Reflections on the Baker-Gammel-Wills (Padé) Conjecture

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Abstract In 1961, Baker, Gammel and Wills formulated their famous conjecture that if a function f is meromorphic in the unit ball, and analytic at 0, then a subsequence of its diagonal Padé approximants converges uniformly in compact subsets to f. This conjecture was disproved in 2001, but it generated a number of related unresolved conjectures. We review their status.

1 Introduction

Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j$$

be a formal power series, with complex coefficients. Given integers $m, n \geq 0$, the (m, n) Padé approximant to f is a rational function

$$[m/n] = P/Q$$

where P, Q are polynomials of degree at most m, n respectively, such that Q is not identically 0, and such that

$$(fQ - P)(z) = O\left(z^{m+n+1}\right). \tag{1.1}$$

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By this, we mean that the coefficients of $1, z, z^2, \ldots, z^{m+n}$ in the formal power series on the left-hand side vanish. In the special case $n = 0$, $[m/0]$ is just the mth partial sum of the power series.

It is easily seen that $[m/n]$ exists: we can reformulate (1.1) as a system of $m+n+1$ homogeneous linear equations in the $(m+1)+(n+1)$ coefficients of the polynomials P and Q. As there are more unknowns than equations, there is a non-trivial solution, and it follows from (1.1) that Q cannot be identically 0 in any non-trivial solution. While P and Q are not separately unique, the ratio $[m/n]$ is.

It was C. Hermite, who gave his student Henri Eugene Padé the approximant to study in the 1890's. Although the approximant was known earlier, by amongst others, Jacobi and Frobenius, it was perhaps Padé's thorough investigation of the structure of the Padé table, namely the array

$$\begin{array}{ccccc} [0/0] & [0/1] & [0/2] & [0/3] & \ldots \\ [1/0] & [1/1] & [1/2] & [1/3] & \ldots \\ [2/0] & [2/1] & [2/2] & [2/3] & \ldots \\ [3/0] & [3/1] & [3/2] & [3/3] & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

that has ensured the approximant bearing his name.

Padé approximants have been applied in proofs of irrationality and transcendence in number theory, in practical computation of special functions, and in analysis of difference schemes for numerical solution of partial differential equations. However, the application which really brought them to prominence in the 1960's and 1970's, was in location of singularities of functions: in various physical problems, for example inverse scattering theory, one would have a means for computing the coefficients of a power series f. One could use just $2n + 1$ of these coefficients to compute the $[n/n]$ Padé approximants to f, and use the poles of the approximant as predictors of the location of poles or other singularities of f. Moreover, under certain conditions on f, which were often satisfied in physical examples, this process could be theoretically justified.

In addition to their wide variety of applications, they are also closely associated with continued fraction expansions, orthogonal polynomials, moment problems, the theory of quadrature, amongst others. See [4] and [8] for a detailed development of the theory, and [10] for their history.

One of the fascinating features of Padé approximants is the complexity of their convergence theory. The convergence properties vary greatly, depending on how one traverses the table. When the denominator degree is kept fixed as n, and the underlying function f is analytic in a ball center 0, except for poles of total multiplicity n, the "column" sequence $\{[m/n]\}_{m=1}^{\infty}$ converges uniformly in compact subsets omitting these poles. This is de Montessus de Ballore's theorem [4], which has been extended and explored in multiple settings .

In this paper, we focus more on the "diagonal" sequence $\{[n/n]\}_{n=1}^{\infty}$. Uniform convergence of diagonal sequences of Padé approximants has been established, for example, for Polya frequency series [3] and series of Stieltjes/Markov/Hamburger [8]. The former have the form

$$f(z) = a_o e^{\gamma z} \prod_{j=1}^{\infty} \frac{1+\alpha_j z}{1-\beta_j z},$$

where $a_0 > 0, \gamma \geq 0$, all $\alpha_j, \beta_j \geq 0$, and

$$\sum_j (\alpha_j + \beta_j) < \infty.$$

The latter have the form

$$f(z) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{1-tz} = \sum_{j=0}^{\infty} z^j \int t^j d\mu(t),$$

and μ is a positive measure supported on the real line, with all finite power moments $\int t^j d\mu(t)$. When μ has non-compact support, the corresponding power series has zero radius of convergence. Nevertheless, the diagonal Padé approximants $\{[n/n]\}_{n=1}^{\infty}$ still converge off the real line to f, at least when μ is a *determinate* measure. The latter means that μ is the only positive measure having moments $\int t^j d\mu(t)$. If μ is supported on $[0,\infty)$ (the so-called Stieltjes case), and is determinate, the diagonal sequence converges uniformly in compact subsets of $\mathbb{C}\backslash(-\infty, 0]$. It is Stieltjes series that often arise in physical applications.

Various modifications of Stieltjes series have also been successfully investigated - for example when μ' has a sign change, or when a rational function is added to the Stieltjes function, or multiplies it. See, for example, [1], [15], [16], [28], [41], [43], [50], [53].

Convergence has also been established for classes of special functions such as hypergeometric functions [4], [8], and q-series, even in the singular case when $|q| = 1$ [17]. For functions with "smooth" coefficients, one expects that their Padé approximants should behave well. For rapidly decaying smooth Taylor series coefficients, this has been established in [31]: if $a_j \neq 0$ for j large enough, and

$$\lim_{j\to\infty} \frac{a_{j-1}a_{j+1}}{a_j^2} = q,$$

where $|q| < 1$, then the full diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ converges locally uniformly in compact subsets of the plane.

In stark contrast to the positive results above, there are entire functions f for which

$$\limsup_{n\to\infty} |[n/n](z)| = \infty$$

for all $z \in \mathbb{C} \setminus \{0\}$, as established by Hans Wallin [55]. Wallin's function is a somewhat extreme example of the phenomenon of *spurious poles*: approximants can have poles which in no way are related to those of the underlying function. This phenomenon was observed in the early days of Padé approximation, in a simpler form, by Dumas [21].

Physicists such as George Baker in the 1960's endeavoured to surmount the problem of spurious poles. They noted that these typically affect convergence only in a small neighborhood, and there were usually very few of these "bad" approximants. Thus, one might compute $[n/n]$, $n = 1, 2, 3, \ldots, 50$, and find a definite convergence trend in 45 of the approximants, with 5 of the 50 approximants displaying pathological behavior. Moreover, the 5 bad approximants could be distributed anywhere in the 50, and need not be the first few. Nevertheless, after omitting the "bad" approximants, one obtained a clear convergence trend. This seemed to be a characteristic of the Padé method, and led to a famous conjecture [7].

Baker-Gammel-Wills Conjecture (1961). *Let f be meromorphic in the unit ball, and analytic at 0. There is an infinite subsequence $\{[n/n]\}_{n \in \mathcal{S}}$ of the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ that converges uniformly in all compact subsets of the unit ball omitting poles of f.*

Thus, there is an infinite sequence of "good" approximants. In the first form of the conjecture, f was required to have a non-polar singularity on the unit circle, but this was subsequently relaxed (cf. [4, p. 188 ff.]). In other forms of the conjecture, f is assumed to be analytic in the unit ball. There is also apparently a cruder form of the conjecture due to Padé himself, dating back to the 1900's; the author must thank J. Gilewicz for this historical information.

A decade after the Baker-Gammel-Wills Conjecture, John Nuttall realized that convergence in measure is a perhaps more appropriate mode of convergence, than uniform convergence. In a short 1970 paper [40], he established the celebrated:

Nuttall's Theorem. *Let f be meromorphic in $\mathbb{C}$, and analytic at 0. Then the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ converges in meas (planar Lebesgue measure) in compact subsets of the plane. That is, given $r, \varepsilon > 0$,*

$$meas \{z : |z| \leq r \text{ and } |f - [n/n]| (z) \geq \varepsilon\} \to 0 \text{ as } n \to \infty.$$

One consequence is that a subsequence converges a.e. In his 1974 paper [55] containing his counterexample, Wallin also gave conditions on the size of the power series coefficients for convergence a.e. of the full diagonal sequence. Nuttall's theorem was soon extended by Pommerenke, using the concept of cap (logarithmic capacity). For a compact set K, we define

$$cap\,(K) = \lim_{n\to\infty}\left(\inf\left\{\|P\|_{L_\infty(K)} : P \text{ a monic polynomial of degree } n\right\}\right)^{1/n},$$

and we extend this to arbitrary sets E as inner capacity:

$$cap\,(E) = \sup\{cap\,(K) : K \subset E,\ K \text{ compact}\}.$$

The capacity of a ball is its radius, and the capacity of a line segment is a quarter of its length. It is a "thinner" set function than planar measure. In fact any set of capacity 0 has Hausdorff dimension 0, and the usual Cantor set has positive logarithmic capacity. The exact value of this for the Cantor set is a well known, and difficult, problem. Those requiring more background can consult [45], [46].

Pommerenke [42] proved:

Pommerenke's Theorem. *Let f be analytic in $\mathbb{C}\backslash E$, and analytic at 0, where $cap\,(E) = 0$. Then, given $r, \varepsilon > 0$*

$$cap\,\{z : |z| \le r \text{ and } |f - [n/n]|\,(z) \ge \varepsilon^n\} \to 0 \text{ as } n \to \infty.$$

Since any countable set has capacity 0, Pommerenke's theorem implies Nuttall's. The two are often combined and called the Nuttall-Pommerenke theorem.

While E above may be uncountable, it cannot include branchpoints. The latter require far deeper techniques, developed primarily by Herbert Stahl in a rigorous form, building on earlier ideas from Nuttall. Stahl showed that one can cut the plane joining the branchpoints in a certain way, yielding a set of minimal capacity, outside which the Padé approximants converge in capacity. This celebrated and deep theory, is expounded in [47], [48], [49], [50], [51]. Stahl's work gave some hope that BGW might be true for algebraic functions, and indeed, he formulated several conjectures [52], one of which is [52, p. 291]:

Stahl's Conjecture for Algebraic Functions. *Let f be an algebraic function, so that for some $m \ge 1$, and polynomials $P_0, P_1, \dots, P_m$, not all 0,*

$$P_0 + P_1 f + P_2 f^2 + \cdots + P_m f^m \equiv 0.$$

Assume also that f is meromorphic in the unit ball. Then there is a subsequence of $\{[n/n]\}_{n=1}^\infty$ that converges uniformly to f in compact subsets of the unit ball, omitting poles of f.

Stahl formulated a more general conjecture, where the unit ball is replaced by the "convergence domain" or "extremal domain" for f. This is a the largest domain inside which $\{[n/n]\}_{n=1}^\infty$ converges in capacity. Stahl's Conjecture was established for a large class of hyperelliptic functions by S. P. Suetin [53]. Some very impressive recent related work, due to Aptekarev, Baratchart, and Yattselev appears in [2], [9], and due to Martinez-Finkelshtein, Rakhmanov, and Suetin, appears in [39]. Deep Riemann-Hilbert techniques play a key role in these papers.

While the positive and negative results of the 1970's cast doubt on the truth of the Baker-Gammel-Wills Conjecture, a counterexample remained elusive. It is very difficult to show pathological behavior of a *full sequence* of Padé approximants. The author looked for a long time for a counterexample among the explicitly known Padé approximants to q-series, in the exceptional case where $|q| = 1$. Of course, q-series are usually considered for $|q| < 1$ or $|q| > 1$.

In [38], E. B. Saff and the author investigated the Padé table and continued fraction for the partial theta function $\sum_{j=0}^{\infty} q^{j(j-1)/2} z^j$ when $| q |= 1$. Subsequently K.A. Driver and the author [17], [19], [18], [20] undertook a detailed study of the Padé table and continued fraction for the more general Wynn's series [57]

$$\sum_{j=0}^{\infty}[\prod_{l=0}^{j-1}(A - q^{l+\alpha})]z^j; \quad \sum_{j=0}^{\infty} \frac{z^j}{\prod_{l=0}^{j-1}(C - q^{l+\alpha})}; \quad \sum_{j=0}^{\infty}[\prod_{l=0}^{j-1} \frac{A - q^{l+\alpha}}{C - q^{l+\gamma}}]z^j.$$

Here A, C, α and γ are suitably restricted parameters. All of these satisfy the Baker-Gammel-Wills Conjecture.

Finally in 2001 [36], the author found a counterexample in the continued fraction of Rogers-Ramanujan. For q not a root of unity, let

$$G_q(z) := \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\cdots(1-q^j)} z^j$$

denote the Rogers-Ramanujan function, and

$$H_q(z) = G_q(z) / G_q(qz).$$

Meromorphic Counterexample. *Let* $q := \exp(2\pi i \tau)$ *where* $\tau := \frac{2}{99+\sqrt{5}}$. *Then* H_q *is meromorphic in the unit ball and analytic at* 0. *There does not exist any subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *that converges uniformly in all compact subsets of* $\mathcal{A} := \{z : |z| < 0.46\}$ *omitting poles of* H_q.

It did not take long for A. P. Buslaev to improve on this, by finding a function analytic in the unit ball, for which the Baker-Gammel-Wills Conjecture, as well as Stahl's conjecture for algebraic functions, both fail [11], [12]. Buslaev was part of the Russian school of Padé approximation, led by A.A. Goncar. One of their important foci was inverse theory: given certain properties of a sequence of Padé approximants formed from a formal power series, what can we deduce about the analytic properties of the underlying function? For example, if a ball contains none of the poles of the approximants, does it follow that the underlying function is analytic there? Some references to their work are [14], [22], [23], [24], [43], [54].

Buslaev's Analytic Counterexample. *Let*

$$f(z) = \frac{-27 + 6z^2 + 3(9+j)z^3 + \sqrt{81(3-(3+j)z^3)^2 + 4z^6}}{2z(9+9z+(9+j)z^2)},$$

where $j = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$*. The branch of the* $\sqrt{}$ *is chosen so that* $f(0) = 0$*. Then for some* $R > 1 > r > 0$*,* f *is analytic in* $\{z : |z| < R\}$*, but for large enough* n*,* $[n/n]$ *has a pole in* $|z| < r$*, and consequently no subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *converges uniformly in all compact subsets of* $\{z : |z| < 1\}$*.*

Buslaev later showed [13] that for q a suitable root of unity, the Rogers-Ramanujan function above, is also a counterexample to both BGW and Stahl's Conjecture. Although this resolves the conjecture, it raises further questions. In both the above counterexamples, uniform convergence fails due to the persistence of spurious poles in a specific compact subset of the unit ball. Moreover, in both the above examples, given any point of analyticity of f in the unit ball, some subsequence converges in some neighborhood of the unit ball. In fact, just two subsequences are enough to provide uniform convergence throughout the unit ball, as pointed out by Baker in [5]. It is perhaps with this in mind that in 2005, George Baker modified his 1961 conjecture [6]:

George Baker's "Patchwork" Conjecture. *Let* f *be analytic in the unit ball, except for at most finitely many poles, none at* 0*. Then there exist a finite number of subsequences of* $\{[n/n]\}_{n=1}^{\infty}$ *such that for any given point of analyticity* z *in the ball, at least one of these subsequences converges pointwise to* $f(z)$*.*

It seems that if true in this form, the convergence would be uniform in some neighborhood of z. Baker also includes poles amongst the permissible z, with the understanding that the corresponding subsequence diverges to ∞.

An obvious question is why we restrict ourselves to uniform convergence of subsequences - perhaps convergence in some other mode is more appropriate? Yes, there is no possible analogue of the Nuttall-Pommerenke theorem for functions with finite radius of meromorphy. Indeed, the author and E.A. Rakhmanov [29], [44] independently showed that there are functions analytic in the unit ball for which the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ does not converge in measure, let alone in capacity. But this does not exclude:

Conjecture on convergence in capacity of a subsequence. *Let* f *be analytic or meromorphic in the unit ball, and analytic at* 0*. There exists a subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *and* $r > 0$ *such that the subsequence converges in* *measure or capacity to* f *in* $\{z : |z| < r\}$.

Notice that we are not even asking for convergence in capacity throughout the unit ball, nor for the r to be independent of f. Weak results in this direction appear in [33], [35], [37]. In [52, p. 289], this was formulated in the stronger form where $r = 1$. Another obvious point is that all the counterexamples involve a function with finite radius of meromorphy. What about entire functions, or functions meromorphic in the whole plane?

Baker-Gammel-Wills Conjecture for functions defined in the plane. *Let* f *be entire, or meromorphic in* $\mathbb{C}$ *and analytic at* 0*. Then there exists* $r > 0$ *and a subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *that converges uniformly to* f *in compact subsets of* $\{z : |z| < r\}$.

This seems like an especially relevant addendum to the 1961 conjecture. A stronger form would be that some subsequence converges uniformly in compact subsets of the plane omitting poles of f. Of course if the r above is independent of f, the stronger form would follow.

Another relevant direction is to restrict the growth of the entire function, and try establish convergence. The best growth condition is due to the author [34], but is very weak:

Theorem. *Assume that the series coefficients* $\{a_n\}$ *of* f *satisfy*

$$\limsup_{n\to\infty} |a_n|^{1/n^2} < 1. \tag{1.2}$$

Then there exists a subsequence of $\{[n/n]\}_{n=1}^{\infty}$ *that converges uniformly in compact subsets of the plane to* f.

In fact, in that paper, the Maclaurin series coefficients were replaced by errors of rational approximation on a disk, center 0, of radius $\sigma > 0$,

$$E_{nn}(\sigma) = \inf\left\{ \|f - R\|_{L_\infty(|z|\le\sigma)} : R \text{ of type } (n, n) \right\}$$

and the hypothesis was

$$\limsup_{n\to\infty} E_{nn}(\sigma)^{1/n^2} < 1,$$

while f was allowed to be meromorphic rather than entire. It seems appropriate to suggest:

Growth Conditions for the truth of BGW. *Find the fastest rate of growth of the coefficients of an entire function that guarantees truth of BGW, or at least find a more general condition than* (1.2).

In an earlier related paper [30], an even weaker result was used to show that the Baker-Gammel-Wills Conjecture is usually true in the sense of category. That is, if we place the topology of locally uniform convergence on the space of all entire functions, the set of entire functions for which the conjecture is false, is a countable union of nowhere dense sets (that is, is of "first category").

One can of course go beyond classical Padé approximants in looking for uniform convergence. For example, one can fix the poles of the approximants, leading to what are called Padé-type approximants. We shall not attempt to survey or reference this very extensive topic. While this avoids spurious poles, one sacrifices the degree of interpolation - and the optimal location of poles becomes an issue.

Another path is to interpolate at multiple points rather than 0, while still leaving the poles free. Here there can still be spurious poles, but one hopes that the freedom in choice of interpolation points, ameliorates this. It can also help to ensure better approximation on non-circular regions. It is a classical result of E. Levin [26], [27] that the best L_2 rational approximant of type (n, n) (that is with numerator, denominator degree $\leq n$) interpolates the approximated function f in at least $2n + 1$ points. As a consequence, for functions analytic in the closed unit ball, there is always a full sequence of diagonal multipoint approximants that converges uniformly in the closed ball to f.

In the special case where one keeps previous interpolation points as one increases the numerator and denominator degree, multipoint Padé approximation is called Newton-Padé approximation. If one allows these interpolation points to depend on the approximated function, then for functions meromorphic in the plane, one can find a full diagonal sequence of Newton-Padé approximants that converges uniformly in compact subsets omitting poles [32].

While Padé approximation may not be such a hot topic as in the period 1970–2000, it is clear that there are significant and challenging problems that are still unresolved, and worthy of the efforts of young researchers.

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