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ON SHARP CONSTANTS IN MARCINKIEWICZ-ZYGMUND AND PLANCHEREL-POLYA INEQUALITIES

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Abstract. The Plancherel-Polya inequalities assert that for $1 < p < \infty$, and entire functions f of exponential type at most π,

$$A_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f|^p \leq B_p \sum_{j=-\infty}^{\infty} |f(j)|^p .$$

The Marcinkiewicz-Zygmund inequalities assert that for $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\frac{A_p'}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right)\right|^p \leq \int_0^1 \left|P\left(e^{2\pi i t}\right)\right|^p dt \leq \frac{B_p'}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right)\right|^p .$$

We show that the sharp constants in both inequalities are the same, that is $A_p = A_p'$ and $B_p = B_p'$. Moreover, the two inequalities are equivalent. We also discuss the case $p \leq 1$.

1. Introduction

The Plancherel-Polya inequalities [5, p. 152] assert that for $1 < p < \infty$, and entire functions f of exponential type at most π,

$$A_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f|^p \leq B_p \sum_{j=-\infty}^{\infty} |f(j)|^p , \tag{1.1}$$

provided either the series or integral is finite. For $0 < p \leq 1$, the left-hand inequality is still true, but the right-hand inequality requires additional restrictions [1], [3], [9]. Of course, A_p, B_p are independent of f. Moreover, a dilation of the variable yields an analogous inequality for f of any given finite type. These inequalities play an important role in sampling theory and applications of Paley-Wiener spaces [5].

The Marcinkiewicz-Zygmund inequalities assert [11, Vol. II, p. 30] that for $p > 1, n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\frac{A_p'}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right)\right|^p \leq \int_0^1 \left|P\left(e^{2\pi i t}\right)\right|^p dt \leq \frac{B_p'}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right)\right|^p . \tag{1.2}$$

Here too, A_p' and B_p' are independent of n and P, and the left-hand inequality is also true for $0 < p \leq 1$ [7]. These inequalities are useful in studying convergence

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of Fourier series, Lagrange interpolation, in number theory, and weighted approximation. They have been extended to many settings, and there are a great many methods to prove them [4], [7], [8].

To the best of this author's knowledge, the sharp constants in (1.1) and (1.2) are unknown, except for the case $p = 2$, where of course $A_2 = B_2 = A_2' = B_2' = 1$ [5, p. 150]. Throughout, we assume that A_p, B_p, A_p', B_p' are the sharp constants, so that A_p and A_p' are as large as possible, while B_p and B_p' are as small as possible. The main result of this paper is:

Theorem 1
For $0 < p < \infty$,

$$A_p = A_p' \tag{1.3}$$

and for $1 < p < \infty$,

$$B_p = B_p'. \tag{1.4}$$

This theorem can be seen as a further example of the longstanding connection between asymptotics for polynomials and entire functions of exponential type [2], [10, Chapters 4, 5]. Indeed, the main idea in the proof is that scaling limits transform polynomials into entire functions of exponential type. We also prove a duality inequality. Let σ_p denote the norm of the Fourier partial sum projection in L_p, that is

(1.5)
$$\sigma_p = \sup\left\{\int_0^1 \left|\sum_{j=0}^{n} c_j e^{2\pi ijt}\right|^p dt : n \geq 1, \{c_j\} \subset \mathbb{C} \text{ and } \int_0^1 \left|\sum_{j=-\infty}^{\infty} c_j e^{2\pi ijt}\right|^p dt \leq 1\right\}.$$

Theorem 2
(a) Let $1 < p < \infty, p \neq 2$, *and* $q = \frac{p}{p-1}$. *Then*

$$\left(B_q^{1/q}\right)^{-1} \leq A_p^{1/p} \leq \left(B_q^{1/q}\right)^{-1} \sigma_p^{1/p}. \tag{1.6}$$

(b) Moreover, for all $p > 1$ *except* $p = 2$,

$$B_p > 1, \tag{1.7}$$

while for $0 < p < \infty$ *except* $p = 2$,

$$A_p < 1. \tag{1.8}$$

In [1, p. 101, Thm. 6.7.15], it is proven that $B_p \leq \frac{4}{\pi} e^{p\pi/2}$.

2. Proofs

Throughout $C, C_1, C_2,\ldots$ denote constants independent of n, m, x, t. We shall use the sinc kernel $S(z) = \frac{\sin \pi z}{\pi z}$, an entire function of exponential type π. We shall also use the fact that for $\lambda > 1$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=-\infty}^{\infty} \left|S\left(\frac{k+x}{m}\right)\right|^\lambda = \int_{-\infty}^{\infty} |S(t)|^\lambda \, dt, \tag{2.1}$$

uniformly for $x \in [0,1]$. This is easily established: if $r \geq 2$,

$$\begin{aligned} &\frac{1}{m} \sup_{x\in[0,1]} \sum_{k:|k|\geq mr}^{\infty} \left| S\left(\frac{k+x}{m}\right)\right|^{\lambda} + \int_{|t|\geq r} |S(t)|^{\lambda}\, dt \\ \leq\ & C\left(\frac{1}{m}\sum_{k:|k|\geq mr}^{\infty}\left(\frac{m}{|k|}\right)^{\lambda} + \int_{|t|\geq r}\frac{1}{|t|^{\lambda}}dt\right) \leq Cr^{1-\lambda}, \end{aligned}$$

so that the tails of both the series and integral are uniformly small for large r, while, as S is uniformly continuous in $\mathbb{R}$, the theory of Riemann sums yields uniformly for $x \in [0,1]$,

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k:|k|\leq mr}\left|S\left(\frac{k+x}{m}\right)\right|^{\lambda} = \int_{-r}^{r}|S(t)|^{\lambda}\,dt.$$

We start with:

Proof that for $p > 0$, $A_p \leq A'_p$

Let $n \geq 1$ and P be a polynomial of degree $\leq n-1$. Choose positive integers m, J such that $Jp > 1$ and $m > J$. Let

$$f(z) = e^{-i\pi\left(1-\frac{1}{n}\right)z} P\left(e^{2\pi iz/n}\right) S\left(\frac{z}{mn}\right)^{J}.$$

It is easily seen that f is entire of exponential type $\leq \pi$. Moreover,

$$\sum_{j=-\infty}^{\infty}|f(j)|^{p} = \sum_{k=-\infty}^{\infty}\sum_{\ell=0}^{n-1}|f(kn+\ell)|^{p} = \sum_{\ell=0}^{n-1}\left|P\left(e^{2\pi i\ell/n}\right)\right|^{p}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+\ell/n}{m}\right)\right|^{Jp},$$

while

$$\begin{aligned} \int_{-\infty}^{\infty}|f(x)|^{p}\,dx &= n\int_{-\infty}^{\infty}\left|P\left(e^{2\pi it}\right)\right|^{p}\left|S\left(\frac{t}{m}\right)\right|^{Jp}dt \\ &= n\int_{0}^{1}\left|P\left(e^{2\pi is}\right)\right|^{p}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+s}{m}\right)\right|^{Jp}ds. \end{aligned}$$

We can then recast the left inequality in (1.1) for this function f as

$$\begin{aligned} &A_p\frac{1}{n}\sum_{\ell=0}^{n-1}\left|P\left(e^{2\pi i\ell/n}\right)\right|^{p}\left(\frac{1}{m}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+\ell/n}{m}\right)\right|^{Jp}\right) \\ \leq\ & \int_{0}^{1}\left|P\left(e^{2\pi is}\right)\right|^{p}\left(\frac{1}{m}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+s}{m}\right)\right|^{Jp}\right)ds. \end{aligned}$$

We now let $m \to \infty$ and use the uniform convergence in (2.1), to obtain

$$A_p\frac{1}{n}\sum_{\ell=0}^{n-1}\left|P\left(e^{2\pi i\ell/n}\right)\right|^{p} \leq \int_{0}^{1}\left|P\left(e^{2\pi is}\right)\right|^{p}ds.$$

Since this holds for any $n \geq 1$ and any polynomial P of degree $\leq n-1$, it follows that $A'_p \geq A_p$. ■

Proof that for $p > 1$, $B_p \geq B'_p$

Let P have degree $\leq n-1$. We proceed as above, but use the right inequality in (1.1), to obtain

$$\begin{aligned}&\int_0^1 \left|P\left(e^{2\pi is}\right)\right|^p \left(\frac{1}{m}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+s}{m}\right)\right|^{Jp}\right) ds\\ \leq\ & B_p \frac{1}{n}\sum_{\ell=0}^{n-1}\left|P\left(e^{2\pi i\ell/n}\right)\right|^p \left(\frac{1}{m}\sum_{k=-\infty}^{\infty}\left|S\left(\frac{k+\ell/n}{m}\right)\right|^{Jp}\right).\end{aligned}$$

Letting $m\to\infty$ yields

$$\int_0^1 \left|P\left(e^{2\pi is}\right)\right|^p ds \leq B_p \frac{1}{n}\sum_{\ell=0}^{n-1}\left|P\left(e^{2\pi i\ell/n}\right)\right|^p.$$

As above, it follows that $B_p \geq B_p'$. ■

The converse inequalities are more difficult:

Proof that for $p>1$, $B_p \leq B_p'$ and hence $B_p = B_p'$

Let f be an entire function of exponential type $\leq \pi$, for which the integral in (1.1) is convergent. Then f admits the sampling series expansion [5, p. 152]

$$f(z) = \sum_{k=-\infty}^{\infty} f(k)\, S(z-k). \tag{2.2}$$

It converges uniformly in compact subsets of the plane. Moreover,

$$\lim_{L\to\infty}\int_{-\infty}^{\infty}\left|f(x) - \sum_{|k|\leq L} f(k)\, S(x-k)\right|^p dx = 0. \tag{2.3}$$

For $|j| \leq [n/2]$, let

$$\ell_{jn}(z) = \frac{1}{n}\frac{z^n-1}{ze^{-2\pi ij/n}-1} \tag{2.4}$$

denote the jth fundamental polynomial of Lagrange interpolation at the nth roots of unity. If n is even, we consider only $j=n/2$, not $j=-n/2$. Let us fix $L\geq 1$, and for $n>2L$, define

$$P_n(z) = \sum_{|j|\leq L} f(j)\,(-1)^j\, \ell_{jn}(z),$$

a polynomial of degree $\leq n-1$. Then

$$\sum_{j=1}^{n}\left|P_n\left(e^{2\pi ij/n}\right)\right|^p = \sum_{|j|\leq L}|f(j)|^p. \tag{2.5}$$

A straightforward calculation shows that for fixed j,

$$\lim_{n\to\infty} \ell_{jn}\left(e^{2\pi it/n}\right) = e^{i\pi t}\,(-1)^j\, S(t-j),$$

uniformly for t in compact sets. Moreover, for $1/3 \geq |s| \geq 2|j|/n$,

$$\left|\ell_{jn}\left(e^{2\pi is}\right)\right| \leq \frac{1}{n\left|\sin\pi(s-j/n)\right|} \leq \frac{1}{2n\left|s-j/n\right|} \leq \frac{1}{n|s|}. \tag{2.6}$$

Thus uniformly for t in compact sets,

$$\lim_{n\to\infty} P_n\left(e^{2\pi it/n}\right) = e^{i\pi t} \sum_{|k|\leq L} f(k)\, S(t-k)$$

while for $|s| \geq 2L/n$,

$$\left|P_n\left(e^{2\pi is}\right)\right| \leq \frac{1}{n|s|} \sum_{|k|\leq L} |f(k)| =: \frac{C_L}{n|s|}.$$

Then, given $r > 2L$,

$$\begin{aligned} n\int_0^1 \left|P_n\left(e^{2\pi is}\right)\right|^p ds &= n\int_{-1/2}^{1/2} \left|P_n\left(e^{2\pi is}\right)\right|^p ds \\ &= \int_{-r}^{r} \left|P_n\left(e^{2\pi it/n}\right)\right|^p dt + O\left(C_L^p n \int_{|s|\geq r/n} \frac{ds}{(n|s|)^p}\right) \\ &= \int_{-r}^{r} \left|\sum_{|k|\leq L} f(k)\, S(t-k)\right|^p dt + O\left(C_L^p r^{1-p}\right). \end{aligned}$$

Using this and (2.5), we may now recast the right-hand inequality in (1.2) as

$$\int_{-r}^{r} \left|\sum_{|k|\leq L} f(k)\, S(t-k)\right|^p dt + O\left(C_L^p r^{1-p}\right) \leq B_p' \sum_{|j|\leq L} |f(j)|^p.$$

Letting $r \to \infty$ gives

$$\int_{-\infty}^{\infty} \left|\sum_{|k|\leq L} f(k)\, S(t-k)\right|^p dt \leq B_p' \sum_{|j|\leq L} |f(j)|^p.$$

We may now let $L \to \infty$, and use (2.3) to deduce

$$\int_{-\infty}^{\infty} |f(t)|^p\, dt \leq B_p' \sum_{j=-\infty}^{\infty} |f(j)|^p.$$

As f was an arbitrary entire function of exponential type $\leq \pi$, we deduce that $B_p \leq B_p'$. Together with the previous proof, this shows that $B_p = B_p'$. ■

Proof that for $p > 1,\ A_p \geq A_p'$ and hence $A_p = A_p'$

Here, we proceed as above, but use the left-hand inequality in (1.2), leading to

$$A_p' \sum_{|j|\leq L} |f(j)|^p \leq \int_{-r}^{r} \left|\sum_{|k|\leq L} f(k)\, S(t-k)\right|^p dt + O\left(C_L^p r^{1-p}\right).$$

Letting $r \to \infty$ gives for $M \leq L$

$$A_p' \sum_{|j|\leq M} |f(j)|^p \leq \int_{-\infty}^{\infty} \left|\sum_{|k|\leq L} f(k)\, S(t-k)\right|^p dt.$$

We now let $L \to \infty$ in the right-hand side, and use (2.3), and then finally let $M \to \infty$ in the left-hand side, giving

$$A'_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

As this holds for any such f, we obtain $A_p \geq A'_p$. The converse inequality $A_p \leq A'_p$ was proved above. ■

The case $p \leq 1$ is more difficult:

Proof that for $0 < p \leq 1$, $A_p \leq A'_p$ and hence $A_p = A'_p$
Let f be entire of exponential type $\leq \pi$, with the integral in (1.1) finite. We note that the left-hand inequality in (1.1) (which is valid even for $p \leq 1$), shows that $\sup_j |f(j)| < \infty$, and hence

$$\sum_{j=-\infty}^{\infty} |f(j)|^{\lambda} < \infty \text{ for all } \lambda > p. \tag{2.7}$$

In particular, f satisfies (1.1) with $p = 2$, and consequently, we still have the sampling series expansion (2.2). Choose a positive integer J such that $Jp \geq 2$, and let $\varepsilon \in (0, 1/2)$. Let $L > 1$,

$$U_k(z) = \frac{1}{k} \frac{z^k - 1}{z - 1}$$

and with $[x]$ denoting the greatest integer $\leq x$, let

$$P_n(z) = \left(\sum_{|j| \leq L} f(j) (-1)^j \ell_{j, n-[\varepsilon n]}(z) \right) U_{\left[\frac{\varepsilon}{J} n\right]}(z)^J,$$

a polynomial of degree $\leq n - 1$. A straightforward calculation shows for as $n \to \infty$,

$$\ell_{j, n-[\varepsilon n]}\left(e^{2\pi i t/n}\right) = (-1)^j e^{i\pi t(1-\varepsilon)} S((1-\varepsilon)t - j) + o(1),$$

uniformly for t in compact sets and any fixed j (cf. [6, Lemma 2.2]). In a similar way, uniformly for t in compact sets,

$$U_{\left[\frac{\varepsilon}{J} n\right]}\left(e^{2\pi i t/n}\right) = e^{i\pi \frac{\varepsilon t}{J}} S\left(\frac{\varepsilon t}{J}\right) + o(1),$$

so uniformly for such t, as $n \to \infty$,

$$\begin{aligned}
\left| P_n\left(e^{2\pi i t/n}\right) \right| &= \left| e^{i\pi t} \left(\sum_{|j| \leq L} f(j) S((1-\varepsilon)t - j) \right) S\left(\frac{\varepsilon t}{J}\right)^J + o(1) \right| \\
&\leq \left| \sum_{|j| \leq L} f(j) S((1-\varepsilon)t - j) \right| \left| S\left(\frac{\varepsilon t}{J}\right)^J \right| + o(1) \\
&\leq |f(t(1-\varepsilon))| + O\left(\sum_{|j| > L} |f(j)| \right) \min\left\{1, \frac{1}{|t|}\right\}^J + o(1),
\end{aligned}$$

recall that $|S(s)| \leq \min\left\{1, \frac{1}{\pi|s|}\right\}$, and (2.7). Then for fixed $r > 4L$, we obtain, as $n \to \infty$,

$$n\int_{-r/n}^{r/n} \left|P_n\left(e^{2\pi is}\right)\right|^p ds \leq \frac{1}{1-\varepsilon}\int_{-\infty}^{\infty} |f(t)|^p\,dt + C\left(\sum_{|j|>L} |f(j)|\right)^p + o(1).$$

Here, as $Jp > 1$, C is independent of r, n, L, but does depend on ε and J. Next, for $\frac{1}{3} \geq |s| \geq r/n$, (2.6) gives

$$\left|P_n\left(e^{2\pi is}\right)\right| \leq \left(\frac{C}{n|s|}\sum_{|j|\leq L}|f(j)|\right)\left(\frac{1}{\frac{\varepsilon}{J}n|s|}\right)^J \leq \frac{C}{r}\left(\frac{1}{n|s|}\right)^J$$

where C is independent of L, s, r, n. Then

$$\begin{aligned} & n\int_{r/n\leq|s|\leq 1/2} \left|P_n\left(e^{2\pi is}\right)\right|^p ds \leq \frac{Cn}{r^p}\int_{r/n\leq|s|\leq 1/2}\left(\frac{1}{n|s|}\right)^{Jp} ds \\ \leq\ & Cr^{-p+1-Jp}. \end{aligned}$$

Thus, as $n \to \infty$,

$$\begin{aligned} & n\int_{-1/2}^{1/2} \left|P_n\left(e^{2\pi is}\right)\right|^p ds \\ \leq\ & \frac{1}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p\,dt + C\left(\sum_{|j|\geq L}|f(j)|\right)^p + Cr^{-p+1-Jp} + o(1). \end{aligned}$$

Here C is independent of r, n, L. Next, for each fixed k, as $n \to \infty$,

$$\left|P_n\left(e^{2\pi ik/n}\right)\right| = \left|\sum_{|j|\leq L} f(j)\,S((1-\varepsilon)k - j)\right|\left|S\left(\frac{\varepsilon k}{J}\right)\right|^J + o(1).$$

The left-hand inequality in (1.2) gives, for any fixed $L \geq M \geq 1$, as $n \to \infty$,

$$\begin{aligned} & A_p'\sum_{|k|\leq M}\left|\sum_{|j|\leq L} f(j)\,S((1-\varepsilon)k-j)\right|^p\left|S\left(\frac{\varepsilon k}{J}\right)\right|^{Jp} \\ \leq\ & \frac{1}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p\,dt + C\left(\sum_{|j|\geq L}|f(j)|\right)^p + Cr^{-p+1-Jp}. \end{aligned}$$

Here C is independent of L, r, n, but depends on ε, J. We now let $r \to \infty$ (with ε still fixed), obtaining for $M \leq L$,

$$\begin{aligned} & A_p'\sum_{|k|\leq M}\left|\sum_{|j|\leq L} f(j)\,S((1-\varepsilon)k-j)\right|^p\left|S\left(\frac{\varepsilon k}{J}\right)\right|^{Jp} \\ \leq\ & \frac{1}{1-\varepsilon}\int_{-\infty}^{\infty}|f(t)|^p\,dt + C\left(\sum_{|j|\geq L}|f(j)|\right)^p. \end{aligned}$$

Next, let $L \to \infty$, which is permissible as $\sum_j |f(j)| < \infty$. We obtain

$$A'_p \sum_{|k| \le M} |f((1-\varepsilon)k)|^p \left| S\left(\frac{\varepsilon k}{J}\right) \right|^{Jp} \le \frac{1}{1-\varepsilon} \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

We now let $\varepsilon \to 0+$, and use the continuity of S, to obtain

$$A'_p \sum_{|k| \le M} |f(k)|^p \le \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

Letting $M \to \infty$, gives

$$A'_p \sum_{k=-\infty}^{\infty} |f(k)|^p \le \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

As f is an arbitrary entire function of exponential type $\le \pi$, we obtain $A_p \ge A'_p$ and hence $A_p = A'_p$ from earlier results. ■

Proof of Theorem 1
This has been proved above. ■

Proof of Theorem 2
(a) We use a duality argument: if P is a polynomial of degree $\le n$,

$$\left(\frac{1}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right) \right|^p \right)^{1/p} = \frac{1}{n} \left| \sum_{j=1}^{n} \overline{c_j} P\left(e^{2\pi i j/n}\right) \right|$$

for some $\{c_j\}$ with $\frac{1}{n} \sum_{j=1}^{n} |c_j|^q = 1$. Equivalently, if R is a polynomial of degree $\le n-1$ with $R\left(e^{2\pi i j/n}\right) = c_j$ for all j,

$$\begin{aligned} \left(\frac{1}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right) \right|^p \right)^{1/p} &= \frac{1}{n} \left| \sum_{j=1}^{n} \overline{R\left(e^{2\pi i j/n}\right)} P\left(e^{2\pi i j/n}\right) \right| \\ &= \left| \int_0^1 \left(\overline{R} P\right)\left(e^{2\pi i t}\right) dt \right|, \end{aligned}$$

by the simple sum

$$\frac{1}{n} \sum_{j=1}^{n} \left(e^{2\pi i j/n}\right)^k = \int_0^1 e^{2\pi i k t} dt, \quad |k| < n. \tag{2.8}$$

We use Hölder's inequality to continue this as

$$\begin{aligned} &\left(\frac{1}{n} \sum_{j=1}^{n} \left| P\left(e^{2\pi i j/n}\right) \right|^p \right)^{1/p} \\ \le\ & \left(\int_0^1 \left| R\left(e^{2\pi i t}\right) \right|^q dt \right)^{1/q} \left(\int_0^1 \left| P\left(e^{2\pi i t}\right) \right|^p dt \right)^{1/p} \\ \le\ & \left(B'_q \frac{1}{n} \sum_{j=1}^{n} \left| R\left(e^{2\pi i j/n}\right) \right|^q \right)^{1/q} \left(\int_0^1 \left| P\left(e^{2\pi i t}\right) \right|^p dt \right)^{1/p}, \end{aligned}$$

by (1.2). As the sum involving $|R|$ is at most 1, we deduce that

$$B_q'^{-p/q}\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p \leq \int_0^1 \left|P\left(e^{2\pi it}\right)\right|^p dt.$$

Thus $B_q'^{-p/q} \leq A_p'$, or

$$B_q'^{-1/q} \leq A_p'^{1/p}. \tag{2.9}$$

Similarly, for some measurable function g, with $\int_0^1 \left|g\left(e^{2\pi it}\right)\right|^q dt \leq 1$,

$$\left(\int_0^1 \left|P\left(e^{2\pi it}\right)\right|^p dt\right)^{1/p} = \int_0^1 \left(P\overline{g}\right)\left(e^{2\pi it}\right) dt = \int_0^1 \left(P\overline{R}\right)\left(e^{2\pi it}\right) dt,$$

where if g has Fourier series $\sum_{j=-\infty}^{\infty} c_j e^{ijt}$, $R(t) = \sum_{j=0}^{n-1} c_j e^{ijt}$, and we are using orthogonality. We now again use quadrature on the unit circle, followed by Holder's inequality, to continue this as

$$\begin{aligned}
&= \frac{1}{n}\left|\sum_{j=1}^{n}\overline{R\left(e^{2\pi ij/n}\right)}P\left(e^{2\pi ij/n}\right)\right| \\
&\leq \left(\frac{1}{n}\sum_{j=1}^{n}\left|R\left(e^{2\pi ij/n}\right)\right|^q\right)^{1/q}\left(\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p\right)^{1/p} \\
&\leq \left(A_q'^{-1}\int_0^1 \left|R\left(e^{2\pi it}\right)\right|^q dt\right)^{1/q}\left(\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p\right)^{1/p} \\
&\leq \left(A_q'^{-1}\sigma_q\int_0^1 \left|g\left(e^{2\pi it}\right)\right|^q dt\right)^{1/q}\left(\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p\right)^{1/p}
\end{aligned}$$

In summary,

$$\int_0^1 \left|P\left(e^{2\pi it}\right)\right|^p dt \leq \left(A_q'^{-1}\sigma_q\right)^{p/q}\frac{1}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p.$$

Thus $B_p' \leq \left(A_q'^{-1}\sigma_q\right)^{p/q}$, or swapping the roles of p, q, $B_q' \leq \left(A_p'^{-1}\sigma_p\right)^{q/p}$. Hence,

$$\left(B_q'^{1/q}\right)^{-1}\sigma_p^{1/p} \geq \left(A_p'\right)^{1/p}.$$

This, (2.9), and Theorem 1 give (1.6).
(b) Fix n and let ℓ_{0n} be as in (2.4). For $s \in [0,1]$ and $p > 0$, let

$$\chi_p(s) = \int_0^1 \left|\ell_{0n}\left(e^{2\pi i(t-s)}\right)\right|^p dt - \frac{1}{n}\sum_{j=0}^{n-1}\left|\ell_{0n}\left(e^{2\pi i(j/n-s)}\right)\right|^p$$

denote the quadrature error for the polynomial $\ell_{0n}\left(ze^{-2\pi is}\right)$. Observe that

$$\chi_p(0) = \int_0^1 \left|\ell_{0n}\left(e^{2\pi it}\right)\right|^p dt - \frac{1}{n}$$

and in particular, $\chi_2(0) = 0$ by (2.8). Inasmuch as $|\ell_{0n}(z)| < 1$ unless $z = 1$, this shows that

$$\chi_p(0) \begin{cases} > 0, & 0 < p < 2 \\ < 0, & p > 2 \end{cases}.$$

Thus for $p \neq 2$, $\chi_p(s)$ is a continuous, not identically vanishing function of s. Also by periodicity,

$$\int_0^1 \chi_p(s)\, ds = 0.$$

It then follows that for some values of s, $\chi_p(s) > 0$, and for others $\chi_p(s) < 0$. The former inequality shows that $B_p > 1$, and the latter $A_p < 1$. ■

References

[1] R.P. Boas, *Entire Functions*, Academic Press, New York, 1954.

[2] D.P. Dryanov, M.A. Qazi, Q.I. Rahman, *Entire Functions of Exponential Type in Approximation Theory*, (in) Constructive Theory of Functions, (ed. B. Bojanov), Darba, Sofia, 2003, pp. 86-135.

[3] C. Eoff, *The Discrete Nature of the Paley-Wiener Spaces*, Proc. Amer. Math. Soc., 123(1995), 505-512.

[4] F. Filbir and H.N. Mhaskar, *Marcinkiewicz-Zygmund measures on manifolds*, Journal of Complexity, 27(2011), 568-596.

[5] B. Ja Levin, *Lectures on Entire Functions*, Translations of Mathematical Monographs, American Mathematical Society, Providence, 1996.

[6] Eli Levin, D. S. Lubinsky, *L_p Christoffel Functions, L_p Universality, and Paley-Wiener Spaces*, manuscript.

[7] D.S. Lubinsky, *Marcinkiewicz-Zygmund Inequalities: Methods and Results*, (in) Recent Progress in Inequalities (ed. G.V. Milovanovic et al.), Kluwer Academic Publishers, Dordrecht, 1998, pp. 213-240.

[8] D.S. Lubinsky, *A Survey of Weighted Polynomial Approximation with Exponential Weights*, Surveys on Approximation Theory, 3(2007), 1-105.

[9] M. Plancherel and G. Polya, *Fonctions entierers et integrales de Fourier multiples*, Coment. Math. Helvet., 10(1937), 110-163.

[10] A.F. Timan, *Theory of Approximation of Functions of a Real Variable,* Dover, New York, 1994.

[11] A. Zygmund, *Trigonometric Series*, Vols. 1,2, Second Paperback edition, Cambridge University Press, Cambridge, 1988.

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