Weights whose Biorthogonal Polynomials admit a Rodrigues Formula

D. S. Lubinsky[1], I. Soran[2]
[1,2]School of Mathematics,
Georgia Institute of Technology,
Atlanta, GA 30332-0160,
USA.
[1] lubinsky@math.gatech.edu
[2] gtg119v@mail.gatech.edu

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Abstract

Let $\alpha > 0$ and $\psi(x) = x^{\alpha}$. Let w be a nonnegative integrable function on an interval I. Let P_n be a polynomial of degree n determined by the biorthogonality conditions

$$\int_I P_n \psi^j w = 0, j = 0, 1, ..., n-1.$$

We determine for which weights w, P_n admits an analogue of the classical Rodrigues formula for orthogonal polynomials, and present the formula whenever it exists. We also provide generating functions and fairly explicit representations for P_n.

1 [1]Introduction and Results

Let I be a real interval and $\psi : I \to \mathbb{R}$ be a strictly increasing continuous function. Let w be a function non-negative and positive a.e. on I for which all the modified moments

$$\omega_{j,k} = \int_I \psi(x)^j x^k w(x)\, dx,\ j, k = 0, 1, 2, ... \tag{1}$$

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exist. Then we may try determine a polynomial P_n of degree n by the biorthogonality conditions

$$\int_I P_n(x)\,\psi(x)^j\,w(x)\,dx = \begin{cases} 0, & j = 0,1,2,...,n-1, \\ I_n \neq 0, & j = n \end{cases}. \tag{2}$$

The fact that ψ is increasing forces P_n to have n simple zeros in I. In turn that easily implies the uniqueness of P_n up to a multiplicative constant. One representation for P_n is a determinantal one:

$$P_n(x) = \frac{\det\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \cdots & \omega_{0,n} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots & \omega_{1,n} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \cdots & \omega_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1,0} & \omega_{n-1,1} & \omega_{n-1,2} & \cdots & \omega_{n-1,n} \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix}}{\det\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \cdots & \omega_{0,n-1} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \cdots & \omega_{1,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-1,0} & \omega_{n-1,1} & \omega_{n-1,2} & \cdots & \omega_{n-1,n-1} \end{bmatrix}},$$

provided the denominator determinant is non-0. Non-vanishing of that determinant is necessary and sufficient for the existence of P_n [3, p. 2ff.]. In our case, we can prove the non-vanishing by contradiction. For if the determinant vanished, we can find real numbers $\{c_k\}_{k=0}^{n-1}$ not all 0 such that for $Q(x) = \sum_{k=0}^{n-1} c_k x^k$,

$$\int_I Q\psi^j w = 0, 0 \leq j \leq n-1.$$

Choosing P to be a polynomial in x of degree $\leq n-1$ such that $P \circ \psi$ has sign changes where Q does gives

$$0 < \int_I QPw = 0,$$

a contradiction. Biorthogonal polynomials of a more general form have been studied in several contexts - see [3].

It was A. Sidi who first considered biorthogonal polynomials of this type, for the weight $w = 1$, the interval $I = (0,1)$, and the special function

$$\psi(x) = \log x,$$

He constructed what are now called the *Sidi polynomials*, in problems of quadrature and convergence acceleration [4], [5], [9], [10], [11]. Sidi's polynomials admit the Rodrigues type formula

$$P_n\left(e^u\right) = e^{-u}\left(\frac{d}{du}\right)^n\left[e^u\left(1-e^u\right)^n\right] \tag{3}$$

and are explicitly given as

$$P_n\left(x\right) := \sum_{j=0}^{n}\binom{n}{j}\left(j+1\right)^n\left(-x\right)^j,$$

Their asymptotic behavior as $n \to \infty$ was investigated in [5]. The zero distribution of more general biorthogonal polynomials has been investigated in [7].

In a recent paper, Herbert Stahl and the first author [6] derived a Rodrigues type formula, and an explicit expression for $P_n(x)$ when $I = (0,1)$, $w = 1$, and $\psi(x) = x^\alpha$, any $\alpha > 0$. These have the form

$$P_n\left(u^{1/\alpha}\right) = u^{1-1/\alpha}\left(\frac{d}{du}\right)^n\left[u^{n-1+1/\alpha}\left(1-u^{1/\alpha}\right)^n\right] \tag{4}$$

and

$$P_n\left(x\right) = \sum_{j=0}^{n}\binom{n}{j}\left[\prod_{k=0}^{n-1}\left(k+\frac{j+1}{\alpha}\right)\right]\left(-x\right)^j. \tag{5}$$

It then seems interesting, in the spirit of classical orthogonal polynomials, to determine for which weights w, there is some type of Rodrigues formula. It is well known that the only weights whose orthogonal polynomials admit Rodrigues formulae are the Jacobi, Laguerre, and Hermite weights. Tricomi [14, pp. 129-133] gives a very readable account of this (in German). A survey of characterizations of classical orthogonal polynomials was given by Al-Salam [1], while the Rodrigues formulae are discussed in [2], [8], [13].

In Tricomi's presentation, one starts with a weight w on an interval I, with corresponding orthogonal polynomials $\{p_n\}_{n=0}^{\infty}$, and looks for a Rodrigues formula

$$p_n\left(x\right) = \frac{1}{w\left(x\right)}\left(\frac{d}{dx}\right)^n\left[w\left(x\right)X\left(x\right)^n\right]. \tag{6}$$

Here X is a polynomial of degree at most 2. While one might look at other forms, it is readily seen that to get a polynomial of degree n from this, X cannot have degree higher than 2. By examining the case $n = 1$, one determines which weights allow such formulae for their orthogonal polynomials. Three cases arise:

(I) X is a polynomial of degree 2.

After extracting a constant, we can then factorize it as

$$X(x) = (x-a)(x-b).$$

In this case, it turns out that apart from a multiplicative constant, w is a Jacobi weight on (a,b):

$$w(x) = (x-a)^{\alpha}(b-x)^{\beta}$$

with $\alpha, \beta > -1$.

(II) X is a polynomial of degree 1.

After extracting a constant, we can then factorize it as

$$X(x) = x-a.$$

In this case, it turns out that apart from a multiplicative constant, w is a Laguerre weight on (a,∞):

$$w(x) = (x-a)^{\alpha} e^{-cx}$$

with $\alpha > -1, c > 0$.

(III) X is a constant polynomial.

In this case, it turns out that apart from a multiplicative constant, w is a Hermite weight on $(-\infty,\infty)$:

$$w(x) = e^{-cx^2+dx}$$

for some $c > 0$, $d \in \mathbb{R}$.

The differential equation satisfied by these three classical weights is called a Pearson differential equation [1, p. 8]; it determines when there is a Rodrigues formula.

The main purpose of this paper is to determine which weights w have biorthogonal polynomials that admit Rodrigues type formulae when $\psi(x) = x^{\alpha}$. Clearly there has to be a modification of (6), and in the search for this, we are guided by (3) and (4). Moreover, for non-integer α, our interval of biorthogonality cannot include the negative real axis. We prove:

Theorem 1
Let $\alpha > 0$ *and*

$$\psi(x) = x^{\alpha}.$$

Let I *be an open interval on which* ψ *is well defined, and let* $w : I \to [0, \infty)$ *be infinitely differentiable and positive a.e. on* I *with all moments in (1) finite. Let* P_n *be a polynomial of degree* n *determined by the biorthogonality conditions*

$$\int_I P_n(x)\,\psi(x)^j\,w(x)\,dx \begin{cases} = 0, & j < n \\ \neq 0, & j = n \end{cases}. \tag{7}$$

(I) If $I = (0,1)$*, then for* $n \geq 0$*,* P_n *admits (up to a constant multiple) the representation*

$$P_n\left(u^{1/\alpha}\right) = \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)} \left(\frac{d}{du}\right)^n \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\left(u\left(1-u^{1/\alpha}\right)\right)^n\right] \tag{8}$$

iff w *is a Jacobi weight*

$$w(x) = x^a (1-x)^b \tag{9}$$

for some $a, b > -1$.
(II) If $I = (0, \infty)$*, then for* $n \geq 0$*,* P_n *admits (up to a constant multiple) the representation*

$$P_n\left(u^{1/\alpha}\right) = \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)} \left(\frac{d}{du}\right)^n \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right) u^n\right] \tag{10}$$

iff w *is a Laguerre weight*

$$w(x) = x^a e^{-cx} \tag{11}$$

for some $a > -1$ *and* $c > 0$.
(III) If $I = (-\infty, \infty)$*, then for* $n \geq 0$*,* P_n *admits (up to a constant multiple) the representation*

$$P_n\left(u^{1/\alpha}\right) = \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)} \left(\frac{d}{du}\right)^n \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\right] \tag{12}$$

iff $\alpha = 1$ *and* w *is a Hermite weight*

$$w(x) = e^{-cx^2+bx} \tag{13}$$

for some $c > 0$ *and* $b \in \mathbb{R}$.

Remarks

(a) In stating the result, we specified the interval in each of the three cases to simplify the formulation. Perhaps the most curious case is $I = (-\infty, \infty)$, in which only $\alpha = 1$ is permissible, reducing to classical orthogonal polynomials. That α needs to be an integer in this case follows from the requirement that $\psi(x) = x^{\alpha}$ is real valued. However, it is surprising that $\alpha = 3, 5, 7, ...$ have biorthogonal polynomials that do not admit Rodrigues type formulae.

(b) We see that our analogues of the polynomial $X(x)$ of degree ≤ 2 in (6) are $X(x) = x\left(1 - x^{1/\alpha}\right)$ for $I = (0, 1)$; $X(x) = x$ for $I = (0, \infty)$; and $X(x) = 1$ for $I = \mathbb{R}$.

(c) In the case $\alpha = 1$, all the Rodrigues formulae above reduce to those for classical orthogonal polynomials.

(d) There is a dual orthogonal relation to (7), namely

$$\int_I P_n\left(u^{1/\alpha}\right) u^j w_1(u)\, du = 0, 0 \leq j < n,$$

where

$$w_1(u) = w\left(u^{1/\alpha}\right) u^{1/\alpha - 1}.$$

(The interval of integration is still I because $\psi(x) = x^{\alpha}$ maps I onto I in the cases when there is a Rodrigues formula).

(d) For the Jacobi and Laguerre case, we can give some explicit representations and also a generating function. We start with the former case. Recall the Pochhammer symbol

$$(c)_n = c(c+1)(c+2)...(c+n-1).$$

Corollary 2

Let $\alpha > 0$ and $n \geq 1$. Let w be a Jacobi weight (9) and P_n be given by (8).

(a) Let $S_{n,j}, -1 \leq j \leq n-1$, be determined by the relations $S_{n,-1}(x) = \frac{1}{x}; S_{n,0}(x) = -\frac{b+n}{\alpha}$ and for $j \geq 1$,

$$S_{n,j}(x) = S_{n,j-1}(x)\left\{\frac{1}{\alpha} - j + x\left[-\frac{b+n-j}{\alpha} + j - \frac{1}{\alpha}\right]\right\} + \frac{1}{\alpha} x(1-x) S'_{n,j-1}(x). \tag{14}$$

Then

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{\left(\frac{a+1}{\alpha}\right)_n}{\left(\frac{a+1}{\alpha}\right)_j} (1-x)^{n-j}\, x S_{n,j-1}(x). \tag{15}$$

(b) The leading coefficient of P_n is

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\left(\frac{a+1}{\alpha}\right)_n}{\left(\frac{a+1}{\alpha}\right)_j}(-1)^{n-j}\left(-\frac{b+n}{\alpha}\right)_j.$$

(c) Let $u\in\mathbb{C}\backslash\left((-\infty,0]\cup[1,\infty)\right)$ and Γ be a positively oriented circle center u^{α}, of small enough radius. Then for $|z|$ sufficiently small, with all branches taken as principal ones,

$$\frac{w(u)}{u^{1-\alpha}}\sum_{n=0}^{\infty}\frac{P_n(u)\,z^n}{n!}=\frac{1}{2\pi i}\int_{\Gamma}\frac{t^{1/\alpha-1}w\left(t^{1/\alpha}\right)}{t\left(1-z\left(1-t^{1/\alpha}\right)\right)-u^{\alpha}}dt. \tag{16}$$

We note that for small enough $|z|$, there is exactly one simple pole of the integrand in (16) inside Γ. It is located at

$$t=u^{\alpha}\left(1+z\left(1-u\right)\right)+O\left(z^2\right).$$

However, it seems impossible to explicitly compute the location of the residue (except in the classical case $\alpha=1$) and hence deduce an explicit generating function from this contour integral. For the Laguerre case, we can obtain a more explicit generating function:

Corollary 3
Let $\alpha>0$ and $n\geq 1$. Let w be a Laguerre weight (11) with $c=1$ and P_n be given by (12).
(a) Let $R_{n,j}, 1\leq j\leq n$, be polynomials determined by the relations

$$R_{n,1}(x)=\frac{a+1}{\alpha}-1+n-\frac{x}{\alpha}$$

and for $j\geq 1$,

$$R_{n,j+1}(x)=\left[\frac{a+1}{\alpha}-1+n-j-\frac{x}{\alpha}\right]R_{n,j}(x)+\frac{x}{\alpha}R_{n,j}'(x). \tag{17}$$

Then

$$P_n(x)=R_{n,n}(x). \tag{18}$$

(b) The leading coefficient of P_n is $(-1/\alpha)^n$.
(c) For $v\in\mathbb{C}$ and $|z|<1$,

$$\sum_{n=0}^{\infty}\frac{P_n(v)\,z^n}{n!}=(1-z)^{-\frac{a+1}{\alpha}}\exp\left(v\left[1-(1-z)^{-1/\alpha}\right]\right). \tag{19}$$

Note that for $\alpha = 1$, the generating function becomes a classical one for Laguerre polynomials, taking account of the different normalization of the Laguerre polynomial L_n [8, p. 202, eqn. (4)].

We prove the results for Jacobi weights, namely Theorem 1(I) and Corollary 2 in Section 2; the results for Laguerre weights, namely Theorem 1(II) and Corollary 3 in Section 3; and the Hermite case is considered in Section 4.

2 *The Jacobi Case*

In this section, we prove Theorem 1 (I) and Corollary 2. We begin with the necessity that w is a Jacobi weight for a Rodrigues formula to hold:

Proof of Necessity that w is a Jacobi weight
Assume that (8) holds. Then for $n = 1$ this gives

$$\begin{aligned} P_1\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)\left(u\left(1-u^{1/\alpha}\right)\right)\right] \\ &= \frac{w'}{w}\left(u^{1/\alpha}\right)\frac{1}{\alpha}u^{1/\alpha}\left(1-u^{1/\alpha}\right)+\frac{1}{\alpha}-\frac{2}{\alpha}u^{1/\alpha}. \end{aligned}$$

Set $x = u^{1/\alpha}$ and use that P_1 is a linear polynomial. We obtain for some constants A and B,

$$A + Bx = \frac{w'}{w}(x)\, x\,(1-x).$$

Dividing by $x(1-x)$ and using partial fractions gives for some constants a and b,

$$\frac{a}{x} + \frac{b}{1-x} = \frac{w'}{w}(x).$$

Integrating shows that w is a Jacobi weight (9), apart from a multiplicative constant. The fact that $a, b > -1$ follows from integrability of w. ■

We turn to the sufficiency part of Theorem 1 (I). We must prove that when w is a Jacobi weight, then P_n given by (8) firstly satisfies the orthogonality conditions, and secondly is a polynomial of degree n.

Proof of the Orthogonality Condition (7)

Let w be a Jacobi weight (9), and P_n be given by (8). Let

$$\begin{aligned} I_j &= \int_0^1 P_n(x)\,(x^\alpha)^j\, w(x)\,dx \\ &= \frac{1}{\alpha}\int_0^1 P_n\left(u^{1/\alpha}\right) u^j w\left(u^{1/\alpha}\right) u^{1/\alpha-1} du \\ &= \frac{1}{\alpha}\int_0^1 u^j \left(\frac{d}{du}\right)^n \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\left[u\left(1-u^{1/\alpha}\right)\right]^n\right] du. \end{aligned}$$

Observe that $u^{1/\alpha-1}w\left(u^{1/\alpha}\right)\left[u\left(1-u^{1/\alpha}\right)\right]^n$ has a zero at 0 of multiplicity $\frac{1}{\alpha}-1+\frac{a}{\alpha}+n > n-1$. Moreover the multiplicity of the zero at 1 is $b+n > n-1$. We integrate by parts j times to obtain

$$I_j = \frac{1}{\alpha}(-1)^j j! \int_0^1 \left(\frac{d}{du}\right)^{n-j} \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\left[u\left(1-u^{1/\alpha}\right)\right]^n\right] du = 0,$$

if $j < n$. When $j = n$, we obtain instead

$$I_n = \frac{1}{\alpha}(-1)^n n! \int_0^1 u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\left[u\left(1-u^{1/\alpha}\right)\right]^n du \neq 0,$$

as the integrand is positive in $(0,1)$. ■

Remark
After a substitution, we see that

$$\begin{aligned} I_n &= (-1)^n n! \int_0^1 x^{a+n\alpha}(1-x)^{b+n}\,dx \\ &= (-1)^n n! \frac{\Gamma(a+n\alpha+1)\,\Gamma(b+n+1)}{\Gamma(a+b+2+n+n\alpha)}. \end{aligned} \tag{20}$$

The most complicated part of the proof is showing that P_n is indeed a polynomial of degree n. This requires:

Lemma 2.1
For $j \geq 1$,

$$\left(\frac{d}{du}\right)^j \left(1-u^{1/\alpha}\right)^{b+n} = \left(1-u^{1/\alpha}\right)^{b+n-j} u^{1/\alpha-j} S_{n,j-1}\left(u^{1/\alpha}\right), \tag{21}$$

where $S_{n,j-1}$ is a polynomial of degree $j-1$, determined by the recursion

$$S_{n,0}(x) = -\frac{b+n}{\alpha}$$

and for $j \geq 1$,

$$S_{n,j}(x) = S_{n,j-1}(x)\left\{\frac{1}{\alpha} - j + \left(-\frac{b+n-j}{\alpha} + j - \frac{1}{\alpha}\right)x\right\} + \frac{1}{\alpha}x(1-x)S'_{n,j-1}(x). \tag{22}$$

The leading coefficient of $S_{n,j}$ *is*

$$\left(-\frac{b+n}{\alpha}\right)_{j+1}. \tag{23}$$

Proof

We use induction on j: first for $j = 1$,

$$\frac{d}{du}\left(1-u^{1/\alpha}\right)^{b+n} = (b+n)\left(1-u^{1/\alpha}\right)^{b+n-1}u^{1/\alpha-1}\left(-\frac{1}{\alpha}\right),$$

so we can take

$$S_{n,0}\left(u^{1/\alpha}\right) = -\frac{b+n}{\alpha}. \tag{24}$$

Now assume that (21) is true for j. We shall prove it for $j+1$. Differentiating (21) gives

$$\begin{aligned}
&\left(\frac{d}{du}\right)^{j+1}\left(1-u^{1/\alpha}\right)^{b+n} \\
=\; & \frac{d}{du}\left[\left(1-u^{1/\alpha}\right)^{b+n-j}u^{1/\alpha-j}S_{n,j-1}\left(u^{1/\alpha}\right)\right] \\
=\; & \left(1-u^{1/\alpha}\right)^{b+n-(j+1)}u^{1/\alpha-(j+1)}\left\{\begin{array}{c} -\frac{b+n-j}{\alpha}u^{1/\alpha}S_{n,j-1}\left(u^{1/\alpha}\right) \\ +(1-u^{1/\alpha})\left(\frac{1}{\alpha}-j\right)S_{n,j-1}\left(u^{1/\alpha}\right) \\ +\frac{1}{\alpha}\left(1-u^{1/\alpha}\right)u^{1/\alpha}S'_{n,j-1}\left(u^{1/\alpha}\right)\end{array}\right\} \\
=\; & \left(1-u^{1/\alpha}\right)^{b+n-(j+1)}u^{1/\alpha-(j+1)}S_{n,j}\left(u^{1/\alpha}\right),
\end{aligned} \tag{25}$$

where $S_{n,j}(x)$ is a polynomial of degree at most j in x determined by the recursion (22). By induction, (21) is true for all $j \geq 1$. Finally, if d_j is the leading coefficient of $S_{n,j}$, we see that $d_0 = -\frac{b+n}{\alpha}$ and for $j \geq 1$,

$$d_j = d_{j-1}\left(-\frac{b+n}{\alpha} + j\right).$$

Iterating this gives (23). ■

The result of the lemma remains true for $j = 0$ if we adopt the convention

$$S_{n,-1}(x) \equiv \frac{1}{x}. \tag{26}$$

We can now complete the sufficiency part of Theorem 1(I):

Proof that P_n given by (8) is a polynomial of degree n
We use Leibniz's formula on (8):

$$\begin{aligned} P_n\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{d}{du}\right)^j \left(1-u^{1/\alpha}\right)^{b+n} \\ &\quad \times \left(\frac{d}{du}\right)^{n-j} \left(u^{n-1+\frac{a+1}{\alpha}}\right) \\ &= \sum_{j=0}^{n} \binom{n}{j} \left(1-u^{1/\alpha}\right)^{n-j} S_{n,j-1}\left(u^{1/\alpha}\right) u^{1/\alpha} \\ &\quad \times \left(n-1+\frac{a+1}{\alpha}\right)\left(n-2+\frac{a+1}{\alpha}\right)\ldots\left(j+\frac{a+1}{\alpha}\right), \end{aligned}$$

by Lemma 2.1, and with the convention (26). Setting $x = u^{1/\alpha}$ gives

$$P_n(x) = \sum_{j=0}^{n} \binom{n}{j} \frac{\left(\frac{a+1}{\alpha}\right)_n}{\left(\frac{a+1}{\alpha}\right)_j} (1-x)^{n-j} \, x S_{n,j-1}(x), \tag{27}$$

a polynomial of degree at most n. To show that P_n must have degree n we use the biorthogonality relations (7). Firstly, those relations imply that P_n has at least n simple zeros in $(0,1)$. For else, we can construct a polynomial Q of degree at most $n-1$ such that $Q \circ \psi$ has sign changes in $(0,1)$ exactly where P_n does, so that (after multiplying Q by ± 1) $P_n Q \circ \psi > 0$ a.e. in $(0,1)$. Then

$$0 < \int_0^1 P_n(x)\, Q \circ \psi(x)\, w(x)\, dx = 0,$$

by (7). This contradiction shows that P_n either has degree n or is identically 0. That the former must be true follows from the second relation in (7). ■

Proof of Corollary 2
(a), (b) These follow readily from (27) and Lemma 2.1.

(c) Let $u \in (0,1)$ and Γ be a positively oriented circle center u of small radius. By Cauchy's integral formula for derivatives, with all branches principal,

$$\begin{aligned} \frac{w\left(u^{1/\alpha}\right)}{u^{1-1/\alpha}} P_n\left(u^{1/\alpha}\right) &= \left(\frac{d}{du}\right)^n \left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right)\left(u\left(1-u^{1/\alpha}\right)\right)^n\right] \\ &= \frac{n!}{2\pi i}\int_\Gamma \frac{t^{1/\alpha-1} w\left(t^{1/\alpha}\right)\left[t\left(1-t^{1/\alpha}\right)\right]^n}{(t-u)^{n+1}} dt. \end{aligned}$$

Then

$$\begin{aligned} \frac{w\left(u^{1/\alpha}\right)}{u^{1-1/\alpha}} \sum_{n=0}^{\infty} \frac{P_n\left(u^{1/\alpha}\right) z^n}{n!} &= \frac{1}{2\pi i}\int_\Gamma \frac{t^{1/\alpha-1} w\left(t^{1/\alpha}\right)}{t-u} \sum_{n=0}^{\infty}\left(\frac{t\left(1-t^{1/\alpha}\right)z}{t-u}\right)^n dt \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{t^{1/\alpha-1} w\left(t^{1/\alpha}\right)}{t-u-t\left(1-t^{1/\alpha}\right)z} dt. \end{aligned}$$

The interchange of series and integral and summation of the geometric series is justified by uniform convergence (for $|z|$ sufficiently small). Replacing u by $u^\alpha \in (0,1)$ then yields (16) for such u. The left-hand side of (16) is an analytic function of $u \in \mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$, with principal choice of branches, provided $|z|$ is sufficiently small. We can see this by using the first contour integral above to bound $\left|\frac{w(u)}{u^{\alpha-1}}\frac{P_n(u)}{n!}\right|$ by C^n uniformly in n and for u in a given compact subset of $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$. The right-hand side is also analytic in that region. In fact we can use analytic continuation and finitely many shifts of the center of Γ, while keeping the radius constant to move the contour from a point in $(0,1)$ to any fixed point in $\mathbb{C}\setminus((-\infty,0]\cup[1,\infty))$. Then (16) follows throughout this region. ■

3 The Laguerre Case

In this section, we prove Theorem 1(II) and Corollary 3. We begin with the necessity that w is a Laguerre weight when there is a Rodrigues formula:

Proof of Necessity that w is a Laguerre weight
Assume that (10) holds. Then for $n = 1$ this gives

$$\begin{aligned} P_1\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)\left[u^{1/\alpha-1} w\left(u^{1/\alpha}\right) u\right] \\ &= \frac{w'}{w}\left(u^{1/\alpha}\right)\frac{1}{\alpha}u^{1/\alpha} + \frac{1}{\alpha}. \end{aligned}$$

Set $x = u^{1/\alpha}$ and use that P_1 is a linear polynomial. We obtain for some constants A and B,

$$A + Bx = \frac{w'}{w}(x)\,x$$

and hence

$$\frac{A}{x} + B = \frac{w'}{w}(x)\,.$$

Integrating shows that w is a Laguerre weight

$$w(x) = x^A e^{Bx},$$

apart from a constant factor. The fact that $A > -1, B < 0$ follows from integrability of w. ■

We turn to the sufficiency part of Theorem 1 (II). We must prove that when w is a Laguerre weight, then P_n given by (10) firstly satisfies the orthogonality conditions, and secondly is a polynomial of degree n.

Proof of the Orthogonality Condition (7)

Let w be a Laguerre weight (11), and P_n be given by (10). Let

$$\begin{aligned} I_j &= \int_0^\infty P_n(x)\,(x^\alpha)^j\,w(x)\,dx \\ &= \frac{1}{\alpha}\int_0^\infty P_n\left(u^{1/\alpha}\right)u^j w\left(u^{1/\alpha}\right)u^{1/\alpha-1}du \\ &= \frac{1}{\alpha}\int_0^\infty u^j\left(\frac{d}{du}\right)^n\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n\right]du. \end{aligned}$$

Observe that $u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n$ has a zero at 0 of multiplicity $\frac{1}{\alpha}-1+\frac{a}{\alpha}+n > n-1$. Moreover $u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n$ decays at ∞ faster than any negative power of u. We integrate by parts j times to obtain

$$I_j = \frac{1}{\alpha}(-1)^j\,j!\int_0^\infty\left(\frac{d}{du}\right)^{n-j}\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n\right]du = 0,$$

if $j \le n-1$. When $j = n$, we obtain instead

$$I_n = \frac{1}{\alpha}(-1)^n\,n!\int_0^\infty u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n du \ne 0,$$

as the integrand is positive. ■

If we assume that $c=1$ in (11), then after a substitution, we see that

$$\begin{aligned} I_n &= (-1)^n n! \int_0^\infty x^{a+n\alpha} e^{-x} dx \\ &= (-1)^n n! \Gamma(a+n\alpha+1). \end{aligned} \tag{28}$$

To show that P_n is indeed a polynomial of degree n, we need:

Lemma 3.1
Let $\Delta \in \mathbb{R}$. For $j \geq 1$,

$$\left(\frac{d}{du}\right)^j \left[u^{\Delta+n} e^{-cu^{1/\alpha}}\right] = u^{\Delta+n-j} e^{-cu^{1/\alpha}} R_{n,j}\left(u^{1/\alpha}\right), \tag{29}$$

where

$$R_{n,1}(x) = \Delta + n - \frac{c}{\alpha} x \tag{30}$$

and for $j \geq 1$, $R_{n,j+1}$ is a polynomial of degree $j+1$ determined by the recursion

$$R_{n,j+1}(x) = R_{n,j}(x)\left\{\Delta + n - j - \frac{c}{\alpha} x\right\} + \frac{x}{\alpha} R'_{n,j}(x). \tag{31}$$

The leading coefficient of $R_{n,j}$ is $\left(-\frac{c}{\alpha}\right)^n$.
Proof
We use induction on j: first for $j=1$,

$$\begin{aligned} &\frac{d}{du}\left[u^{\Delta+n} e^{-cu^{1/\alpha}}\right] \\ &= u^{\Delta+n-1} e^{-cu^{1/\alpha}} \left[\Delta + n - \frac{c}{\alpha} u^{1/\alpha}\right] \\ &= u^{\Delta+n-1} e^{-cu^{1/\alpha}} R_{n,1}\left(u^{1/\alpha}\right), \end{aligned}$$

where $R_{n,1}$ is a polynomial of degree 1 given by (30). Now assume that (29) is true for j. We shall prove it for $j+1$. Differentiating (29) gives

$$\begin{aligned} &\left(\frac{d}{du}\right)^{j+1} \left[u^{\Delta+n} e^{-cu^{1/\alpha}}\right] \\ &= \frac{d}{du}\left[u^{\Delta+n-j} e^{-cu^{1/\alpha}} R_{n,j}\left(u^{1/\alpha}\right)\right] \\ &= u^{\Delta+n-(j+1)} e^{-cu^{1/\alpha}} \left\{\begin{array}{c} (\Delta+n-j) R_{n,j}\left(u^{1/\alpha}\right) \\ -\frac{c}{\alpha} u^{1/\alpha} R_{n,j}\left(u^{1/\alpha}\right) \\ +\frac{1}{\alpha} u^{1/\alpha} R'_{n,j}\left(u^{1/\alpha}\right) \end{array}\right\} \\ &= u^{\Delta+n-(j+1)} e^{-cu^{1/\alpha}} R_{n,j+1}\left(u^{1/\alpha}\right), \end{aligned}$$

where $R_{n,j+1}(x)$ is a polynomial of degree $j+1$ in x determined by the recursion (31). By induction, (29) is true for all $j \geq 1$. ■

The result of the lemma remains true for $j = 0$ if we set

$$R_{n,0}(x) \equiv 1. \tag{32}$$

We can now complete the sufficiency part of Theorem 1(II):

Proof that P_n given by (10) is a polynomial of degree n
We use Lemma 3.1 on P_n given by (10), with w a Laguerre weight as in (11) and $\Delta = \frac{a+1}{\alpha} - 1$:

$$\begin{aligned} P_n\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)^n\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n\right] \\ &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)^n\left[u^{(a+1)/\alpha-1+n}e^{-cu^{1/\alpha}}\right] \\ &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}u^{(a+1)/\alpha-1}e^{-cu^{1/\alpha}}R_{n,n}\left(u^{1/\alpha}\right) = R_{n,n}\left(u^{1/\alpha}\right). \end{aligned} \tag{33}$$

That P_n must have degree n follows from $I_n \neq 0$, as in the proof of the Jacobi case. More simply the lemma shows that the leading coefficient of $P_n = R_{n,n}$ is $(-c/\alpha)^n$. ■

Proof of Corollary 3
(a), (b) follow from (33), Lemma 3.1, with $\Delta = \frac{a+1}{\alpha} - 1$ and the fact that we chose $c = 1$.
(c) Let $u \in (0, \infty)$. By Cauchy's integral formula for derivatives,

$$\begin{aligned} \frac{w\left(u^{1/\alpha}\right)}{u^{1-1/\alpha}}P_n\left(u^{1/\alpha}\right) &= \left(\frac{d}{du}\right)^n\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)u^n\right] \\ &= \frac{n!}{2\pi i}\int_\Gamma \frac{t^{1/\alpha-1}w\left(t^{1/\alpha}\right)t^n}{(t-u)^{n+1}}dt. \end{aligned}$$

Here, as usual, Γ is a circle center u of sufficiently small radius. Then for $|z|$ sufficiently small,

$$\begin{aligned}\frac{w\left(u^{1/\alpha}\right)}{u^{1-1/\alpha}}\sum_{n=0}^{\infty}\frac{P_n\left(u^{1/\alpha}\right)z^n}{n!} &= \frac{1}{2\pi i}\int_{\Gamma}\frac{t^{1/\alpha-1}w\left(t^{1/\alpha}\right)}{t-u}\sum_{n=0}^{\infty}\left(\frac{tz}{t-u}\right)^n dt \\ &= \frac{1}{2\pi i}\int_{\Gamma}\frac{t^{1/\alpha-1}w\left(t^{1/\alpha}\right)}{t-u-tz}dt.\end{aligned}$$

The integrand has a simple pole at $t=u/\left(1-z\right)$. By the residue theorem, we continue this as

$$=\left(1-z\right)^{-1}\left(\frac{u}{1-z}\right)^{1/\alpha-1}w\left(\left(\frac{u}{1-z}\right)^{1/\alpha}\right).$$

Rearranging this gives

$$\sum_{n=0}^{\infty}\frac{P_n\left(u^{1/\alpha}\right)z^n}{n!}=\left(1-z\right)^{-\frac{a+1}{\alpha}}\exp\left(u^{1/\alpha}\left[1-\left(1-z\right)^{-1/\alpha}\right]\right).$$

All the algebraic manipulations of the multivalued functions are valid for $u\in(0,\infty)$ and $|z|$ small enough. Replacing $u^{1/\alpha}$ by v and noting that the left-hand side is the Maclaurin series in z (for fixed v) of the right-hand side, we obtain for all $v\in(0,\infty)$ and $|z|<1$,

$$\sum_{n=0}^{\infty}\frac{P_n\left(v\right)z^n}{n!}=\left(1-z\right)^{-\frac{a+1}{\alpha}}\exp\left(v\left[1-\left(1-z\right)^{-1/\alpha}\right]\right).$$

To extend this to v off the positive real axis, we observe that

$$P_n\left(v\right)=\left(\frac{d}{dz}\right)^n\left\{\left(1-z\right)^{-\frac{a+1}{\alpha}}\exp\left(v\left[1-\left(1-z\right)^{-1/\alpha}\right]\right)\right\}_{|z=0}.$$

By analyticity with respect to v of both sides of this relation, it persists for all complex v. Then (19) also follows for all complex v. ■

4 The Hermite Case

In this section we prove Theorem 1(III). The main thing to be proved is that w must be a Hermite weight and α must equal 1, for a Rodrigues formula to hold. One immediate observation is that α must be an integer. For if α is non-integral, then $\psi\left(x\right)=x^{\alpha}$ is not real valued on the negative real axis.

Of course if α is an even integer, then ψ is not increasing, but we shall show that even allowing for this, there is still no Rodrigues formula. So in the sequel, we assume that α is a positive integer.

Proof of Necessity that w is the Hermite weight
Assume that (12) holds. Then for $n = 1$ this gives

$$\begin{aligned} P_1\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)\right] \\ &= \frac{w'}{w}\left(u^{1/\alpha}\right)\frac{1}{\alpha}u^{1/\alpha-1} + \left(\frac{1}{\alpha}-1\right)\frac{1}{u}. \end{aligned} \tag{34}$$

Setting $x = u^{1/\alpha}$ gives

$$P_1(x) = \frac{w'}{w}(x)\frac{1}{\alpha}x^{1-\alpha} + \left(\frac{1}{\alpha}-1\right)x^{-\alpha}.$$

Next since P_1 is a linear polynomial, we obtain for some constants A and B,

$$Ax^{\alpha-1} + Bx^{\alpha} + \frac{\alpha-1}{x} = \frac{w'}{w}(x). \tag{35}$$

Integrating gives

$$w(x) = |x|^{\alpha-1}\exp\left(\frac{A}{\alpha}x^{\alpha} + \frac{B}{\alpha+1}x^{\alpha+1}\right).$$

To show that $\alpha = 1$, we use the Rodrigues formula for $n = 2$. First note that differentiating (35) gives

$$\frac{w''}{w}(x) - \left(\frac{w'}{w}(x)\right)^2 = A(\alpha-1)x^{\alpha-2} + B\alpha x^{\alpha-1} - \frac{\alpha-1}{x^2}. \tag{36}$$

Next, (12) gives

$$\begin{aligned} P_2\left(u^{1/\alpha}\right) &= \frac{u^{1-1/\alpha}}{w\left(u^{1/\alpha}\right)}\left(\frac{d}{du}\right)^2\left[u^{1/\alpha-1}w\left(u^{1/\alpha}\right)\right] \\ &= \left(\frac{1}{\alpha}-1\right)\left(\frac{1}{\alpha}-2\right)u^{-2} + \frac{3}{\alpha}\left(\frac{1}{\alpha}-1\right)u^{1/\alpha-2}\frac{w'}{w}\left(u^{1/\alpha}\right) \\ &\quad + \frac{1}{\alpha^2}\left(u^{1/\alpha-1}\right)^2\frac{w''}{w}\left(u^{1/\alpha}\right). \end{aligned}$$

Setting $x = u^{1/\alpha}$ gives

$$\begin{aligned} P_2(x) &= \left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right) x^{-2\alpha} \\ &+\frac{3}{\alpha}\left(\frac{1}{\alpha} - 1\right) x^{1-2\alpha} \frac{w'}{w}(x) + \frac{1}{\alpha^2}\left(x^{1-\alpha}\right)^2 \frac{w''}{w}(x). \end{aligned}$$

Substituting in (35) and (36) and gathering terms gives

$$\begin{aligned} P_2(x) &= x^{-2\alpha}\left\{\left(\frac{1}{\alpha} - 1\right)\left(\frac{1}{\alpha} - 2\right) + \frac{3}{\alpha}\left(\frac{1}{\alpha} - 1\right)(\alpha - 1) - \frac{\alpha - 1}{\alpha^2} + \frac{(\alpha - 1)^2}{\alpha^2}\right\} \\ &+x^{-\alpha}\left\{\frac{3}{\alpha}\left(\frac{1}{\alpha} - 1\right) A + \frac{\alpha - 1}{\alpha^2} A + \frac{2}{\alpha^2}(\alpha - 1) A\right\} \\ &+x^{1-\alpha}\left\{\frac{3}{\alpha}\left(\frac{1}{\alpha} - 1\right) B + \frac{B}{\alpha} + \frac{2}{\alpha^2} B(\alpha - 1)\right\} \\ &+\left(\frac{A}{\alpha}\right)^2 + \frac{2AB}{\alpha^2} x + \left(\frac{B}{\alpha}\right)^2 x^2. \end{aligned}$$

We continue this as

$$P_2(x) = 0x^{-2\alpha} + 0x^{-\alpha} + \frac{B}{\alpha^2} x^{1-\alpha} + \left(\frac{A}{\alpha}\right)^2 + \frac{2AB}{\alpha^2} x + \left(\frac{B}{\alpha}\right)^2 x^2.$$

Here if $\alpha \neq 1$, then $\alpha \geq 2$, and the condition that P_2 be a polynomial of degree ≤ 2 forces $B = 0$, and then

$$P_2(x) = \left(\frac{A}{\alpha}\right)^2,$$

a constant. Since the orthogonality condition (7) forces P_2 to have at least two zeros, we deduce that $A = 0$. Then

$$w(x) = |x|^{\alpha-1},$$

which is not integrable over the real line. So we need $\alpha = 1$. ■

Proof of sufficiency for w the Hermite weight and $\alpha = 1$
We have to show that for

$$w(x) = \exp\left(Ax + Bx^2\right),$$

with $B < 0$,

$$P_n(x) = \frac{1}{w(x)}\left(\frac{d}{dx}\right)^n w(x)$$

is an orthogonal polynomial of degree n. This is of course classical and can be found in Tricomi [14, pp. 129-133] for general A. For the case $A = 0, B = -1$ (which the general case becomes after a linear transformation), the proof is in numerous texts, for example [2], [8], [13]. ■

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