1}\left(\xi_{n-1}\right) \\ = & -\pi\int_b^a \sin\pi\left(\Delta_{n,n-1}+t\right)\ dt+o\left(|a-b|\right). \end{aligned}$$

Proof

(a) As the asymptotic (1.15) holds uniformly for z in compact subsets of the plane, we can differentiate it to obtain (2.3).

(b) Now

$$\begin{aligned} & \left(p_n\left(x_n+\frac{\tau_n}{n}a\right)-p_n\left(x_n+\frac{\tau_n}{n}b\right)\right)/p_n\left(\xi_n\right) \\ = & \int_b^a p_n'\left(x_n+\frac{\tau_n}{n}t\right)\frac{\tau_n}{n}dt/p_n\left(\xi_n\right). \end{aligned}$$

Note that this is meaningful even for complex a, b, with the integral being taken over the directed line segment from b to a. Using (2.1) and (2.3), we continue this as

$$\begin{aligned} & \int_b^a \frac{p_n'\left(\xi_n+\frac{\tau_n}{n}\left(\Delta_{n,n}+t\right)\right)\frac{\tau_n}{n}}{p_n\left(\xi_n\right)}dt \\ = & \int_b^a \left(-\pi\sin\pi\left(\Delta_{n,n}+t\right)+o\left(1\right)\right)dt \\ = & -\pi\int_b^a \sin\pi\left(\Delta_{n,n}+t\right)\ dt+o\left(|a-b|\right). \end{aligned}$$

(c) Using (2.2),

$$\begin{aligned} & \left(p_{n-1}\left(x_n+\frac{\tau_n}{n}a\right)-p_{n-1}\left(x_n+\frac{\tau_n}{n}b\right)\right)/p_{n-1}\left(\xi_{n-1}\right) \\ = & \int_b^a p_{n-1}'\left(x_n+\frac{\tau_n}{n}t\right)\frac{\tau_n}{n}dt/p_{n-1}\left(\xi_{n-1}\right) \\ = & \int_b^a \frac{p_{n-1}'\left(\xi_{n-1}+\frac{\tau_{n-1}}{n-1}\left(\Delta_{n,n-1}+\chi_n t\right)\right)}{p_{n-1}\left(\xi_{n-1}\right)}\frac{\tau_{n-1}}{n-1}\chi_n dt \\ = & \int_b^a \left(-\pi\sin\left(\pi\left(\Delta_{n,n-1}+\chi_n t\right)\right)+o\left(1\right)\right)dt \\ = & -\pi\int_b^a \sin\pi\left(\Delta_{n,n-1}+t\right)\ dt+o\left(|a-b|\right). \end{aligned}$$

■

Proof of Theorem 1.3

We apply (1.15) and (b), (c) of Lemma 2.1. Now if $a \neq b$,

$$\begin{aligned}
& \frac{\tau_n}{np_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)}K_n\left(x_n+\frac{\tau_n}{n}a, x_n+\frac{\tau_n}{n}b\right) \\
= & \frac{\gamma_{n-1}}{\gamma_n}\frac{\left[p_n\left(x_n+\frac{\tau_n}{n}a\right)-p_n\left(x_n+\frac{\tau_n}{n}b\right)\right]p_{n-1}\left(x_n+\frac{\tau_n}{n}b\right)}{(a-b)\,p_n\left(\xi_n\right)p_{n-1}\left(\xi_{n-1}\right)} \\
& +\frac{\gamma_{n-1}}{\gamma_n}\frac{p_n\left(x_n+\frac{\tau_n}{n}b\right)\left[p_{n-1}\left(x_n+\frac{\tau_n}{n}b\right)-p_{n-1}\left(x_n+\frac{\tau_n}{n}a\right)\right]}{(a-b)\,p_n\left(\xi_n\right)p_{n-1}\left(\xi_{n-1}\right)} \\
= & \frac{\gamma_{n-1}}{\gamma_n}\left[\frac{-\pi}{a-b}\int_b^a \sin\pi\left(\Delta_{n,n}+t\right)\,dt+o(1)\right]\left[\cos\pi\left(\Delta_{n,n-1}+b\chi_n\right)+o(1)\right] \\
& +\frac{\gamma_{n-1}}{\gamma_n}\left[\cos\pi\left(\Delta_{n,n}+b\right)+o(1)\right]\left[\frac{\pi}{a-b}\int_b^a \sin\pi\left(\Delta_{n,n-1}+t\right)\,dt+o(1)\right]
\end{aligned}$$

by (1.15) and (b), (c) of Lemma 2.1. Note that because of the uniformity of the limits, this holds in a confluent form even if $a=b$. We continue this, using $\chi_n = 1+o(1)$, as

$$\begin{aligned}
= & \frac{\gamma_{n-1}}{\gamma_n}\frac{\pi}{a-b}\int_a^b\left[\sin\pi\left(\Delta_{n,n}+t\right)\cos\pi\left(\Delta_{n,n-1}+b\right)-\cos\pi\left(\Delta_{n,n}+b\right)\sin\pi\left(\Delta_{n,n-1}+t\right)\right]dt \\
& +o\left(\frac{\gamma_{n-1}}{\gamma_n}\right).
\end{aligned}$$

(2.4)

Next, we expand the integrand using double angle formulae, in a straightforward but tedious fashion:

$$\begin{aligned}
& \sin\pi\left(\Delta_{n,n}+t\right)\cos\pi\left(\Delta_{n,n-1}+b\right)-\cos\pi\left(\Delta_{n,n}+b\right)\sin\pi\left(\Delta_{n,n-1}+t\right) \\
= & \left[\sin\pi\Delta_{n,n}\cos\pi t+\cos\pi\Delta_{n,n}\sin\pi t\right]\left[\cos\pi\Delta_{n,n-1}\cos\pi b-\sin\pi\Delta_{n,n-1}\sin\pi b\right] \\
& -\left[\cos\pi\Delta_{n,n}\cos\pi b-\sin\pi\Delta_{n,n}\sin\pi b\right]\left[\sin\pi\Delta_{n,n-1}\cos\pi t+\cos\pi\Delta_{n,n-1}\sin\pi t\right] \\
= & \cos\pi t\cos\pi b\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right)+\sin\pi t\sin\pi b\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right) \\
= & \cos\left(\pi\left(t-b\right)\right)\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right).
\end{aligned}$$

We can then continue (2.4) as

$$\begin{aligned}
& \frac{\gamma_{n-1}}{\gamma_n}\frac{\pi}{a-b}\int_a^b\left[\cos\left(\pi\left(t-b\right)\right)\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right)\right]dt+o\left(\frac{\gamma_{n-1}}{\gamma_n}\right) \\
= & \frac{\gamma_{n-1}}{\gamma_n}\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right)\frac{1}{a-b}\left(-\sin\pi\left(a-b\right)\right)+o\left(\frac{\gamma_{n-1}}{\gamma_n}\right) \\
= & -\pi\frac{\gamma_{n-1}}{\gamma_n}\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right)\mathbb{S}\left(a-b\right)+o\left(\frac{\gamma_{n-1}}{\gamma_n}\right).
\end{aligned}$$

In summary, uniformly for a,b in compact subsets of the plane,

$$\begin{aligned}&\frac{\tau_n}{np_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)}K_n\left(x_n+\frac{\tau_n}{n}a,x_n+\frac{\tau_n}{n}b\right)\\ &= -\pi\frac{\gamma_{n-1}}{\gamma_n}\sin\pi\left(\Delta_{n,n}-\Delta_{n,n-1}\right)\mathbb{S}\left(a-b\right)+o\left(\frac{\gamma_{n-1}}{\gamma_n}\right). \qquad (2.5)\end{aligned}$$

Next observe from (2.1), (1.13), and (1.14), that

$$x_n=\xi_n+\Delta_{n,n}\frac{\tau_n}{n}=\xi_{n-1}+\Delta_{n,n-1}\frac{\tau_n}{n}+o\left(\frac{1}{n}\right)$$

$$\Rightarrow\frac{\tau_n}{n}\left[\Delta_{n,n}-\Delta_{n,n-1}\right]=\xi_{n-1}-\xi_n+o\left(\frac{1}{n}\right).$$

As τ_n is bounded below, this allows us to reformulate (2.5) as

$$\begin{aligned}&\frac{\tau_n}{n}K_n\left(x_n+\frac{\tau_n}{n}a,x_n+\frac{\tau_n}{n}b\right)\\ &= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\left\{\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]\mathbb{S}\left(a-b\right)+o\left(1\right)\right\}. \qquad (2.6)\end{aligned}$$

In particular, setting $a=b=0$,

$$\begin{aligned}&\frac{\tau_n}{n}K_n\left(x_n,x_n\right)\\ &= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\left\{\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]+o\left(1\right)\right\}, \qquad (2.7)\end{aligned}$$

so that (2.6) can be recast as

$$\begin{aligned}&\frac{\tau_n}{n}K_n\left(x_n+\frac{\tau_n}{n}a,x_n+\frac{\tau_n}{n}b\right)\\ &= \frac{\tau_n}{n}K_n\left(x_n,x_n\right)\mathbb{S}\left(a-b\right)+o\left(\frac{\gamma_{n-1}}{\gamma_n}\left|p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\right|\right),\end{aligned}$$

giving (1.17). If (1.19) holds, then $\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]$ is bounded away from 0, so we can reformulate (2.6) as

$$\begin{aligned}&\frac{\tau_n}{n}K_n\left(x_n+\frac{\tau_n}{n}a,x_n+\frac{\tau_n}{n}b\right)\\ &= -\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]\left\{\mathbb{S}\left(a-b\right)+o\left(1\right)\right\}\end{aligned}$$

and (2.7) as

$$\begin{aligned}&\frac{\tau_n}{n}K_n\left(x_n,x_n\right)\\=\ &-\pi\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(\xi_{n-1}\right)p_n\left(\xi_n\right)\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]\left\{1+o\left(1\right)\right\}.\end{aligned}$$

Together these give (1.18). Finally if (1.20) holds, then we see from (2.6) that necessarily $\sin\left[\pi\frac{n}{\tau_n}\left(\xi_{n-1}-\xi_n\right)\right]$ is bounded away from 0 and again (1.18) follows. ■

3. Proof of Theorem 1.1

Recall that y_{jn} is the zero of p_n' in $(x_{j+1,n},x_{jn})$. We begin with:

Lemma 3.1
Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous.
(a) Uniformly for $y_{jn}\in I$,

$$\lim_{n\to\infty}n\left(x_{jn}-y_{jn}\right)\omega\left(x_{jn}\right)=\frac{1}{2}=\lim_{n\to\infty}n\left(y_{jn}-x_{j+1,n}\right)\omega\left(x_{jn}\right); \tag{3.1}$$

$$\lim_{n\to\infty}n\left(x_{jn}-x_{j+1,n}\right)\omega\left(x_{jn}\right)=1; \tag{3.2}$$

$$\lim_{n\to\infty}n\left(y_{jn}-y_{j+1,n}\right)\omega\left(x_{jn}\right)=1. \tag{3.3}$$

(b) Uniformly for $y_{jn}\in I$,

$$\frac{\gamma_{n-1}}{\gamma_n}\left|p_{n-1}\left(y_{j,n-1}\right)p_n\left(y_{jn}\right)\right|\left|\sin\left[\pi n\omega\left(y_{jn}\right)\left(y_{j,n-1}-y_{jn}\right)\right]+o\left(1\right)\right|\sim1. \tag{3.4}$$

(c) Fix $A>0$. Uniformly for $n\geq1$ and $x\in I$,

$$\left\|p_n\right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\sim\left|p_n\left(y_{jn}\right)\right|, \tag{3.5}$$

where $y_{jn}\in\left[x-\frac{A}{n},x+\frac{A}{n}\right]$ or is the closest zero of p_n' to this interval.

Proof

(a) First note that uniformly for $y_{jn}\in I$ and z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{p_n\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)}{p_n\left(y_{jn}\right)}=\cos\pi z. \tag{3.6}$$

This was proved in [10] and is Theorem A above. Next, Theorem 2.1 in [17] shows that (3.2) holds uniformly for $x_{jn} \in I$. In particular $x_{jn} - x_{j+1,n} = O\left(\frac{1}{n}\right)$ uniformly for $x_{jn} \in I$. Write

$$x_{jn} = y_{jn} + \frac{z_n}{n\omega\left(y_{jn}\right)},$$

so that $z_n > 0$ and $z_n = O(1)$. From (3.6), we have

$$0 = \frac{p_n\left(x_{jn}\right)}{p_n\left(y_{jn}\right)} = \cos \pi z_n + o(1)$$

so necessarily for some non-negative integer j_n,

$$z_{jn} = j_n + \frac{1}{2} + o(1).$$

If $j_n \geq 1$ for infinitely many n, then Hurwitz' Theorem shows that there would be other zeros of p_n between x_{jn} and y_{jn}, which contradicts that $y_{jn} \in (x_{j+1,n}, x_{jn})$. So $j_n = 0$ for n large enough, which gives the first limit in (3.1). Note too that $\omega\left(x_{jn}\right)/\omega\left(y_{jn}\right) = 1 + o(1)$ by continuity of ω. The second is similar. Both (3.2) and (3.3) follow from (3.1), though as noted, (3.2) appears in [17].

(b) Because of (3.6), we can apply Theorem 1.3 and results from its proof. In that theorem, we set $x_n = y_{jn}$, $\tau_n = \frac{1}{\omega(y_{jn})}$; $\xi_n = y_{jn}$; so that $\xi_{n-1} = y_{j,n-1}$. Note that (1.13), (1.14), (1.16) are satisfied because of the spacing estimates in Lemma 3.1, and the continuity of ω. From (2.7),

$$\begin{aligned} & \frac{1}{n\omega\left(y_{jn}\right)} K_n\left(y_{jn}, y_{jn}\right) \\ = \quad & -\pi \frac{\gamma_{n-1}}{\gamma_n} p_{n-1}\left(y_{j,n-1}\right) p_n\left(y_{jn}\right) \left\{\sin\left[\pi n \omega\left(y_{jn}\right)\left(y_{j,n-1} - y_{jn}\right)\right] + o(1)\right\}. \end{aligned}$$

(3.7)

Next, Theorem 2.2 in [17] establishes that uniformly for $t \in I$,

$$\lim_{n\to\infty} \frac{1}{n} K_n(t,t)\,\mu'(t) = \omega(t).$$

Since ω is positive and continuous in I as is μ', we then obtain (3.4) from (3.7).

(c) This follows directly from the limit in (3.6) and the fact that $|p_n(y_{jn})|$ is the maximum of $|p_n|$ in $[x_{j+1,n}, x_{jn}]$. ■

Proof that Theorem 1.1(a)⇔(b)

This follows directly from Lemma 3.1(a). ■

Proof that Theorem 1.1(b)⇒(c).
First note that as supp[μ] is compact [5, p. 41],

$$\frac{\gamma_{n-1}}{\gamma_n} \leq C. \tag{3.8}$$

Our hypothesis (1.2), as well as (3.4) give that uniformly for $y_{jn} \in I$,

$$\frac{\gamma_{n-1}}{\gamma_n} \left| p_{n-1}\left(y_{j,n-1}\right) p_n\left(y_{jn}\right)\right| \sim 1. \tag{3.9}$$

Then (3.5) gives uniformly for $x \in I$,

$$\frac{\gamma_{n-1}}{\gamma_n} \left\| p_{n-1} \right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \left\| p_n \right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \sim 1. \tag{3.10}$$

Let $I_{jn} = [y_{j+1,n}, y_{jn}]$ for all j, n. We similarly obtain from (3.6) and (3.9) and our spacing that

$$\frac{\gamma_{n-1}}{\gamma_n} \left(\int_{I_{j,n-1}} p_{n-1}^2 d\mu \right)^{1/2} \left(\int_{I_{jn}} p_n^2 d\mu \right)^{1/2} \geq \frac{C}{n}.$$

Here we are also using that μ' is positive and continuous in I. Adding over $y_{jn} \in I$, and using that there are necessarily $\geq Cn$ such y_{jn}, because of the spacing, we obtain

$$\frac{\gamma_{n-1}}{\gamma_n} \sum_{y_{jn} \in I} \left(\int_{I_{j,n-1}} p_{n-1}^2 d\mu \right)^{1/2} \left(\int_{I_{jn}} p_n^2 d\mu \right)^{1/2} \geq C.$$

Cauchy-Schwarz' inequality gives

$$\frac{\gamma_{n-1}}{\gamma_n} \left(\int p_{n-1}^2 d\mu \int p_n^2 d\mu \right)^{1/2} \geq C$$

so that

$$\frac{\gamma_{n-1}}{\gamma_n} \geq C.$$

Together with (3.8), this gives

$$a_n = \frac{\gamma_{n-1}}{\gamma_n} \sim 1. \tag{3.11}$$

So from (3.10), uniformly in $x \in I$,

$$\left\| p_{n-1} \right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \left\| p_n \right\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \sim 1. \tag{3.12}$$

■

Proof that Theorem 1.1(c)⇒(d).
This is immediate. ■

Proof that Theorem 1.1(d)⇒(b).
From (3.4), (3.11), and our assumed bound (1.4),

$$\left|\sin\left[\pi n\omega\left(y_{jn}\right)\left(y_{j,n-1}-y_{jn}\right)\right]+o\left(1\right)\right|\geq C.$$

This yields

$$dist\left(n\omega\left(y_{jn}\right)\left(y_{jn}-y_{j,n-1}\right),\mathbb{Z}\right)\geq C.$$

■

Proof of the bound (1.5)
From the recurrence relation and (3.11),

$$\begin{aligned} & \left\|\left(x-b_{n}\right)p_{n}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\left\|p_{n}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \\ \leq\ & C\left(\left\|p_{n+1}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\left\|p_{n}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}+\left\|p_{n-1}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\left\|p_{n}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\right) \\ \leq\ & C, \end{aligned}$$

by (1.4). Then also uniformly in $x\in I$,

$$\left\|\left(x-b_{n}\right)p_{n}^{2}\right\|_{L_{\infty}\left[x-\frac{A}{n},x+\frac{A}{n}\right]}\leq C$$

and we obtain (1.5). ■

4. Proof of Theorem 1.2

We begin with:

Lemma 4.1
Let μ be a regular measure on $\mathbb{R}$ with compact support. Let I be a closed subinterval of the support and assume that in some open interval containing I, μ is absolutely continuous, while μ' is positive and continuous. Assume (1.1). Let $A>0$.
(a) Let $L\geq 1$. There exists n_0 such that uniformly for $n\geq n_0$, for $x_{jn}\in I$ and $|k-j|\leq L$,

$$dist\left(n\omega\left(x_{jn}\right)\left(x_{k,n-1}-x_{jn}\right),\mathbb{Z}\right)\geq C. \tag{4.1}$$

(b) Let

$$\delta_{jn}:=n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j-1,n-2}\right). \tag{4.2}$$

There exist $n_0,\eta_0>0$ such that uniformly for $n\geq n_0$, and for $x_{jn}\in I$,

$$\left|\delta_{jn}\right|\leq 1-\eta_{0}. \tag{4.3}$$

(c) There exist $n_0,C_1>0$ such that uniformly for $n\geq n_0$ and for $x_{jn}\in I$, we have

$$\left|x_{jn}-b_{n-1}\right|\sim\left\|p_{n-2}\right\|_{L_{\infty}\left[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}\right]}^{2}\left|\delta_{jn}\right|. \tag{4.4}$$

Here if $x_{jn} - b_{n-1} = 0$, *both sides are* 0.

Proof

(a) Using the spacing (3.2),

$$\begin{aligned} & dist\left(n\omega\left(x_{jn}\right)\left(x_{k,n-1}-x_{jn}\right),\mathbb{Z}\right) \\ = \ & dist\left(n\omega\left(x_{jn}\right)\left(x_{j,n-1}-x_{jn}\right),\mathbb{Z}\right)+o\left(1\right) \end{aligned}$$

so (1.1) gives the result.

(b) The interlacing of the zeros of successive orthogonal polynomials shows that both x_{jn} and $x_{j-1,n-2}$ lie in the interval $(x_{j,n-1}, x_{j-1,n-1})$. Even more, the bounds given in (a) show that for n large enough, both x_{jn} and $x_{j-1,n-2}$ lie in the interval $\left(x_{j,n-1}+\frac{C_1}{n\omega(x_{jn})}, x_{j-1,n-1}-\frac{C_1}{n\omega(x_{jn})}\right)$ for some $C_1 > 0$. Then

$$\begin{aligned} & \left|\delta_{jn}\right| = \left|n\omega\left(x_{jn}\right)\left(x_{jn}-x_{j-1,n-2}\right)\right| \\ \leq \ & n\omega\left(x_{jn}\right)\left(x_{j,n-1}-x_{j+1,n-1}\right)-2C_1 = 1-2C_1+o\left(1\right), \end{aligned}$$

by (3.2).

(c) From the recurrence relation,

$$\left(x_{jn}-b_{n-1}\right)p_{n-1}\left(x_{jn}\right)=a_{n-1}p_{n-2}\left(x_{jn}\right). \tag{4.5}$$

We now examine the behavior of the left and right-hand side as $n \to \infty$. By (3.1) to (3.3), the local asymptotic (3.6), and the fact that $x_{jn} - y_{j,n-1} = O\left(\frac{1}{n}\right)$,

$$\begin{aligned} \frac{p_{n-1}\left(x_{jn}\right)}{p_{n-1}\left(y_{j,n-1}\right)} &= \cos\pi\left(n\omega\left(y_{j,n-1}\right)\left(x_{jn}-y_{j,n-1}\right)\right)+o\left(1\right) \\ &= \cos\pi\left(n\omega\left(y_{j,n-1}\right)\left(x_{jn}-x_{j,n-1}+x_{j,n-1}-y_{j,n-1}\right)\right)+o\left(1\right) \\ &= \cos\pi\left(n\omega\left(y_{j,n-1}\right)\left(x_{jn}-x_{j,n-1}\right)+\frac{1}{2}\right)+o\left(1\right) \\ &= -\sin\pi\left(n\omega\left(y_{j,n-1}\right)\left(x_{jn}-x_{j,n-1}\right)\right)+o\left(1\right) \end{aligned}$$

so using our original condition (1.1), we obtain for some threshold n_0 that is independent of j, and for $n \geq n_0$,

$$\left|p_{n-1}\left(x_{jn}\right)\right| \sim \left|p_{n-1}\left(y_{j,n-1}\right)\right|. \tag{4.6}$$

Next, in analyzing the term on the right in (4.5), we use the differentiated form of (3.6): uniformly for $y_{jn} \in I$ and z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{p_n'\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)}{n\omega\left(y_{jn}\right)p_n\left(y_{jn}\right)}=-\pi\sin\pi z. \tag{4.7}$$

Then noting that we can replace n by $n \pm 2$ in the term involving z, we see that

$$\begin{aligned}\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})} &= \int_{(x_{j-1,n-2}-y_{j-1,n-2})n\omega(y_{j-1,n-2})}^{(x_{jn}-y_{j-1,n-2})n\omega(y_{j-1,n-2})} \frac{p'_{n-2}\left(y_{j-1,n-2}+\frac{t}{n\omega(y_{j-1,n-2})}\right)}{n\omega(y_{j-1,n-2})\,p_{n-2}(y_{j-1,n-2})}dt \\ &= \int_{(x_{j-1,n-2}-y_{j-1,n-2})n\omega(y_{j-1,n-2})}^{(x_{jn}-y_{j-1,n-2})n\omega(y_{j-1,n-2})} (-\pi\sin\pi t + o(1))\,dt.\end{aligned}$$

Here the lower limit of integration is

$$(x_{j-1,n-2}-y_{j-1,n-2})\,n\omega(y_{j-1,n-2}) = \frac{1}{2}+o(1),$$

(by (3.1)), so we can continue the above as

$$\begin{aligned}\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})} &= \int_0^{(x_{jn}-x_{j-1,n-2})n\omega(y_{j-1,n-2})} \left(-\pi\sin\left(\pi\left(t+\frac{1}{2}\right)\right)+o(1)\right)dt + o(\delta_{jn}) \\ &= \int_0^{(x_{jn}-x_{j-1,n-2})n\omega(y_{j-1,n-2})} (-\pi\cos\pi t + o(1))\,dt + o(\delta_{jn}) \\ &= -\sin\pi\delta_{jn} + o(\delta_{jn}).\end{aligned}$$

Here we are also using that $\omega(y_{j-1,n-2})/\omega(x_{jn}) \to 1$ as $n \to \infty$ by continuity of ω in the interior of I. Next, from (b), $|\delta_{jn}| \le 1-\varepsilon$, so $|\sin\pi\delta_{jn}| \sim |\delta_{jn}|$ and we can continue this as

$$\frac{p_{n-2}(x_{jn})}{p_{n-2}(y_{j-1,n-2})} = -(\sin\pi\delta_{jn})(1+o(1)).$$

It is possible here that $\delta_{jn}=0$, but in such a case both sides are 0. Combining this with (4.5), (4.6) and (3.11) gives uniformly in j and n, for $n \ge n_0$,

$$|x_{jn}-b_{n-1}|\,|p_{n-1}(y_{j,n-1})| \sim |p_{n-2}(y_{j-1,n-2})|\,|\sin\pi\delta_{jn}| \sim |p_{n-2}(y_{j-1,n-2})|\,|\delta_{jn}|.$$

Here by our local limits and (1.3),

$$|p_{n-1}(y_{j,n-1})| = \|p_{n-1}\|_{L_\infty[x_{j+1,n-1},x_{j,n-1}]} \sim \|p_{n-2}\|^{-1}_{L_\infty\left[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}\right]}.$$

A related assertion holds for $p_{n-2}(y_{j-1,n-2})$. We deduce that

$$|x_{jn}-b_{n-1}| \sim \|p_{n-2}\|^2_{L_\infty\left[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}\right]}\,|\delta_{jn}|.$$

Again if $x_{jn}=b_{n-1}$, $\delta_{jn}=0$. ■

Proof that Theorem 1.2(a)⇔(c)

If first (1.6) holds, then $|\delta_{jn}| \ge C\,|x_{jn}-b_{n-1}|$ and (4.4) gives

$$C\,|\delta_{jn}| \ge \|p_{n-2}\|^2_{L_\infty\left[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}\right]}\,|\delta_{jn}|,$$

which forces

$$\|p_{n-2}\|^2_{L_\infty\left[x_{jn}-\frac{A}{n},x_{jn}+\frac{A}{n}\right]} \le C_1,$$

uniformly in $x_{jn} \in I$, provided no $\delta_{jn} = 0$. Since $\delta_{jn} = 0$ can occur for at most one j, namely when $x_{jn} = b_{n-1}$ (as follows from the recurrence relation), that exceptional interval can be covered by others with A large enough. So we have (1.8).

Conversely, suppose we have (1.8). Then from (4.4),

$$|x_{jn} - b_{n-1}| \le C\,|\delta_{jn}|\,,$$

so that we have (1.6). ■

Proof that Theorem 1.2(b)⇔(c)

It is immediate that (b)⇒(c). For the converse we note that if (c) holds, then from Theorem 1.1(c),

$$\|p_{n-1}\|_{L_\infty\left[x-\frac{A}{n},x+\frac{A}{n}\right]} \ge C$$

uniformly for $x \in I$. This together with (1.8), gives (1.7). ■

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