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ZERO DISTRIBUTION OF MÜNTZ EXTREMAL POLYNOMIALS IN $L_p[0,1]$

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ABSTRACT. Let $\{\lambda_j\}_{j=0}^{\infty}$ be a sequence of distinct positive numbers. Let $1 \leq p \leq \infty$ and $T_{n,p} = T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}(x)$ denote the L_p extremal Müntz polynomial in $[0,1]$ with exponents $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$. We investigate the zero distribution of $\{T_{n,p}\}_{n=1}^{\infty}$. In particular, we show that if

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \alpha > 0,$$

then the normalized zero counting measure of $T_{n,p}$ converges weakly as $n \to \infty$ to

$$\frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^{\alpha}(1-t^{\alpha})}} dt,$$

while if $\alpha = 0$ or ∞, the limiting measure is a Dirac delta at 0 or 1 respectively.

1. Introduction and Results

Let $\lambda_1, \lambda_2, \ldots$ be a sequence of distinct positive numbers. An expression of the form

$$\sum_{j=0}^{n} c_j x^{\lambda_j} \tag{1.1}$$

is called a Müntz polynomial. The name refers, of course, to the famous theorem of Müntz that if $\inf_j \lambda_j > 0$, these polynomials are dense in L_p spaces iff

$$\sum_{j=0}^{\infty} \frac{1}{\lambda_j} = \infty.$$

Müntz polynomials share many of the properties of ordinary algebraic polynomials. The most fundamental is that a polynomial of the form (1.1) has at most n distinct zeros in $(0,\infty)$, or is identically zero.

Müntz extremal polynomials are generalizations of classical orthogonal and Chebyshev polynomials. They have been investigated by amongst others, Borwein and Erdelyi [2], Milovanovic and his coworkers [3]. Let $1 \leq p \leq \infty$. We denote

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by $T_{n,p}(x) = T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}(x)$ the linear combination of $\{x^{\lambda_j}\}_{j=0}^{n}$ with coefficient of x^{λ_n} equal to 1, satisfying

$$\|T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}\|_{L_p[0,1]} = \min_{c_0 \cdots c_{n-1}} \|x^{\lambda_n} - \sum_{j=0}^{n-1} c_j x^{\lambda_j}\|_{L_p[0,1]}. \tag{1.2}$$

It is known that $T_{n,p}$ exists and is unique, has exactly n distinct (and simple) zeros in $(0,1)$, and the zeros of $T_{n,p}$ and $T_{n+1,p}$ interlace. Moreover, if we swap λ_n with some λ_j, the extremal polynomial changes only by a non-zero multiplicative constant. Thus when dealing with a fixed n, and studying zeros of extremal polynomials, we may assume that $\{\lambda_j\}_{j=0}^{n}$ are in increasing order. However, we shall not need to assume that $\{\lambda_j\}_{j=0}^{\infty}$ is increasing. Concerning the zeros as $n \to \infty$, an important result of Borwein [2, Thm. 4.1.1, p. 155] asserts that the corresponding Müntz polynomials are dense iff the maximum spacing between successive zeros of $T_{n,p}$ has limit 0 as $n \to \infty$. Saff and Varga [6] studied the related zero distribution of lacunary incomplete polynomials.

In this paper, we study the asymptotic zero distribution of $\{T_{n,p}\}_{n=1}^{\infty}$. Let ν_n denote the normalized zero counting measure of $T_{n,p}$, so that

$$\nu_n([a,b]) = \frac{1}{n} \times \text{Number of zeros of } T_{n,p} \text{ in } [a,b].$$

In the case of polynomials, where $\lambda_j = j$, $j \geq 0$, it is a classical result [5, pp. 169–170], [7, Thm. 3.4.1, p. 84 and Thm. 3.6.1, p. 98] that for $0 \leq a < b \leq 1$,

$$\lim_{n\to\infty} \nu_n([a,b]) = \int_a^b \frac{dx}{\pi\sqrt{x(1-x)}}.$$

Equivalently we write

$$d\nu_n \xrightarrow{*} \frac{dx}{\pi\sqrt{x(1-x)}}, \qquad n \to \infty$$

and say that $d\nu_n$ converges weakly to the arcsine distribution on $[0,1]$. This type of result has been studied in detail for the case $p = 2$ of orthogonal polynomials, and when there is a weight w in the norm in (1.2). The monograph of Stahl and Totik [7] gives a comprehensive account, while the monograph of Andrievskii and Blatt [1] considers discrepancy, or rate of convergence, to the limiting distribution.

In a loose sense, our conclusion is that when $\lim_{n\to\infty} \lambda_n/n$ exists, all the possible zero distributions are those provided by

$$\lambda_j = \alpha j, \qquad j \geq 0$$

for some $\alpha \in [0, \infty]$. Extremal polynomials for these exponents are essentially L_p extremal polynomials with the substitution of variable $x = t^\alpha$. Accordingly, we define for $0 < \alpha < \infty$, a probability measure on $(0,1)$,

$$d\mu_\alpha(t) = \frac{\alpha}{\pi} \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}}\, dt. \tag{1.3}$$

For $\alpha = 0$, we set

$$d\mu_0 = d\delta_0, \tag{1.4}$$

a unit mass at 0, and for $\alpha = \infty$, we set

$$d\mu_\infty = d\delta_1, \tag{1.5}$$

a unit mass at 1. We prove:

Theorem 1.1. *Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and $\{\lambda_j\}_{j=0}^{\infty}$ denote a sequence of distinct positive numbers with*

$$\lim_{j\to\infty} \frac{\lambda_j}{j} = \alpha. \tag{1.6}$$

Then if $0 \leq a \leq b \leq 1$,

$$\lim_{n\to\infty} \nu_n\left([a,b]\right) = \mu_\alpha\left([a,b]\right), \tag{1.7}$$

that is,

$$d\nu_n \stackrel{*}{\longrightarrow} d\mu_\alpha, \qquad n \to \infty.$$

Remarks . (a) An interesting feature of the theorem is that asymptotic zero distribution has no relation to the density of Müntz polynomials – in stark contrast to the Borwein-Erdelyi result on spacing. Thus if $\lambda_n = n \log n, n \geq 2$, then the corresponding Müntz polynomials are dense, while the asymptotic zero distribution is a Dirac delta at 1. If $\lambda_n = n^2, n \geq 0$, then the limiting zero distribution is still a Dirac delta at 1, but the corresponding Müntz polynomials are not dense.
(b) We can somewhat weaken the hypothesis (1.6): roughly speaking we can ignore $o(n)$ of the exponents in $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$. To make this more precise, assume $\alpha < \infty$. We write

$$\lim_{j\to\infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha \tag{1.8}$$

if for each $\varepsilon \in (0,1)$, there exists for large enough n, a set

$$S_{n,\varepsilon} \subset \{0,1,2,\ldots,n\} \tag{1.9}$$

with at most εn elements such that

$$j \in \{0,1,2,\ldots,n\} \backslash S_{n,\varepsilon} \Rightarrow \left|\frac{\lambda_j}{j} - \alpha\right| < \varepsilon. \tag{1.10}$$

In the case $\alpha = \infty$, we replace this by for each $K > 0$, there exists for large enough n, a set $S_{n,\varepsilon} \subset \{0,1,2,\ldots,n\}$ with at most εn elements such that

$$j \in \{0,1,2,\ldots,n\} \backslash S_{n,\varepsilon} \Rightarrow \frac{\lambda_j}{j} > K.$$

Theorem 1.2. *Let $1 \leq p \leq \infty$, $0 \leq \alpha \leq \infty$, and $\{\lambda_j\}_{j=0}^{\infty}$ denote a sequence of distinct positive numbers with*

$$\lim_{j\to\infty \ a.e.} \frac{\lambda_j}{j} = \alpha. \tag{1.11}$$

Then the conclusion (1.7) *of Theorem* 1.1 *persists.*

We shall also show that one cannot ignore more than $o(n)$ exponents in $\{\lambda_j\}_{j=0}^{n}$ without affecting the zero distribution:

Theorem 1.3. *Let $1 \leq p \leq \infty$ and $\varepsilon \in (0,1)$. Let $\{\lambda_j\}_{j=0}^{\infty}$, $\{\gamma_j\}_{j=0}^{\infty}$, $\{\rho_j\}_{j=0}^{\infty}$ denote sequences of distinct positive numbers with*

$$\lim_{j\to\infty} \frac{\gamma_j}{j} = 0; \quad \lim_{j\to\infty} \frac{\rho_j}{j} = \infty. \tag{1.12}$$

Assume also that for large enough n, there is the disjoint union

$$\{\lambda_j\}_{j=0}^{n} := \{\gamma_j\}_{j=0}^{k(n)} \cup \{\rho_j\}_{j=0}^{\ell(n)}, \tag{1.13}$$

where

$$\lim_{n\to\infty} \frac{k(n)}{n} = \varepsilon.$$

Then

$$d\nu_n \xrightarrow{*} \varepsilon\, d\mu_0 + (1-\varepsilon)\, d\mu_\infty, \qquad n \to \infty. \tag{1.14}$$

We are not sure if this result generalizes to the case where 0 and ∞ are replaced in (1.12) by other limits. What is clear is that for a general choice of $\{\lambda_j\}_{j=0}^{\infty}$, the asymptotic zero distribution can be quite complicated, and there need not be a weak limit. For example, by adjoining sufficiently large blocks of exponents $\{\alpha j\}_{j=n_1}^{n_2}$, one may construct $\{\lambda_n\}_{n=0}^{\infty}$, such that every $\mu_\alpha, \alpha \in [0,\infty]$, is a weak limit of some subsequence of $\{\nu_n\}$. We prove the results in the next section.

2. Proofs

We begin with some notation. We abbreviate $T_{n,p}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ as $T_{n,p}\{\lambda_0 \cdots \lambda_n\}$. Let $Z_p(\lambda_0 \cdots \lambda_n)[a,b]$ denote the total number of zeros of $T_{n,p}\{\lambda_0 \cdots \lambda_n\}(x)$ in $[a,b]$. We say that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a *refinement* of $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ if

$$\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\} \subset \{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}.$$

The main tools of proof are interlacing properties of successive Chebyshev polynomials, monotonicity properties with respect to the exponents, and zero distribution for the specific choice $\{\alpha j\}_{j=0}^{\infty}$.

Lemma 2.1. *Let $\{\gamma_j\}_{j=0}^{m}$ be distinct positive numbers and $\{\lambda_j\}_{j=0}^{n}$ be distinct positive numbers.*
(a) *Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ is a refinement of $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then for $[a,b] \subset [0,1]$,*

$$|Z_p(\lambda_0 \cdots \lambda_n)[a,b] - Z_p(\gamma_0 \cdots \gamma_m)[a,b]| \le 2(m-n). \tag{2.1}$$

(b) *Suppose that $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ have ℓ exponents in common. Then for $[a,b] \subset [0,1]$,*

$$|Z_p(\lambda_0 \cdots \lambda_n)[a,b] - Z_p(\gamma_0 \cdots \gamma_k)[a,b]| \le 2(n+k+2-2\ell). \tag{2.2}$$

Proof. (a) We may rewrite $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_m\}$ as $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m\}$. Since any subset of $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_m}\}$ is a Chebyshev system on $[\varepsilon, 1]$ for any $0 < \varepsilon < 1$, the zeros of $T_{n,p}\{\lambda_0 \cdots \lambda_j\}(x)$ and $T_{n,p}\{\lambda_0 \cdots \lambda_{j+1}\}(x)$ interlace [4, Corollary 1.1, p. 2]. It then follows that for every interval $[a,b]$,

$$|Z_p(\lambda_0 \cdots \lambda_j)[a,b] - Z_p(\lambda_0 \cdots \lambda_{j+1})[a,b]| \le 2.$$

Applying this for $j = n, n+1, \ldots, m$ gives (2.1).
(b) We may find a refinement of both $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$ consisting of $n+k+2-\ell$ elements. Applying (a) to the refinement and each of the sets $\{\gamma_0, \gamma_1, \gamma_2, \ldots, \gamma_k\}$ and $\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n\}$, and then combining the two inequalities gives the result. □

Apart from interlacing, we shall also use the lexicographic property:

Lemma 2.2. *Let* $\{\lambda_j\}_{j=0}^n$ *be a sequence of distinct positive numbers and* $\{\gamma_j\}_{j=0}^n$ *be a sequence of distinct positive numbers with*

$$\lambda_j \leq \gamma_j, \quad 0 \leq j \leq n. \tag{2.3}$$

Then for $0 \leq a \leq 1$*,*

$$Z_p(\lambda_0 \cdots \lambda_n)[a,1] \leq Z_p(\gamma_0 \cdots \gamma_n)[a,1]. \tag{2.4}$$

Proof. We may assume that the two sets have n exponents in common. For then, one can apply the result for this special case n times, using monotonicity each time. Let $0 < \varepsilon < 1$. Then in $[\varepsilon, 1]$, the combined set of powers $\{x^{\lambda_j}\}_{j=0}^n \cup \{x^{\gamma_j}\}_{j=0}^n$ (with duplicates deleted, and exponents placed in increasing order) is a Descartes system. If $T_{n,p}^{\varepsilon}\{\lambda_0 \cdots \lambda_n\}(x)$ and $T_{n,p}^{\varepsilon}\{\gamma_0 \cdots \gamma_n\}(x)$ denote the corresponding Müntz extremal polynomials on $[\varepsilon, 1]$, it is known that the zeros of $T_{n,p}^{\varepsilon}\{\lambda_0 \cdots \lambda_n\}(x)$ lie to the left of those of $T_{n,p}^{\varepsilon}\{\gamma_0 \cdots \gamma_n\}(x)$, in the sense that the jth smallest zero of the former Müntz polynomial is $\leq$ the jth smallest zero of the latter Müntz polynomial. For $p = \infty$, a proof of this is given in the book of Borwein and Erdelyi [2, Thm. 3.3.4, pp. 116–117]. For $1 < p \leq \infty$, a proof is given in Pinkus and Ziegler [4, Thm. 5.1, p. 13], while when $p = 1$, we can apply the remarks there (or a continuity argument involving $p \to 1+$). As $\varepsilon \to 0+$, $T_{n,p}^{\varepsilon}\{\gamma_0 \cdots \gamma_n\}(x)$ must converge uniformly to $T_{n,p}\{\gamma_0 \cdots \gamma_n\}(x)$ because of uniqueness of $T_{n,p}\{\gamma_0 \cdots \gamma_n\}(x)$, and the fact that the extremal error increases as $[\varepsilon, 1]$ grows to $[0, 1]$. Hence the zeros of $T_{n,p}\{\lambda_0 \cdots \lambda_n\}(x)$ lie to the left of those of $T_{n,p}\{\gamma_0 \cdots \gamma_n\}(x)$ and (2.4) follows. □

The next result asserts essentially that if for "most" indices j, we have $\lambda_j \leq \gamma_j$, then the asymptotic proportion of zeros in $[a, 1]$ of extremal polynomials with exponents $\{\lambda_j\}$ does not exceed that for $\{\gamma_j\}$.

Lemma 2.3. *Let* $\{\lambda_j\}_{j=0}^\infty$ *and* $\{\gamma_j\}_{j=0}^\infty$ *be sequences of distinct positive numbers with the following property: for each* $\varepsilon > 0$*, there exists for large enough* n*, a set*

$$S_{n,\varepsilon} \subset \{0, 1, 2, \ldots, n\} \tag{2.5}$$

with at most εn *elements such that*

$$j \in \{0, 1, 2, \ldots, n\} \backslash S_{n,\varepsilon} \Rightarrow \lambda_j \leq \gamma_j. \tag{2.6}$$

Then for $0 \leq a \leq 1$*,*

$$\limsup_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] \leq \limsup_{n\to\infty} \frac{1}{n} Z_p(\gamma_0 \cdots \gamma_n)[a,1] \tag{2.7}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[a,1] \leq \liminf_{n\to\infty} \frac{1}{n} Z_p(\gamma_0 \cdots \gamma_n)[a,1]. \tag{2.8}$$

Proof. Let us fix $\varepsilon > 0$, n large, and $S_{n,\varepsilon}$ be as in the statement. We define for the given n, a modified set of exponents $\{\lambda_j^*\}_{j=0}^n$ by

$$\lambda_j^* = \begin{cases} \lambda_j, & j \in \{0, 1, 2, \ldots, n\} \backslash S_{n,\varepsilon} \\ \gamma_j, & j \in S_{n,\varepsilon}. \end{cases}$$

Then

$$\lambda_j^* \leq \gamma_j, \quad 0 \leq j \leq n.$$

By the previous lemma, for $0 \leq a \leq 1$,

$$Z_p\left(\lambda_0^* \cdots \lambda_n^*\right)[a,1] \leq Z_p\left(\gamma_0 \cdots \gamma_n\right)[a,1].$$

Also $\left\{\lambda_j^*\right\}_{j=0}^n$ and $\left\{\lambda_j\right\}_{j=0}^n$ have at least $1 + n(1-\varepsilon)$ elements in common, so by Lemma 2.1(b),

$$\left|Z_p\left(\lambda_0^* \cdots \lambda_n^*\right)[a,1] - Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1]\right| \leq 4\varepsilon n + 4.$$

Combining these inequalities gives

$$Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] \leq Z_p\left(\gamma_0 \cdots \gamma_n\right)[a,1] + 4\varepsilon n + 4.$$

Dividing by n and letting $n \to \infty$ gives

$$\limsup_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] \leq \limsup_{n\to\infty} \frac{1}{n} Z_p\left(\gamma_0 \cdots \gamma_n\right)[a,1] + 4\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, (2.7) follows. Similarly, (2.8) follows. □

Next, we study the zero distribution for the comparison sequence $\{\alpha j\}_{j=0}^{\infty}$:

Lemma 2.4. *Let* $\alpha \in (0,\infty)$ *and*

$$\gamma_j = \alpha j, \qquad j \geq 0. \tag{2.9}$$

Then for $0 \leq a < b \leq 1$,

$$\lim_{n\to\infty} \frac{1}{n} Z_p\left(\gamma_0 \cdots \gamma_n\right)[a,b] = \mu_\alpha\left([a,b]\right). \tag{2.10}$$

Proof. Suppose first $p < \infty$. Let $T_{n,p}^*$ denote the monic (ordinary) polynomial of degree n satisfying

$$\int_0^1 \left|T_{n,p}^*(x)\right|^p \frac{1}{\alpha} x^{1/\alpha - 1} dx = \min_{\deg(P) \leq n-1} \int_0^1 \left|x^n - P(x)\right|^p \frac{1}{\alpha} x^{1/\alpha-1} dx.$$

The substitution $x = t^\alpha$ gives

$$\int_0^1 \left|T_{n,p}^*(t^\alpha)\right|^p dt = \min_{\deg(P) \leq n-1} \int_0^1 \left|t^{\alpha n} - P(t^\alpha)\right|^p dt.$$

It follows from uniqueness that

$$T_{n,p}^*(t^a) = T_{n,p}\left\{\gamma_0 \cdots \gamma_n\right\}(t). \tag{2.11}$$

We see then that the total multiplicity of zeros of $T_{n,p}\left\{\gamma_0 \cdots \gamma_n\right\}$ in $[a,b]$ is the total multiplicity of zeros of $T_{n,p}^*$ in $[a^\alpha, b^\alpha]$. Since the weight $\frac{1}{\alpha} x^{1/\alpha-1}$ is positive a.e. in $[0,1]$, classical results assert that the limiting zero distribution of $\left\{T_{n,p}^*\right\}_{n=0}^{\infty}$ is the arcsine distribution [1, Cor. 5.7, p. 261]. Hence as $n \to \infty$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \times \text{Number of zeros of } T_{n,p} \text{ in } [a,b] \\
&\quad = \lim_{n\to\infty} \frac{1}{n} \times \text{Number of zeros of } T_{n,p}^* \text{ in } [a^\alpha, b^\alpha] \\
&\quad = \int_{a^\alpha}^{b^\alpha} \frac{dx}{\pi\sqrt{x(1-x)}} = \frac{\alpha}{\pi} \int_a^b \frac{t^{\alpha-1}}{\sqrt{t^\alpha(1-t^\alpha)}} dt = \int_a^b d\mu_\alpha(t).
\end{aligned}$$

□

Proof of Theorem 1.2. Our hypothesis is

$$\lim_{j\to\infty \text{ a.e.}} \frac{\lambda_j}{j} = \alpha.$$

Assume first that $0 < \alpha < \infty$. Let $\varepsilon \in (0, \alpha)$. We then obtain for large enough n, from (1.10),

$$j \in \{0, 1, 2, \ldots, n\} \backslash S_{n,\varepsilon} \Rightarrow (\alpha - \varepsilon)\, j \le \lambda_j \le (\alpha + \varepsilon)\, j.$$

Applying Lemma 2.3, with $\gamma_j = (\alpha + \varepsilon)\, j$, $j \ge 0$, we deduce that

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1]\\ &\quad\le \limsup_{n\to\infty} \frac{1}{n} Z_p\left(0, (\alpha+\varepsilon), 2(\alpha+\varepsilon), \ldots, n(\alpha+\varepsilon)\right)[a,1]\end{aligned}$$

and similarly applying Lemma 2.3 to $(\alpha - \varepsilon)\, j$, $j \ge 0$, and λ_j, $j \ge 0$ (with roles swapped),

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{n} Z_p\left(0, (\alpha-\varepsilon), 2(\alpha-\varepsilon), \ldots, n(\alpha-\varepsilon)\right)[a,1]\\ &\quad\le \liminf_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1].\end{aligned}$$

Applying Lemma 2.4 with $\gamma_j = (\alpha \pm \varepsilon)\, j$, $j \ge 0$, gives

$$\begin{aligned}&\int_a^1 d\mu_{\alpha-\varepsilon}(t) \le \liminf_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1]\\ &\qquad\le \limsup_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] \le \int_a^1 d\mu_{\alpha+\varepsilon}(t).\end{aligned}$$

Letting $\varepsilon \to 0+$, and using dominated convergence gives

$$\lim_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] = \int_a^1 d\mu_\alpha(t).$$

This gives the result when $[a, b] = [a, 1]$. For general $[a, b]$, we use

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,b]\\ &\quad= \lim_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] - \lim_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)(b,1]\\ &\quad= \int_a^1 d\mu_\alpha(t) - \int_b^1 d\mu_\alpha(t).\end{aligned}$$

Note that because μ_α is absolutely continuous, the number of zeros in a neighborhood of the point b is negligible in the sense of asymptotic distribution. Finally, if $\alpha = 0$, the arguments above give for $0 < a \le 1$,

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1]\\ &\quad\le \limsup_{n\to\infty} \frac{1}{n} Z_p\left(0, \varepsilon, 2\varepsilon, \ldots, n\varepsilon\right)[a,1] = \int_a^1 d\mu_\varepsilon(t).\end{aligned}$$

Letting $\varepsilon \to 0+$ (and using some straightforward estimates) gives

$$\lim_{n\to\infty} \frac{1}{n} Z_p\left(\lambda_0 \cdots \lambda_n\right)[a,1] = 0 = \int_a^1 d\mu_0(t).$$

Since $\frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,1] = 1$, we obtain

$$\lim_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,1] = 1 = \int_0^1 d\mu_0(t).$$

The case $\alpha = \infty$ is similar. □

Proof of Theorem 1.1. This is a special case of Theorem 1.2. □

Proof of Theorem 1.3. Let $0 < a < b < 1$. Because of (1.13) and interlacing properties, to the left of each zero of $T_{n,p}\left\{\gamma_0 \cdots \gamma_{k(n)}\right\}(x)$ in $[0,a]$, there is a zero of $T_{n,p}\{\lambda_0 \cdots \lambda_n\}(x)$. Moreover,

$$Z_p(\lambda_0 \cdots \lambda_n)[0,a] \geq Z_p\left(\gamma_0 \cdots \gamma_{k(n)}\right)[0,a]$$

so applying Theorem 1.1 to $\{\gamma_j\}_{j=0}^{\infty}$,

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,a] &\geq \liminf_{n\to\infty} \frac{k(n)}{n} \frac{1}{k(n)} Z_p\left(\gamma_0 \cdots \gamma_{k(n)}\right)[0,a] \\ &= \varepsilon \int_0^a d\mu_0 = \varepsilon. \end{aligned} \tag{2.12}$$

Similarly,

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[b,1] &\geq \liminf_{n\to\infty} \frac{\ell(n)}{n} \frac{1}{\ell(n)} Z_p(\rho_0 \cdots \ell(n))[b,1] \\ &= (1-\varepsilon) \int_b^1 d\mu_\infty = 1-\varepsilon. \end{aligned} \tag{2.13}$$

Then it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)(a,b) \\ &\quad \leq 1 - \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,a] - \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[b,1] \leq 0. \end{aligned}$$

So for $0 < a < b < 1$,

$$\lim_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)(a,b) = 0.$$

Next, by (2.12) and (2.13),

$$\begin{aligned} \varepsilon &\leq \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,a] \\ &\leq \limsup_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,a] \\ &\leq 1 - \liminf_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)(a,1] \leq \varepsilon, \end{aligned}$$

so

$$\lim_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[0,a] = \varepsilon.$$

Similarly,

$$\lim_{n\to\infty} \frac{1}{n} Z_p(\lambda_0 \cdots \lambda_n)[b,1] = 1-\varepsilon.$$

It follows that as $n \to \infty$,

$$d\nu_n \xrightarrow{*} \varepsilon d\delta_0 + (1-\varepsilon) d\delta_1.$$

□

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