Smallest Eigenvalues of Hankel Matrices for Exponential Weights

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Abstract

We obtain the rate of decay of the smallest eigenvalue of the Hankel matrices $\left(\int_I t^{j+k} W^2(t)\, dt\right)_{j,k=0}^n$ for a general class of even exponential weights $W^2 = \exp(-2Q)$ on an interval I. More precise asymptotics for more special weights have been obtained by many authors.

Remark 1 *Running Title: Smallest Eigenvalues of Hankel Matrices*

1 The Result

Let $I = (-d, d)$ where $0 < d \leq \infty$. Let $Q : I \to [0, \infty)$ be continuous and $W^2 = \exp(-2Q)$ be such that all the moments

$$\int_I t^j W^2(t)\, dt, j = 0, 1, 2, ...,$$

exist. Form the positive definite Hankel matrix

$$H_n = \left(\int_I t^{j+k} W^2(t)\, dt \right)_{j,k=0}^{n}$$

and denote its smallest eigenvalue by λ_n. The focus of this paper is the rate of decay of the smallest eigenvalue λ_n of H_n.

Many authors have investigated the asymptotic behaviour of λ_n as $n \to \infty$. For example, Widom, and Wilf investigated the behaviour of λ_n for weights on a finite interval satisfying the Szegö condition [13]. For the Hermite weight $W(x) = \exp\left(-\frac{1}{2}x^2\right)$, Szegö [11] established the asymptotic

$$\lambda_n = 2^{\frac{13}{4}} \pi^{\frac{3}{2}} e n^{\frac{1}{4}} \exp\left(-2(2n)^{1/2}\right)(1 + o(1)),$$

with similar results for Laguerre weights. The first author, Berg and Ismail [2] showed that λ_n remains bounded away from 0 iff the moment problem for W^2 is indeterminate. Moreover, the first author and Lawrence [3] established asymptotic behaviour of λ_n for weights on $(0, \infty)$ such as $\exp\left(-x^\beta\right), \beta > 0$. Beckermann has explored condition numbers for Hankel matrices [1].

It is well known that λ_n is given by the Rayleigh quotient:

$$\lambda_n = \min\left\{ \frac{\overline{X}^T H_n X}{\overline{X}^T X} : X \in \mathbb{C}^{n+1} \backslash \{\underline{0}\} \right\}.$$

Corresponding to any of these vectors $X = (x_0, x_1, x_2, ...x_n)^T$, we can define a polynomial

$$P(z) = \sum_{j=0}^{n} x_j z^j.$$

Using the definition of H_n, we see that we can recast the Rayleigh quotient in the form

$$\lambda_n = \min\left\{ \frac{\int_I |P|^2 W^2}{\frac{1}{2\pi}\int_{-\pi}^{\pi} |P(e^{i\theta})|^2 d\theta} : \deg(P) \leq n \right\}. \quad (1)$$

This extremum property, very similar to the extremal property of Christoffel functions, is the basis for the analysis in this paper.

Before we define our class of weights, which is the even case of the weights in [7], we need the notion of a quasi-increasing function. A function $g : (0, d) \to (0, \infty)$ is said to be *quasi-increasing* if there exists $C > 0$ such that

$$g(x) \leq Cg(y), 0 < x \leq y < d.$$

Of course, any increasing function is quasi-increasing.

Definition 1.1 General Exponential Weights
Let $W = e^{-Q}$ where $Q : I \to [0, \infty)$ is even and satisfies the following properties:
(a) Q' is continuous in I and $Q(0) = 0$;
(b) Q'' exists and is positive in $I \backslash \{0\}$;
(c)

$$\lim_{t \to d-} Q(t) = \infty;$$

(d) The function

$$T(t) := \frac{tQ'(t)}{Q(t)}, t \neq 0$$

is quasi-increasing in $(0, d)$, with

$$T(t) \geq \Lambda > 1, t \in (0, d);$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{\mid Q'(x) \mid} \leq C_1 \frac{\mid Q'(x) \mid}{Q(x)}, \text{ a.e. } x \in (0, d).$$

Then we write $W \in \mathcal{F}(C^2)$.

The simplest case of the above definition is when $I = \mathbb{R}$ and T is bounded. This is the so called Freud case, for the boundedness of T forces Q to be of at most polynomial growth. A typical example is

$$Q(x) = |x|^{\alpha}, x \in \mathbb{R},$$

where $\alpha > 1$. A more general example satisfying the requirements of Definition 1.1 is

$$Q(x) = \exp_{\ell}(|x|^{\alpha}) - \exp_{\ell}(0),$$

where $\alpha > 1$ and $\ell \geq 0$. Here we set $\exp_0(x) := x$ and for $\ell \geq 1$,

$$\exp_\ell(x) = \underbrace{\exp(\exp(\exp \ldots \exp(x))))}_{\ell \text{ times}}$$

is the ℓth iterated exponential.

An example on the finite interval $I = (-1, 1)$ is

$$Q(x) = \exp_\ell((1 - x^2)^{-\alpha}) - \exp_\ell(1), x \in (-1, 1),$$

where $\alpha > 0$ and $\ell \geq 0$.

In analysis of exponential weights, an important role is played by the Mhaskar-Rakhmanov-Saff number $a_u \in (0, d), u > 0$, which is the unique root of the equation

$$u = \frac{2}{\pi} \int_0^1 \frac{a_u s Q'(a_u s)}{\sqrt{1 - s^2}} ds.$$

One of the features that motivates their importance is the Mhaskar-Saff identity [9]

$$\| PW \|_{L_\infty(I)} = \| PW \|_{L_\infty[-a_n, a_n]},$$

valid for all polynomials P of degree $\leq n$. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t and polynomials P of degree at most n. We write $C = C(\lambda), C \neq C(\lambda)$ to indicate dependence on, or independence of, a parameter λ. The same symbol does not necessarily denote the same constant in different occurrences. Given sequences of real numbers (c_n) and (d_n) we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that

$$C_1 \leq c_n / d_n \leq C_2$$

for the relevant range of n. Similar notation is used for functions and sequences of functions. We shall prove:

Theorem 1.2

Let W be even and $W \in \mathcal{F}(C^2)$. Then for $n \geq 1$,

$$\lambda_n \sim \sqrt{\frac{n}{a_n}} \exp\left(-2 \int_0^n \log\left[\frac{1}{a_s} + \sqrt{1 + \frac{1}{a_s^2}}\right] ds\right). \tag{2}$$

One may recast this estimate in a number of other ways: for example,

$$\lambda_n \sim \sqrt{\frac{n}{a_n}} \exp\left(-2\int_0^n \text{arc sinh}\left(\frac{1}{a_s}\right)\, ds\right).$$

An integration by parts shows that

$$\lambda_n \sim \sqrt{\frac{n}{a_n}} \exp\left(-2\left\{n\log\left[\frac{1}{a_n}+\sqrt{1+\frac{1}{a_n^2}}\right]+\int_0^{a_n}\frac{b_t}{t\sqrt{1+t^2}}dt\right\}\right), \qquad (3)$$

where b_t is the inverse function of a_t, that is

$$b_{a_t} = b\left(a_t\right) = t, t > 0.$$

(For this, one also needs $\lim_{s\to 0+} s\log\frac{1}{a_s} = 0$, which follows from the convergence of $\int_0^1 \log\frac{2}{a_s}ds$, see below). Another form, which is the initial form in our proof, is

$$\lambda_n \sim \sqrt{\frac{n}{a_n}} \exp\left(2\left[V^{\sigma_n}(i) - c_n\right]\right), \qquad (4)$$

where V^{σ_n} is an equilibrium potential, and c_n is an equilibrium constant - we shall define these at the end of this section.

Example
Let $\alpha > 1$ and

$$Q(x) = |x|^{\alpha}, x \in \mathbb{R}.$$

Here

$$a_u = C_\alpha u^{1/\alpha}, u > 0,$$

where [9]

$$C_\alpha = \left(\frac{2^{\alpha-2}\Gamma(\alpha/2)^2}{\Gamma(\alpha)}\right)^{1/\alpha}.$$

Using (2), the Maclaurin series expansion [5, p. 51]

$$\log\left(x+\sqrt{1+x^2}\right) = \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{2^{2k}(k!)^2(2k+1)}x^{2k+1}, |x| \le 1,$$

and some straightforward estimations, we obtain

$$\lambda_n \sim n^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \exp\left(-2n \sum_{k=0}^{\left[\frac{\alpha-1}{2}\right]} (-1)^k \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \frac{a_n^{-2k-1}}{1-\frac{2k+1}{\alpha}}\right),$$

provided α is not an odd integer. If α is an odd integer, we obtain instead

$$\lambda_n \sim n^{\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \exp\left(\begin{array}{c} -2n \sum_{k=0}^{\left[\frac{\alpha-3}{2}\right]} (-1)^k \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} \frac{a_n^{-2k-1}}{1-\frac{2k+1}{\alpha}} \\ -2(\log n)(-1)^{\frac{\alpha-1}{2}} \frac{(\alpha-1)!}{2^{\alpha-1}\left(\frac{\alpha-1}{2}!\right)^2 \alpha} C_\alpha^{-\alpha} \end{array}\right).$$

In both estimates, $[x]$ denotes the greatest integer $\leq x$. In particular, for the Hermite weight $\alpha = 2$, this gives

$$\lambda_n \sim n^{\frac{1}{4}} \exp\left(-4\sqrt{n}\right),$$

which accords with Szegö's result, if we recall that $Q(x) = \frac{1}{2}x^2$ in his formulation.

This paper is organised as follows: in Section 2, we establish a general lower bound for λ_n, using the same methods that were used in [7] to establish lower bounds for Christoffel functions. In Section 3, we establish upper bounds for λ_n by discretizing a potential. Then in Section 4, we complete the proof.

Throughout the paper, we assume that $W \in \mathcal{F}(C^2)$. (In fact, with more work, our results hold for the class $\mathcal{F}(Dini)$ in [7], but in terms of weights defined by explicit formulas, the difference is insubstantial). For each $t > 0$, it is known that there is a non-negative density function σ_t on $[-a_t, a_t]$ with total mass t,

$$\int_{-a_t}^{a_t} \sigma_t(s)\, ds = t, \tag{5}$$

satisfying the equilibrium condition

$$\int_{-a_t}^{a_t} \log \frac{1}{|x-s|} \sigma_t(s)\, ds + Q(x) = c_t, x \in [-a_t, a_t]. \tag{6}$$

We call σ_t the *equilibrium density* of mass t, c_t the *equilibrium constant* for t, and

$$V^{\sigma_t}(z) = \int_{-a_t}^{a_t} \log \frac{1}{|z-s|} \sigma_t(s)\, ds$$

the corresponding *equilibrium potential*. One representation for σ_t is

$$\sigma_t(x) = \frac{\sqrt{a_t^2 - x^2}}{\pi^2} \int_{-a_t}^{a_t} \frac{Q'(s) - Q'(x)}{s - x} \frac{ds}{\sqrt{a_t^2 - s^2}}, x \in (-a_t, a_t). \quad (7)$$

and one for c_t is

$$c_t = \int_0^t \log \frac{2}{a_s} ds.$$

See [7, Chapter 2].

2 Lower Bounds for λ_n

The result of this section is:

Lemma 2.1
Let $0 < \eta < \frac{\pi}{2}$. *Then*

$$\lambda_n^{-1} \leq C_1 \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta + \sup_{\theta \in [0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) \right\}. \quad (8)$$

Here C_1 *depends on* η, *not on* n.

Throughout we fix n and set

$$\Delta = [-a_n, a_n].$$

Given $x \notin \Delta$, we use $g_\Delta(z, x)$ to denote the Green's function for $\overline{\mathbb{C}} \backslash \Delta$ with pole at x, so that $g_\Delta(z,x) + \log | z - x |$ is harmonic as a function of z in $\overline{\mathbb{C}} \backslash \Delta$ and vanishes on Δ. When $x \in \Delta$, we set $g_\Delta(z,x) \equiv 0$, and when $x = \infty$, the Green's function is denoted by $g_\Delta(z)$. We also let

$$\phi(z) = z + \sqrt{z^2 - 1}, z \in \mathbb{C} \backslash [-1, 1]$$

denote the conformal map of $\mathbb{C} \backslash [-1, 1]$ onto the exterior $\{w : |w| > 1\}$ of the unit ball. Then the Green's function for $\overline{\mathbb{C}} \backslash \Delta$ with pole at ∞ admits the representation

$$g_\Delta(z) = \log \left| \phi\left(\frac{z}{a_n}\right) \right|.$$

For further orientation on the potential theory we use, see [10] or [7]. We also use $H[f]$ to denote the Hilbert transform of a function $f \in L_1(\mathbb{R})$, so that

$$H[f](z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{f(t)}{t-z} dt,$$

where the integral must be taken in a Cauchy principal value sense if z is real.

Proof of Lemma 2.1

We use the extremal property (1), in the form

$$\lambda_n^{-1} = \sup \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 d\theta / \int_I |PW|^2,$$

where the sup is taken over all monic polynomials P of degree $\leq n$. Acordingly let P be a monic polynomial of degree $m \leq n$. If ν denotes a measure of total mass m that places unit mass at each zero of P, then $\log|P|$ admits the representation

$$\log|P(z)| = \int \log|z-t|\, d\nu(t).$$

Form

$$G(z) = \int (\log|z-t| + g_\Delta(z,t))\, d\nu(t) + V^{\sigma_n}(z) - c_n + (n-m)\, g_\Delta(z).$$

Since $\log|z-t| + g_\Delta(z,t)$ is bounded and has finite limit as $z \to t$, we see that G is harmonic in $\mathbb{C}\backslash\Delta$. Moreover, since as $z \to \infty$, $V^{\sigma_n}(z) = -n\log|z| + o(1)$ and $g_\Delta(z) = \log\frac{2}{a_n} + \log|z| + o(1)$,

$$\lim_{|z|\to\infty} G(z) = \int g_\Delta(\infty,t)\, d\nu(t) - c_n + (n-m)\log\frac{2}{a_n} =: G(\infty).$$

Thus G is harmonic in $\overline{\mathbb{C}}\backslash\Delta$, and hence has a single valued harmonic conjugate there, $\widetilde{G}(z)$ say. Hence the function

$$f(z) := \exp(G(z) + i\widetilde{G}(z))/\phi\left(\frac{z}{a_n}\right)$$

is analytic in $\overline{\mathbb{C}}\backslash\Delta$, with a simple zero at ∞. Cauchy's integral formula for the exterior of a segment gives for $z \notin \Delta$,

$$f(z) = \frac{1}{2\pi i}\int_\Delta \frac{f_+(x) - f_-(x)}{x-z} dx = \frac{1}{2}\left[H[f_+] - H[f_-](z)\right], \tag{9}$$

where $f_\pm$ denote boundary values of f on Δ from the upper and lower half planes. Note that we set $f_\pm = 0$ outside Δ. Next,

$$|f_\pm(x)| = \exp(G(x)) = |PW|(x), x \in (-a_n, a_n), \tag{10}$$

by (6). Moreover, as the Green's function g_Δ is non-negative,

$$|f(z)| = \exp(G(z)) / \left|\phi\left(\frac{z}{a_n}\right)\right| \geq |P(z)| \exp(V^{\sigma_n}(z) - c_n) / \left|\phi\left(\frac{z}{a_n}\right)\right|, z \notin \Delta. \tag{11}$$

The representation (9) of f gives for $z \notin \Delta$,

$$\begin{aligned} |f(z)| &\leq \frac{1}{2\pi}\left(\int_\Delta (f_+(x) - f_-(x))^2 dx\right)^{1/2} \left(\int_\Delta \frac{dx}{|x-z|^2}\right)^{1/2} \\ &\leq \frac{1}{\pi}\left(\int_\Delta |PW|^2\right)^{1/2} \left(\frac{\pi}{|\operatorname{Im} z|}\right)^{1/2}. \end{aligned}$$

Combining this and (11) gives

$$\begin{aligned} &\frac{1}{2\pi}\left[\int_{-\pi+\eta}^{-\eta} + \int_\eta^{\pi-\eta}\right] \left|P\left(e^{i\theta}\right)\right|^2 d\theta \\ \leq\ &\left(\int_\Delta |PW|^2\right)\left(\frac{1}{\pi \sin\eta}\right) \frac{1}{2\pi}\left[\int_{-\pi+\eta}^{-\eta} + \int_\eta^{\pi-\eta}\right] \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) \left|\phi\left(\frac{e^{i\theta}}{a_n}\right)\right|^2 d\theta \\ \leq\ &\frac{\left(1+\frac{2}{a_n}\right)^2}{\pi \sin\eta} \left(\int_I |PW|^2\right) \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta. \end{aligned} \tag{12}$$

The rest of the integral is more difficult. First, note that since (11) holds and $V^{\sigma_n}(\overline{z}) = V^{\sigma_n}(z) = V^{\sigma_n}(-z)$,

$$\begin{aligned} &\frac{1}{2\pi}\left[\int_{-\eta}^{\eta} + \int_{\pi-\eta}^{\pi+\eta}\right] \left|P\left(e^{i\theta}\right)\right|^2 d\theta \\ \leq\ &\sup_{\theta\in[0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) \frac{\left(1+\frac{2}{a_n}\right)^2}{2\pi}\left[\int_{-\eta}^{\eta} + \int_{\pi-\eta}^{\pi+\eta}\right] \left|f\left(e^{i\theta}\right)\right|^2 d\theta \\ \leq\ &\sup_{\theta\in[0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) \frac{\left(1+\frac{2}{a_n}\right)^2}{4\pi} \int_{-\pi}^{\pi} \left(\left|H[f_+]\left(e^{i\theta}\right)\right|^2 + \left|H[f_-]\left(e^{i\theta}\right)\right|^2\right) d\theta, \end{aligned}$$

by (9). Now since $f_{\pm}$ are continuous on Δ, and are 0 off Δ, they are trivially in $L_2(\mathbb{R})$, and then $H[f_{\pm}]$ belong to the Hardy 2−space of the upper-half plane [6, p. 128]. Next, normalized Lebesgue measure on the semi-circular arc of the unit circle in the upper-half plane is a Carleson measure with respect to that upper-half plane. So we can use Carleson's inequality to replace the integral over the upper and lower halves of the unit circle, by an integral along the real axis:

$$\int_{-\pi}^{\pi} \left|H[f_{\pm}]\left(e^{i\theta}\right)\right|^2 d\theta \leq C \int_{-\infty}^{\infty} |H[f_{\pm}](x)|^2 \, dx$$

with C independent of f. For a discussion of Carleson's inequality and Carleson measures, see [4] or [6]. Then

$$\begin{aligned} &\frac{1}{2\pi}\left[\int_{-\eta}^{\eta} + \int_{\pi-\eta}^{\pi+\eta}\right] \left|P\left(e^{i\theta}\right)\right|^2 d\theta \\ \leq\ & C\left\{\sup_{\theta\in[0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right)\right\} \int_{-\infty}^{\infty} \left(|H[f_+](x)|^2 + |H[f_-](x)|^2\right) dx, \end{aligned}$$

with C independent of f, P (and n). As the Hilbert transform is an isometry of $L_2(\mathbb{R})$, we obtain from (10)

$$\begin{aligned} &\frac{1}{2\pi}\left[\int_{-\eta}^{\eta} + \int_{\pi-\eta}^{\pi+\eta}\right] \left|P\left(e^{i\theta}\right)\right|^2 d\theta \\ \leq\ & 2C\left\{\sup_{\theta\in[0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right)\right\} \int_I |PW|^2 . \end{aligned}$$

Adding this and (12) gives, for all monic polynomials of degree$\leq n$,

$$\begin{aligned} &\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^2 d\theta \Big/ \int_I |PW|^2 \\ \leq\ & C_1\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta + \sup_{\theta\in[0,\eta]} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right)\right\}. \end{aligned}$$

Now the extremal property (1) gives the lemma. ■

We note that this lemma holds more generally than for our class of weights: Q does not need to be even or satisfy any smoothness restrictions. With minor modifications, the lemma holds for any exponential weight W for which the equilibrium measure is supported on a single interval.

3 Upper Bounds for λ_n

In this section, we use Totik's method of discretisation of a potential [12] to obtain a polynomial that gives an upper bound to match the lower bound in the previous section. The details are similar to those in [7, Chapter 7].

Theorem 3.1
There exist C_1 and C_2 and for large enough n, a polynomial P_n of degree $\leq n$ such that for $|z| = 1$ with $\arg(z) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$|P_n(z)| \geq C_1 \exp\left(-\left[V^{\sigma_n}(z) - c_n\right]\right) \tag{13}$$

and

$$\int_I |PW|^2 \leq C_2. \tag{14}$$

Moreover for such n,

$$\lambda_n^{-1} \geq C_3 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta, \tag{15}$$

with C_3 independent of n.

Throughout, we let σ_n^* denote the density σ_n contracted to $[-1,1]$ so that

$$\sigma_n^*(s) = \frac{a_n}{n}\sigma_n(a_n s), s \in (-1,1),$$

and

$$\int_{-1}^{1} \sigma_n^* = 1. \tag{16}$$

For a fixed n, we determine points

$$-1 = t_0 < t_1 < t_2 < ... < t_n = 1$$

and intervals

$$I_j = [t_j, t_{j+1}), 0 \leq j \leq n-1; |I_j| = t_{j+1} - t_j$$

with

$$\int_{I_j} \sigma_n^* = \frac{1}{n}, 0 \leq j \leq n-1.$$

Moreover, we use Totik's idea [12] of the "weight point" or "centre of mass"

$$\xi_j = \int_{t_j}^{t_{j+1}} s\sigma_n^*(s)\,ds \Big/ \int_{t_j}^{t_{j+1}} \sigma_n^*(s)\,ds \in (t_j, t_{j+1}),$$

so that

$$\int_{t_j}^{t_{j+1}} \left(s - \xi_j\right) \sigma_n^*(s)\,ds = 0. \tag{17}$$

We define

$$R_n(z) = \prod_{j=0}^{n-1} \left(z - \xi_j\right),$$

and will prove:

Lemma 3.2
There exists a positive integer L such that for large enough n, and $\frac{2}{a_n} \geq |u| \geq \frac{1}{2a_n}$ with $\arg(u) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$|R_n(u)| \exp\left(nV^{\sigma_n^*}(u)\right) \geq C_1, \tag{18}$$

and

$$|R_n(x)| \exp\left(nV^{\sigma_n^*}(x)\right) \leq C_2 \left(1 - x^2\right)^{-L}, x \in (-1, 1). \tag{19}$$

Later on, if I is unbounded, we shall "damp down" R_n on I by multiplying with another polynomial so that we obtain (14). For the proof of Lemma 3.2, we need properties of the discretisation points:

Lemma 3.3
(a) Uniformly in n and $1 \leq j \leq n-2$,

$$\sigma_n^*(t_j) \sim \sigma_n^*(s) \sim \sigma_n^*(t_{j+1}) \sim \frac{1}{n|I_j|} \sim \frac{1}{n|I_{j+1}|}, s \in [t_j, t_{j+1}]. \tag{20}$$

(b) Moreover, if $j = 0$,

$$\sigma_n^*(s) \leq C\sigma_n^*(t_{j+1}) \sim \frac{1}{n|I_j|} \sim \frac{1}{n|I_{j+1}|}, s \in [t_j, t_{j+1}] \tag{21}$$

with an analogous assertion if $j = n-1$.
(c) There exists $C > 0$ such that for $n \geq 1$, and $u, v \in (-1, 1)$ with

$$|u - v| \leq \left(1 - u^2\right)^5, \tag{22}$$

we have

$$\sigma_n^*(u)/\sigma_n^*(v) \le C.$$

Proof

(a), (b) These are Lemma 7.16 in [7, p. 194].

(c) Note that the class of weights $\mathcal{F}(C^2)$ we treat here lies in the class $\mathcal{F}\left(Lip\frac{1}{2}\right)$ in [7] (see [7, p. 13]) and hence we may apply Theorem 6.3(b) in [7, pp. 147-148] with $\psi(u) = u^{1/2}$. We obtain for $n \ge 1$ and $u, v \in (-1, 1)$,

$$|\sigma_n^*(u) - \sigma_n^*(v)| \le \frac{C}{\sqrt{1-|v|}} \left(\frac{|u-v|}{1-\max\{|u|, |v|\}}\right)^{1/4}.$$

Moreover, from Theorem 6.1(b) in [7, p. 146],

$$\sigma_n^*(v) \ge C\sqrt{1-v^2}.$$

Then subject to (22), we obtain

$$1 - |u| \sim 1 - |v| \sim 1 - u^2,$$

so

$$\left|1 - \frac{\sigma_n^*(u)}{\sigma_n^*(v)}\right| \le \frac{C}{(1-|u|)^{5/4}} |u-v|^{1/4} \le C.$$

■

Proof of (19) of Lemma 3.2

We see that

$$\log|R_n(u)| + nV^{\sigma_n^*}(u) = -\sum_{j=0}^{n-1} \int_{I_j} \log\left|\frac{u-s}{u-\xi_j}\right| (n\sigma_n^*(s))\, ds =: -\sum_{j=0}^{n-1} \Gamma_j. \tag{23}$$

Now we proceed in five steps:

Step 1: An inequality for Γ_j

Fix $u \in [-1, 1]$ and choose j_0 such that $u \in I_{j_0}$. Since $|I_j| \sim |I_{j\pm1}|$ (by Lemma 3.3), we claim that there exists $\tau \in (0, 1)$, independent of u, j and n, such that for $|j - j_0| \ge 2$,

$$s \in I_j \Rightarrow \frac{\xi_j - s}{u - \xi_j} \ge -\tau. \tag{24}$$

To see this, suppose for example that $j \leq j_0 - 2$, so that I_j is to the left of I_{j_0-1}. Then $u - \xi_j > 0$ and

$$\frac{\xi_j - s}{u - \xi_j} \geq \frac{\xi_j - t_{j+1}}{t_{j_0} - \xi_j} \geq \frac{t_j - t_{j+1}}{t_{j_0} - t_j} \geq -\frac{|I_j|}{|I_j| + |I_{j+1}|}.$$

(In the third inequality, we use the fact that the ratio decreases as we decrease ξ_j). So (24) holds in this case. The case where $j \geq j_0 + 2$ is similar. Next, a Taylor series expansion gives

$$\begin{aligned} \log\left|\frac{u-s}{u-\xi_j}\right| &= \log\left(1 + \frac{\xi_j - s}{u - \xi_j}\right) \\ &= \frac{\xi_j - s}{u - \xi_j} - \frac{1}{2}\frac{1}{(1+r)^2}\left(\frac{\xi_j - s}{u - \xi_j}\right)^2, \end{aligned}$$

where r is between 0 and $\frac{\xi_j - s}{u - \xi_j}$. As $r \geq -\tau$,

$$\log\left|\frac{u-s}{u-\xi_j}\right| \geq \frac{\xi_j - s}{u - \xi_j} - \frac{1}{2(1-\tau)^2}\left(\frac{|I_j|}{dist\,(u, I_j)}\right)^2.$$

Then the definition (17) of ξ_j gives

$$\Gamma_j \geq -\frac{1}{2(1-r)^2}\left(\frac{|I_j|}{dist\,(u, I_j)}\right)^2.$$

Step 2: Γ_j with I_j far from I_{j_0}
Consider those j with $|j - j_0| \geq 2$ and

$$dist\,(u, I_j) \geq \left(1 - u^2\right)^5.$$

Let $\mathcal{S}$ denote the set of all such indices j. Here the first restriction on j ensures that

$$dist\,(u, I_j) \geq C\,|I_j|$$

and then using the bound on Γ_j from Step 1,

$$\begin{aligned} \sum_{j \in \mathcal{S}} \Gamma_j &\geq -C \sum_{j \in \mathcal{S}} \frac{|I_j|}{dist\,(u, I_j)} \\ &\geq -C \int_{\left\{s \in [0,1]: |s-u| \geq C_1 (1-u^2)^5\right\}} \frac{ds}{|s-u|} \\ &\geq -C \log\left|\frac{1-u^2}{2}\right|. \end{aligned}$$

Step 3: Γ_j with I_j close, but not too close, to I_{j_0}
Consider those j with $|j-j_0| \geq 2$ and

$$dist\,(u, I_j) < \left(1-u^2\right)^5.$$

Let $\mathcal{T}$ denote the set of all such indices j. Note that from Lemma 3.3(a), (b), and then (c), uniformly for such j, and some $k \in \{j, j+1\}$,

$$\frac{|I_j|}{|I_{j_0}|} \leq C \frac{\sigma_n^*(u)}{\sigma_n^*(t_k)} \leq C.$$

Then

$$\begin{aligned}
\sum_{j\in\mathcal{T}} \Gamma_j &\geq -C\,|I_{j_0}| \sum_{j\in\mathcal{T}} \frac{|I_j|}{dist\,(u, I_j)^2} \\
&\geq -C\,|I_{j_0}| \int_{\left\{s:|s-u|\geq C_1 \left|I_{j_0}\right|\right\}} \frac{ds}{|s-u|^2} \\
&\geq -C.
\end{aligned}$$

Step 4: Γ_j with I_j very close to I_{j_0}
Now we deal with the at most 3 remaining terms Γ_j with $|j-j_0| \leq 1$. Here we can apply Lemma 3.3 to obtain, for some constants C_1, C_2 and C_3 (independent of j, j_0, u and n),

$$\begin{aligned}
\Gamma_j &= \int_{I_j} \log \left| \frac{u-s}{u-\xi_j} \right| \left(n\sigma_n^*(s)\right) ds \\
&\geq \int_{I_j} \log \left| \frac{u-s}{C_1\,|I_j|} \right| \left(n\sigma_n^*(s)\right) ds \\
&\geq \frac{C_2}{|I_j|} \int_{I_j} \log \left| \frac{u-s}{C_1\,|I_j|} \right| ds \\
&\geq C_2 \int_{-C_3}^{C_3} \log \left| \frac{v}{C_1} \right| dv \geq -C_4.
\end{aligned}$$

Thus

$$\sum_{j:|j-j_0|\leq 1} \Gamma_j \geq -C_5.$$

Step 5: Finish the Proof of (19)
Combining (23) and all the estimates above gives for $u \in (-1,1)$,

$$\log|R_n(u)| + nV^{\sigma_n^*}(u) \leq C - C\log\left(1-u^2\right).$$

■

Proof of (18) of Lemma 3.2
We use the Γ_j defined above. For $s \in I_j$ and $u \in \mathbb{C}$,

$$\begin{aligned}
\log \left| \frac{u-s}{u-\xi_j} \right| &= \frac{1}{2} \log \left| 1 + \frac{\xi_j - s}{u-\xi_j} \right|^2 \\
&= \frac{1}{2} \log \left(1 + \left| \frac{\xi_j - s}{u-\xi_j} \right|^2 + 2\,\mathrm{Re} \left(\frac{\xi_j - s}{u-\xi_j} \right) \right) \\
&\leq \frac{1}{2} \left| \frac{\xi_j - s}{u-\xi_j} \right|^2 + \mathrm{Re} \left(\frac{1}{u-\xi_j} \right) (\xi_j - s),
\end{aligned}$$

so integrating over I_j and using (17) gives

$$\begin{aligned}
\Gamma_j &\leq \frac{|I_j|^2}{2\left|u-\xi_j\right|^2} \int_{I_j} n\sigma_n^* (s)\, ds + 0 \\
&\leq \frac{1}{2} \left[\frac{|I_j|}{dist\,(u, I_j)} \right]^2.
\end{aligned}$$

Suppose now that for some $C > 0$,

$$\chi (u) := \sup \left\{ |I_j| : dist\,(u, I_j) \leq \frac{1}{8} \right\} \leq C \left|\mathrm{Im}\, u\right|. \tag{25}$$

Then

$$\begin{aligned}
&\sum_{j: dist(u, I_j) \leq \frac{1}{8}} \Gamma_j \\
&\leq \frac{\chi(u)}{2} \sum_{j: dist(u, I_j) \leq \frac{1}{8}} \frac{|I_j|}{dist\,(u, I_j)^2} \\
&\leq C_1 \chi(u) \sum_{j: dist(u, I_j) \leq \frac{1}{8}} \frac{|I_j|}{(\mathrm{Im}\, u)^2 + dist\,(\mathrm{Re}\, u, I_j)^2} \\
&\leq C_2 \chi(u) \int_{-\infty}^{\infty} \frac{ds}{(\mathrm{Im}\, u)^2 + \left|\mathrm{Re}\, u - s\right|^2} \leq C_3,
\end{aligned} \tag{26}$$

by (25). Moreover,

$$\sum_{j:dist(u,I_j)>\frac{1}{8}} \Gamma_j$$
$$\leq \ C_4 \sum_{j:dist(u,I_j)>\frac{1}{8}} |I_j| \leq C_5.$$

Combining this, (23) and (26) gives

$$|R_n(u)| \exp\left(nV^{\sigma_n^*}(u)\right) \geq C_6,$$

provided (25) holds. Now we show that (25) does hold if

$$\frac{2}{a_n} \geq |u| \geq \frac{1}{2a_n} \text{ and } \arg(u) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]. \tag{27}$$

We consider two subcases:
(I) I is a finite interval
In this case $a_n \to d < \infty$ as $n \to \infty$. Then the condition (27) ensures that $|\operatorname{Im} u| \geq C$, with C independent of u and n. Hence (25) is immediate.
(II) $I = (-\infty, \infty)$
In this case $a_n \to \infty, n \to \infty$, and (27) implies that $|u| \leq \frac{1}{8}$ for large enough n. Then for $n \geq n_0$,

$$dist(u, I_j) \leq \frac{1}{8} \Rightarrow I_j \subset \left(-\frac{1}{3}, \frac{1}{3}\right).$$

(The threshhold n_0 does not depend on u, j, j_0, n). Since (see (7.89) and (7.84) in [7, pp. 187-188]),

$$I_j \subset \left(-\frac{1}{3}, \frac{1}{3}\right) \Rightarrow |I_j| \sim \frac{1}{n},$$

(with constants in the $\sim$ relation independent of n), and since $|\operatorname{Im} u| \sim \frac{1}{a_n}$, we see that (25) reduces to

$$\frac{1}{n} \leq \frac{C}{a_n},$$

which is true as

$$a_n = o(n).$$

(See (3.30) in [7, p. 72] and note that in the even case $\delta_n = a_n$). ■

From this we deduce:

Lemma 3.4
Let L be as in Lemma 3.2. There exist polynomials R_n^* of degree $\leq n+2L$ such that for $\frac{1}{2} \leq |z| \leq 2$ with $\arg(z) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$|R_n^*(z)| \exp\left(V^{\sigma_n}(z) - c_n\right) \geq C_1 \tag{28}$$

and

$$|R_n^* W| \leq C_2 \text{ in } I. \tag{29}$$

Proof
Observe that

$$\begin{aligned} V^{\sigma_n}(a_n u) &= \int_{-a_n}^{a_n} \log \frac{1}{|a_n u - t|} \sigma_n(t)\, dt \\ &= n \int_{-1}^{1} \log \frac{1}{|a_n u - a_n s|} \sigma_n^*(s)\, ds \\ &= n \log \frac{1}{a_n} + n V^{\sigma_n^*}(u). \end{aligned}$$

We set

$$R_n^*(z) := (1 - \left(a_n^{-1} z\right)^2)^L R_n\left(a_n^{-1} z\right) \exp\left(c_n - n \log \frac{1}{a_n}\right),$$

where L is as in Lemma 3.2. We see that

$$\begin{aligned} &|R_n^*(a_n u)| \exp\left(V^{\sigma_n}(a_n u) - c_n\right) \\ = \;& \left|1 - u^2\right|^L |R_n(u)| \exp\left(V^{\sigma_n^*}(u)\right) \end{aligned}$$

and (28) follows from (18), on setting $z = a_n u$. (Note that $|1-u^2|$ is bounded below). Next, for $x \in [-1, 1]$, from (6)

$$\begin{aligned} &|R_n^* W|(a_n x) \\ = \;& |R_n^*(a_n x)| \exp\left(V^{\sigma_n}(a_n x) - c_n\right) \\ = \;& \left(1 - x^2\right)^L |R_n(x)| \exp\left(n V^{\sigma_n^*}(x)\right) \leq C, \end{aligned}$$

by (19). Then

$$\| R_n^* W \|_{L_\infty(I)} = \| R_n^* W \|_{L_\infty[-a_n, a_n]} \leq C.$$

■

Although the sup-norm of $R_n^* W$ is bounded, all we can deduce from this last lemma is that the L_2 norm over I is $O(a_n)$. This is a problem if $a_n \to \infty, n \to \infty$. To fix this, we multiply R_n^* by a polynomial of degree $O(a_n)$ that behaves like $(1+x^2)^{-1}$ on $[-a_n, a_n]$. But that would give a polynomial of degree $n + O(a_n)$, rather than n. To avoid this, we show that the polynomials R_m^* with $m = n - O(a_n)$ still satisfy the conclusions of the previous lemma, and for this we need:

Lemma 3.5
Let $K > 0$. Assume that

$$\lim_{n \to \infty} a_n = \infty.$$

Assume that for $n \geq 1$, we are given an integer $m = m(n) \leq n$ with

$$n - m = O(a_n), n \to \infty.$$

Then for $|u| \leq K$,

$$(V^{\sigma_n}(u) - c_n) - (V^{\sigma_m}(u) - c_m) \geq -C.$$

Proof
We use [7, p. 46, eqn. (2.34)]

$$c_n = \int_0^n \log \frac{2}{a_s} ds$$

and [7, p. 46, eqn. (2.35)]

$$\sigma_n(t) = \int_0^n \gamma_{\Delta_s}(t) \, ds,$$

where γ_{Δ_s} is the equilibrium density for the interval $\Delta_s = [-a_s, a_s]$, so that

$$\gamma_{\Delta_s}(t) = \begin{cases} \frac{1}{\pi\sqrt{a_s^2 - t^2}}, & t \in (-a_s, a_s), \\ 0, & \text{otherwise} \end{cases}.$$

The Green's function for $\mathbb{C}\backslash\Delta_s$ with pole at ∞ has the representations

$$\begin{aligned} g_{\Delta_s}(u) &= \int \log|u-t|\,\gamma_{\Delta_s}(t)\,dt + \log\frac{2}{a_s} \\ &= \log\left|\frac{u}{a_s} + \sqrt{\left(\frac{u}{a_s}\right)^2 - 1}\right|. \end{aligned}$$

Then we see that

$$\begin{aligned} & -V^{\sigma_n}(u) + c_n \\ =& \int \log|u-t| \left(\int_0^n \gamma_{\Delta_s}(t)\,ds\right) dt + \int_0^n \log\frac{2}{a_s} ds \\ =& \int_0^n g_{\Delta_s}(u)\,ds. \end{aligned} \tag{30}$$

So,

$$(V^{\sigma_n}(u) - c_n) - (V^{\sigma_m}(u) - c_m) = -\int_m^n g_{\Delta_s}(u)\,ds.$$

Here for $s \in [m,n]$,

$$\begin{aligned} g_{\Delta_s}(u) &= \log\left|\frac{u}{a_s} + \sqrt{\left(\frac{u}{a_s}\right)^2 - 1}\right| \\ &\le \log\left(\left|\frac{u}{a_s}\right| + \sqrt{\left|\frac{u}{a_s}\right|^2 + 1}\right) \\ &\le \log\left(1 + 2\left|\frac{u}{a_s}\right|\right) \le 2\frac{K}{a_m}. \end{aligned}$$

Thus

$$-\int_m^n g_{\Delta_s}(u)\,ds \ge -C\frac{n-m}{a_m} \ge -C\frac{a_n}{a_m} \ge -C_1.$$

The last relation follows as $m \sim n \Rightarrow a_m \sim a_n$ (see (3.27) in [7, p. 72]). ■

We turn to the

Proof of Theorem 3.1

If (a_n) is bounded, then we can just choose $P_n = R_n^*$ and the assertions (13) and (14) of Theorem 3.1 follow from the corresponding ones in Lemma 3.4. Now we consider the case where (a_n) is unbounded. For $n \geq 1$, let $\ell = \ell(n)$ denote the greatest integer $\leq a_n - 2L$. By Corollary 2 in [8], there exist for large enough n, polynomials S_ℓ of degree $\leq \ell$ with

$$S_\ell(x) \sim \frac{1}{1+x^2}, x \in [-2a_n, 2a_n]$$

and

$$|S_\ell(z)| \geq C, |z| = \frac{1}{2}.$$

Then we set

$$P_n(z) = R_{n-\ell}^*(z) S_\ell\left(\frac{z}{2}\right),$$

a polynomial of degree $\leq n$. Then in $[-a_n, a_n]$, (29) gives

$$|P_n(x)| W(x) \leq \frac{C}{1+(x/2)^2},$$

so

$$\int_{-a_n}^{a_n} |P_n W|^2 \leq C.$$

Restricted range inequalities (see Theorem 4.2 in [7, p. 96]) then give (14). Moreover, (28) and Lemma 3.5 with $m = n - \ell$ give for $|z| = 1$ with $\arg(z) \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$\begin{aligned} |P_n(z)| &\geq C\left|R_{n-\ell}^*\right|(z) \geq C \exp\left(-\left[V^{\sigma_{n-\ell}}(z) - c_{n-\ell}\right]\right) \\ &\geq C_1 \exp\left(-\left[V^{\sigma_n}(z) - c_n\right]\right). \end{aligned}$$

So we have (13). Finally, the extremal property (1) of λ_n gives (15). ■

4 Proof of Theorem 1.2

If we combine Lemma 2.1 and Theorem 3.1, we see that the following three assertions together give Theorem 1.2:
(I)

$$\frac{1}{2\pi} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta \sim \sqrt{\frac{a_n}{n}} \exp\left(-2\left[V^{\sigma_n}(i) - c_n\right]\right). \quad (31)$$

(II) Given $0 < \eta < \frac{\pi}{2}$, there exists $C > 0$ such that

$$\exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) / \exp\left(-2\left[V^{\sigma_n}(i) - c_n\right]\right) \leq \exp\left(-C\frac{n}{a_n}\right), \quad (32)$$

uniformly for $n \geq 1$ and $\theta \in [-\eta, \eta] \cup [\pi - \eta, \pi + \eta]$.

(III)

$$V^{\sigma_n}(i) - c_n = -\int_0^n \log\left(\frac{1}{a_s} + \sqrt{1 + \frac{1}{a_s^2}}\right) ds. \quad (33)$$

(Recall that $n/a_n \to \infty$ as $n \to \infty$).

Proof of (I), (II)

Observe that as σ_n is even,

$$\begin{aligned} V^{\sigma_n}\left(e^{i\theta}\right) - V^{\sigma_n}(i) &= \int_0^{a_n} \log\left|\frac{t^2+1}{t^2 - e^{2i\theta}}\right| \sigma_n(t)\, dt \\ &= \frac{1}{2}\int_0^{a_n} \log\left(1 + \frac{4t^2\cos^2\theta}{\left|t^2 - e^{2i\theta}\right|^2}\right) \sigma_n(t)\, dt. \end{aligned}$$

Here for all θ and t,

$$\frac{4t^2\cos^2\theta}{\left|t^2 - e^{2i\theta}\right|^2} \geq \frac{4t^2\cos^2\theta}{\left(t^2+1\right)^2} \ (\leq 1)$$

while for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$, we have $\sim$ uniformly in θ, t, instead of just $\geq$. Then we obtain for all $\theta \in [-\pi, \pi]$,

$$V^{\sigma_n}\left(e^{i\theta}\right) - V^{\sigma_n}(i) \geq C\left(\cos^2\theta\right) \int_0^{a_n} \frac{t^2}{\left(t^2+1\right)^2} \sigma_n(t)\, dt \quad (34)$$

and for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$V^{\sigma_n}\left(e^{i\theta}\right) - V^{\sigma_n}(i) \sim \left(\cos^2\theta\right) \int_0^{a_n} \frac{t^2}{\left(t^2+1\right)^2} \sigma_n(t)\, dt. \quad (35)$$

In all cases, the constants are independent of n, θ. Now we need the estimates

$$\sigma_n(t) \leq \frac{Cn}{\sqrt{a_n^2 - t^2}}, t \in (-a_n, a_n)$$

and

$$\sigma_n(t) \sim \frac{n}{\sqrt{a_n^2 - t^2}} \sim \frac{n}{a_n}, t \in \left(-\frac{1}{2}a_n, \frac{1}{2}a_n\right).$$

These estimates follow from Theorem 1.11 in [7, pp. 17-18]. Let us substitute these bounds in (34) and (35). Some straightforward estimation gives for all $\theta \in [-\pi, \pi]$,

$$V^{\sigma_n}\left(e^{i\theta}\right) - V^{\sigma_n}(i) \geq C\frac{n}{a_n}\left(\cos^2\theta\right). \tag{36}$$

and for $\theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$,

$$V^{\sigma_n}\left(e^{i\theta}\right) - V^{\sigma_n}(i) \sim \frac{n}{a_n}\left(\cos^2\theta\right) \tag{37}$$

(For $\theta = \frac{\pi}{2}$, we interpret 0/0 as 1). Now (36) directly gives (32). Moreover, this last relation gives for some C_1, C_2, C_3,

$$\begin{aligned} & \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \exp\left(-2\left[V^{\sigma_n}\left(e^{i\theta}\right) - c_n\right]\right) d\theta / \exp\left(-2\left[V^{\sigma_n}(i) - c_n\right]\right) \\ \geq\ & \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \exp\left(-C_1\frac{n}{a_n}\cos^2\theta\right) d\theta \\ \geq\ & \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \exp\left(-C_2\frac{n}{a_n}\left(\theta - \frac{\pi}{2}\right)^2\right) d\theta \geq C_3\sqrt{\frac{a_n}{n}}. \end{aligned}$$

Similarly (37) gives a matching upper bound, and so we have (I) also. ■

Proof of (III)

From (30),

$$c_n - V^{\sigma_n}(i) = \int_0^n g_{\Delta_s}(i)\, ds.$$

Since g_{Δ_s} admits the representation

$$g_{\Delta_s}(z) = \log\left|\frac{z}{a_s} + \sqrt{\left(\frac{z}{a_s}\right)^2 - 1}\right|,$$

we obtain (33). ■

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