SCALING LIMITS FOR MIXED KERNELS

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ABSTRACT. Let μ and ν be measures supported on $(-1,1)$ with corresponding orthonormal polynomials $\{p_n^\mu\}$ and $\{p_n^\nu\}$ respectively. Define the mixed kernel

$$K_n^{\mu,\nu}(x,y)=\sum_{j=0}^{n-1}p_j^\mu(x)\,p_j^\nu(y).$$

We establish scaling limits such as

$$\begin{aligned}&\lim_{n\to\infty}\frac{\pi\sqrt{1-\xi^2}\sqrt{\mu'(\xi)\,\nu'(\xi)}}{n}K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)\\ =\ & S\left(\frac{\pi(a-b)}{2}\right)\cos\left(\frac{\pi(a-b)}{2}+B(\xi)\right),\end{aligned}$$

where $S(t)=\frac{\sin t}{t}$ is the sinc kernel, and $B(\xi)$ depends on μ,ν and ξ. This reduces to the classical universality limit in the bulk when $\mu=\nu$. We deduce applications to the zero distribution of $K_n^{\mu,\nu}$, and asymptotics for its derivatives.

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1. INTRODUCTION[1]

Let μ denote a positive measure on the real line, with infinitely many points in its support, and all finite power moments. For $n\geq 0$, let

$$p_n^\mu(x)=\gamma_n^\mu x^n+...$$

denote the nth orthonormal polynomial for μ, so that

$$\int p_n^\mu p_m^\mu d\mu=\delta_{mn}.$$

The nth reproducing kernel for μ is

$$K_n^\mu(x,y)=\sum_{j=0}^{n-1}p_j^\mu(x)\,p_j^\mu(y).$$

If μ has support $[-1,1]$, then under appropriate conditions on μ, there is the universality limit in the bulk,

$$\lim_{n\to\infty}\frac{\pi\sqrt{1-\xi^2}}{n}\mu'(\xi)\,K_n^\mu\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)=S(\pi(a-b)),$$

where $S(t)=\frac{\sin t}{t}$ is the sinc kernel, $\xi\in(-1,1)$, and a,b may be any complex numbers. This limit arises in the theory of random matrices, and is known to be true in various formulations, in a wide array of settings [2], [4], [5], [9], [12], [13],

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[18], [20]. It has applications to spacing of eigenvalues of random matrices, and zero distribution of orthogonal polynomials, amongst other things.

Let us recall how the connection between reproducing kernels and random matrices begins. One starts with a probability distribution on the eigenvalues $(x_1, x_2, ..., x_n)$ of $n \times n$ Hermitian matrices, of the form

$$\mathcal{P}(x_1, x_2, ..., x_n) = Z_0 \left(\prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \right) \prod d\mu(x_j),$$

where Z_0 is a normalizing constant. By using column operations on a Vandermonde determinant, we see that

(1.1)
$$\prod_{1 \leq j < k \leq n} (x_k - x_j) = \det \left[x_j^{k-1} \right]_{1 \leq j,k \leq n} = \left(\gamma_0^\mu \gamma_1^\mu ... \gamma_{n-1}^\mu \right)^{-1} \det \left[p_{k-1}^\mu (x_j) \right]_{1 \leq j,k \leq n}.$$

Next, we use the fact that square matrices and their transposes have the same determinant, to obtain

$$\prod_{1 \leq j < k \leq n} (x_k - x_j)^2 = \left(\gamma_0^\mu \gamma_1^\mu ... \gamma_{n-1}^\mu \right)^{-2} \det \left[K_n^\mu (x_j, x_k) \right]_{1 \leq j,k \leq n}.$$

Thus, $P(x_1, x_2, ..., x_n)$ can be expressed in terms of reproducing kernels for the measure μ. That there is a similar expression for the $m-$point correlation function is far deeper [4].

Now let ν be another measure supported on the real line, with nth orthonormal polynomial $p_n^\nu(x) = \gamma_n^\nu x^n + ...$ and nth reproducing kernel $K_n^\nu(x, t)$. We define the mixed kernel

$$K_n^{\mu,\nu}(x, y) = \sum_{j=0}^{n-1} p_j^\mu(x) p_j^\nu(y).$$

Suppose that we use (1.1) first for the measure μ and then for the measure ν, and only then take transposes, and multiply. We land up with the representation

$$\prod_{1 \leq j < k \leq n} (x_k - x_j)^2 = \left(\gamma_0^\mu \gamma_1^\mu ... \gamma_{n-1}^\mu \gamma_0^\nu \gamma_1^\nu ... \gamma_{n-1}^\nu \right)^{-1} \left(\det \left[K_n^{\mu,\nu}(x_j, x_k) \right]_{1 \leq j,k \leq n} \right).$$

Thus there might be some interest in analysing scaling limits of the mixed kernel $K_n^{\mu,\nu}$, and that is the focus of this paper. Note that $K_n^{\mu,\nu}$ is the representing kernel of the linear operator L given by

$$L \left(\sum_{j=0}^{n-1} a_j p_j^\mu \right) = \sum_{j=0}^{n-1} a_j p_j^\nu.$$

This linear operator is an isometry between the weighted L^2 spaces $L^2(\mu)$ and $L^2(\nu)$.

In the sequel, we use the notation

$$w(\mu, \nu, \xi) = \pi \sqrt{1 - \xi^2} \sqrt{\mu'(\xi) \nu'(\xi)}.$$

A simple, but useful, special case, is where we have pointwise asymptotics for both p_n^μ and p_n^ν. This is outlined in:

Proposition 1.1
Assume that μ and ν are positive measures with support $(-1,1)$. Assume that I is a subinterval of $(-1,1)$, in which μ and ν are absolutely continuous, and that as $n\to\infty$, uniformly for $\xi=\cos\theta\in I$, both

$$\sqrt{\frac{\pi}{2}}\left(1-\xi^2\right)^{1/4}\mu'(\xi)^{1/2}p_n^{\mu}(\xi)=\cos\left(n\theta+\rho_{\mu}(\xi)\right)+o(1)\,; \tag{1.2}$$

$$\sqrt{\frac{\pi}{2}}\left(1-\xi^2\right)^{1/4}\nu'(\xi)^{1/2}p_n^{\nu}(\xi)=\cos\left(n\theta+\rho_{\nu}(\xi)\right)+o(1)\,; \tag{1.3}$$

where ρ_μ and ρ_ν are continuous functions in I. Let J be a compact subinterval of the interior of I. Then uniformly for $\xi=\cos\theta\in J$ and a,b in compact subsets of $\mathbb{R}$,

$$\begin{aligned}&\lim_{n\to\infty}\frac{w(\mu,\nu,\xi)}{n}K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)\\ =\;& S\left(\frac{\pi(b-a)}{2}\right)\cos\left(\frac{\pi(b-a)}{2}+\rho_\mu(\xi)-\rho_\nu(\xi)\right).\end{aligned} \tag{1.4}$$

If in addition, there exists $C>0$ such that $\mu'\geq C$ and $\nu'\geq C$ in I, this also holds uniformly for a,b in compact subsets of $\mathbb{C}$.

Here are two examples where the proposition is applicable:

Example 1.2
Assume that μ and ν have support $[-1,1]$, and satisfy Szegő's condition, so that

$$\int_{-1}^{1}\frac{\log\mu'(x)}{\sqrt{1-x^2}}dx>-\infty \text{ and } \int_{-1}^{1}\frac{\log\nu'(x)}{\sqrt{1-x^2}}dx>-\infty. \tag{1.5}$$

Assume in addition, that in a subinterval I of $(-1,1)$, μ and ν are absolutely continuous, that μ',ν' are bounded above and below by positive constants, and for $\xi\in I$,

$$\int_I\left|\frac{\mu'(t)-\mu'(\xi)}{t-\xi}\right|^2dt<\infty,$$

with the integral converging uniformly in $\xi\in I$, and a similar condition on ν. Then it follows from results of Freud [6, p. 246, Table II, entry(a)], that the asymptotics (1.2) and (1.3) hold uniformly in I (we note that it is stated only pointwise in the table there). There are earlier results, under more restrictive conditions on μ, in the books of Geronimus [8, p. 200] and Szegő [19, p. 298]. In this example, as follows from [19, p. 299, eqn. (12.2.3)], $\rho_\mu(\xi)$ is given for $\xi=\cos\theta$ by

$$\begin{aligned}\rho_\mu(\xi)&=\frac{1}{4\pi}PV\int_{-\pi}^{\pi}\log\mu'(\cos t)\cot\frac{\theta-t}{2}dt\\ &=\frac{1}{2\pi}PV\int_{-1}^{1}\frac{\log\mu'(t)}{t-\xi}\sqrt{\frac{1-\xi^2}{1-t^2}}dt,\end{aligned} \tag{1.6}$$

where PV denotes the Cauchy principal value integral. That this is continuous follows from the assumed uniform convergence above. So

$$\begin{aligned} B(\xi) &:= \rho_\mu(\xi) - \rho_\nu(\xi) = \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \log \frac{\mu'(\cos t)}{\nu'(\cos t)} \cot \frac{\theta - t}{2} dt \\ &= \frac{1}{2\pi} PV \int_{-1}^{1} \frac{\log(\mu'(t)/\nu'(t))}{t - \xi} \sqrt{\frac{1-\xi^2}{1-t^2}} dt. \qquad (1.7) \end{aligned}$$

The appearance of this conjugate function type expression suggests that for our scaling limits to hold, we need μ'/ν' to satisfy a Szegő condition, though quite possibly we do not need μ, ν' to individually satisfy a Szegő condition. In such a case, the principal value integral defining $B(\xi)$ will exist for a.e. ξ.

Example 1.3
A generalized Jacobi weight has the form

$$\mu'(x) = h(x) \left(\prod_{j=1}^{m} |x - t_j|^{\tau_j} \right) (1-x)^\alpha (1+x)^\beta,$$

where $\alpha, \beta, \tau_1, ..., \tau_m > -1$, and $-1 < t_1 < ... < t_m < 1$, while h satisfies a Dini-Lipschitz condition, that is

$$\int_0^1 \frac{\omega(h; [-1,1]; t)}{t} dt < \infty.$$

Here

$$\omega(h; [-1,1]; t) = \sup \{ |h(x) - h(y)| : x, y \in [-1,1], |x - y| \le t \}, t > 0,$$

is the modulus of continuity of h on $[-1,1]$. Badkov [1, p. 38, Cor. 12] proved that for such measures μ, defined by their absolutely continuous component, the asymptotic (1.2) holds, uniformly for x in a compact subinterval of $(-1,1) \setminus \{t_1, t_2, ..., t_m\}$. Again, ρ_μ may be expressed in the form (1.6), and the Dini-Lipschitz condition can be used to prove the continuity of ρ_μ. So if both μ', ν' are generalized Jacobi weights, the result (1.4) holds in any compact subinterval of $(-1,1)$ excluding all the zeros/singularities of μ', ν', and the formula (1.7) persists.

Our main result is that under far more general conditions, the scaling limit holds in linear Lebesgue measure, meas:

Theorem 1.4
Let μ and ν be measures on $(-1,1)$ that satisfy the Szegő condition (1.5) and let B be as in (1.7). Let $\varepsilon, R > 0$. For $n \ge 1$, let $\mathcal{H}_n$ denote the set of $\xi \in (-1,1)$ for which

$$\sup_{|a|,|b| \le R} \left| \frac{w(\mu,\nu,\xi)}{n} K_n^{\mu,\nu}\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n} \right) - S\left(\frac{\pi(a-b)}{2} \right) \cos\left(\frac{\pi(a-b)}{2} + B(\xi) \right) \right| > \varepsilon. \qquad (1.8)$$

Then

$$\text{meas}\left(\mathcal{H}_n\right) \to 0 \text{ as } n \to \infty. \tag{1.9}$$

Note that a, b are allowed to be complex in (1.8). When $\mu = \nu$, our proof here gives a simpler proof of a special case of the main result of [13]. There universality was shown to hold in measure for arbitrary compactly supported measures.

Corollary 1.5
Let μ and ν be measures on $(-1,1)$ that satisfy the Szegő condition (1.5), and let B be as in (1.7). Let I be a closed subinterval of $(-1,1)$ and assume that in some neighborhood of I, and for some $C > 1$,

$$C \geq \mu' \geq C^{-1} \text{ and } C \geq \nu' \geq C^{-1}. \tag{1.10}$$

Then for any $R, p > 0$,

$$\lim_{n\to\infty} \int_I \sup_{|a|,|b|\leq R} \left| \frac{w(\mu,\nu,\xi)}{n} K_n^{\mu,\nu}\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n}\right) \right.$$
$$\left. -S\left(\frac{\pi(a-b)}{2}\right)\cos\left(\frac{\pi(a-b)}{2} + B(\xi)\right)\right|^p d\xi = 0.$$

(1.11)

We can also deduce asymptotics for derivatives of $K_n^{\mu,\nu}$. Given non-negative integers ℓ, m, define the differentiated kernel involving ℓth derivatives of p_j^μ and mth derivatives of p_j^ν :

$$K_n^{\mu,\nu(\ell,m)}(x,y) = \sum_{j=0}^{n-1} \left(p_j^\mu\right)^{(\ell)}(x)\left(p_j^\nu\right)^{(m)}(x). \tag{1.12}$$

Define for $\ell, m \geq 0$,

$$\tau_{\ell,m} = \begin{cases} (-1)^{\frac{\ell-m}{2}} \frac{1}{\ell+m+1}, & \ell+m \text{ even} \\ 0, & \ell+m \text{ odd} \end{cases}, \tag{1.13}$$

and

$$\rho_{\ell,m} = \begin{cases} (-1)^{\frac{\ell-m+1}{2}} \frac{1}{\ell+m+1}, & \ell+m \text{ odd} \\ 0, & \ell+m \text{ even} \end{cases}. \tag{1.14}$$

Corollary 1.6
Assume the hypotheses of Corollary 1.5. Let $\ell, m \geq 0$. Then for any $p > 0$,

$$\lim_{n\to\infty} \int_I \left| \frac{w(\mu,\nu,\xi)}{n} \left(\frac{\sqrt{1-\xi^2}}{n}\right)^{\ell+m} K_n^{\mu,\nu(\ell,m)}(\xi,\xi) \right.$$
$$\left. -\left[\tau_{\ell,m}\cos B(\xi) - \rho_{\ell,m}\sin B(\xi)\right]\right|^p d\xi = 0.$$

(1.15)

If in addition, we have the uniform limits as in Proposition 1.1, then this limit holds pointwise at ξ.

Finally, uniform scaling limits imply results on zero asymptotics, as first noted by Eli Levin [10]:

Corollary 1.7
Let μ and ν be measures supported on $(-1,1)$, and assume that at a given ξ, for which $w(\mu,\nu,\xi)>0$, we have the limit (1.4), holding uniformly for a,b in compact subets of the plane. Fix $a\in\mathbb{C}$. Let j be a non-zero integer. Then for large enough n, as a function of v, $K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},v\right)$ has a zero $v_{n,j}$ that satisfies

$$\lim_{n\to\infty} n\left(v_{n,j}-\xi\right)=(a+2j)\,\pi\sqrt{1-\xi^2}. \tag{1.16}$$

Moreover, it has a zero $\hat{v}_{n,j}$ that satisfies

$$\lim_{n\to\infty} n\left(\hat{v}_{n,j}-\xi\right)=\left(a+2j+1-\frac{2}{\pi}\left[\rho_\mu(\xi)-\rho_\nu(\xi)\right]\right)\pi\sqrt{1-\xi^2}. \tag{1.17}$$

In addition, for large enough n, the only zeros of this function for v in a ball center ξ, of radius $O\left(\frac{1}{n}\right)$, have this form.

Remark
An interesting special case occurs when the hypotheses of Corollary 1.7 hold, and

$$\frac{\mu'(t)}{\nu'(t)}=\left(\frac{1+t}{1-t}\right)^{2k+1}, t\in(-1,1), \tag{1.18}$$

with k an integer. In this case

$$B(\xi)=\frac{2k+1}{2\pi}PV\int_{-1}^{1}\frac{\log\frac{1+t}{1-t}}{t-\xi}\sqrt{\frac{1-\xi^2}{1-t^2}}dt.$$

One can show that

$$B(\xi)=(2k+1)\frac{\pi}{2}, \tag{1.19}$$

using classical identities for Szegő functions. Indeed if we consider the Jacobi "weight" on the circle, $f(\theta)=\left(\frac{1+\cos\theta}{1-\cos\theta}\right)^{2k+1}$, its Szegő function [19, p. 277, eqn. (10.2.13)] is $\left(\frac{1+z}{1-z}\right)^{2k+1}$, and for $z=e^{i\theta}$, this equals $\left(i\cot\frac{\theta}{2}\right)^{2k+1}=i(-1)^k\left(\cot\frac{\theta}{2}\right)^{2k+1}$. The argument of this is $(2k+1)\frac{\pi}{2}$, and using [19, p. 279, eqn. (10.3.9)], (1.19) follows. Then (1.4) becomes

$$\lim_{n\to\infty}\frac{w(\mu,\nu,\xi)}{n}K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)=(-1)^{k+1}\frac{\sin^2\left(\frac{\pi(b-a)}{2}\right)}{\frac{\pi(b-a)}{2}},$$

uniformly for a,b in compact subsets of the plane. The curious feature is that as a function of b, the right-hand side changes sign only at $b=a$, where there is a simple zero. All other zeros are double. Those zeros attract zeros of the left-hand side of total multiplicity 2. More intriguing is that this suggests the conjecture that perhaps

$$(x-y)\,K_n^{\mu,\nu}(x,y)\,(-1)^{k+1}\geq 0 \text{ for all real } x,y,$$

at least for some special μ,ν related by (1.18).

In the sequel, $C,C_1,C_2,...$ denote positive constants independent of n,a,b,u,v,z and possibly other specified parameters. The same symbol does not necessarily denote the same constant in different occurrences. We prove Proposition 1.1 in Section 2, Theorem 1.4 in Section 3, and Corollaries 1.5-1.7 in Section 4.

2. Proof of Proposition 1.1

We first give the

Proof of Proposition 1.1 for real a, b
Now if $x = \cos\theta \in I$ and $y = \cos\phi \in I$, our assumed asymptotic, and elementary trigonometry show that

$$\begin{aligned}
&\frac{w(\mu,\nu,\xi)}{2n} K_n^{\mu,\nu}(x,y) \\
= \;& \frac{1}{n}\sum_{j=0}^{n-1} \cos\left(j\theta + \rho_\mu(x)\right)\cos\left(j\phi + \rho_\nu(y)\right) + o(1) \\
= \;& \frac{1}{2n}\sum_{j=0}^{n-1} \cos\left(j(\theta+\phi) + \rho_\mu(x) + \rho_\nu(y)\right) \\
& + \frac{1}{2n}\sum_{j=0}^{n-1} \cos\left(j(\theta-\phi) + \rho_\mu(x) - \rho_\nu(y)\right) + o(1) \\
= \;& \frac{1}{2n}\left\{\begin{array}{c} \cos\left(\rho_\mu(x)+\rho_\nu(y)\right)\sum_{j=0}^{n-1}\cos\left(j(\theta+\phi)\right) \\ -\sin\left(\rho_\mu(x)+\rho_\nu(y)\right)\sum_{j=0}^{n-1}\sin\left(j(\theta+\phi)\right) \\ +\cos\left(\rho_\mu(x)-\rho_\nu(y)\right)\sum_{j=0}^{n-1}\cos\left(j(\theta-\phi)\right) \\ -\sin\left(\rho_\mu(x)-\rho_\nu(y)\right)\sum_{j=0}^{n-1}\sin\left(j(\theta-\phi)\right) \end{array}\right\} + o(1).
\end{aligned}$$

(2.1)

Next, if t is real, we note the identities

$$\begin{aligned}
\frac{1}{n}\sum_{j=0}^{n-1}\cos jt &= \frac{S\left(\frac{nt}{2}\right)}{S\left(\frac{t}{2}\right)}\cos\left((n-1)\frac{t}{2}\right); \\
\frac{1}{n}\sum_{j=0}^{n-1}\sin jt &= \frac{S\left(\frac{nt}{2}\right)}{S\left(\frac{t}{2}\right)}\sin\left((n-1)\frac{t}{2}\right).
\end{aligned}$$

These follow easily by summing the finite geometric series $\sum_{j=0}^{n-1} e^{ijt}$ and taking real and imaginary parts, and recalling that $S(t) = \frac{\sin t}{t}$. We now apply these to the trigonometric sums in (2.1). First, however, write $\xi = \cos\theta_0$ for some $\theta_0 \in (0,\pi)$, and

$$\begin{aligned}
x &= x_n = \xi + \frac{a\pi\sqrt{1-\xi^2}}{n} = \cos\theta_n; \\
y &= y_n = \xi + \frac{b\pi\sqrt{1-\xi^2}}{n} = \cos\phi_n.
\end{aligned}$$

Then

$$\cos\theta_n - \cos\theta_0 = x - \xi = \frac{a\pi\sqrt{1-\xi^2}}{n}$$

$$\Rightarrow \left(-\sin\theta_0\right)\left(\theta_n-\theta_0\right)\left(1+o\left(1\right)\right)=\frac{a\pi\sqrt{1-\xi^2}}{n}$$
$$\Rightarrow \theta_n-\theta_0=-\frac{a\pi}{n}\left(1+o\left(1\right)\right).$$

Similarly

$$\phi_n-\theta_0=-\frac{b\pi}{n}\left(1+o\left(1\right)\right).$$

Both these relations hold uniformly in $\xi\in J$, and a,b in compact subsets of the real line. Then

$$\begin{aligned}&\frac{1}{n}\sum_{j=0}^{n-1}\cos\left(j\left(\theta_n-\phi_n\right)\right)\\ =\ &\frac{S\left(\frac{n(\theta_n-\phi_n)}{2}\right)}{S\left(\frac{(\theta_n-\phi_n)}{2}\right)}\cos\left(\left(n-1\right)\frac{\left(\theta_n-\phi_n\right)}{2}\right)\\ =\ &S\left(\frac{b-a}{2}\pi\right)\cos\left(\frac{b-a}{2}\pi\right)+o\left(1\right).\end{aligned}$$

Similarly,

$$\begin{aligned}&\frac{1}{n}\sum_{j=0}^{n-1}\sin\left(j\left(\theta_n-\phi_n\right)\right)\\ =\ &S\left(\frac{b-a}{2}\pi\right)\sin\left(\frac{b-a}{2}\pi\right)+o\left(1\right).\end{aligned}$$

Both hold uniformly for $\xi\in J$ and a,b in compact subsets of the real line. Next,

$$\begin{aligned}&\frac{1}{n}\sum_{j=0}^{n-1}\cos\left(j\left(\theta_n+\phi_n\right)\right)\\ =\ &\frac{S\left(\frac{n(\theta_n+\phi_n)}{2}\right)}{S\left(\frac{(\theta_n+\phi_n)}{2}\right)}\cos\left(\left(n-1\right)\frac{\left(\theta_n+\phi_n\right)}{2}\right)\\ =\ &\frac{S\left(n\theta_0\left(1+o\left(1\right)\right)\right)}{S\left(\theta_0\left(1+o\left(1\right)\right)\right)}O\left(1\right)=O\left(\frac{1}{n}\right),\end{aligned}$$

uniformly for $\xi\in J$, as $\theta_0\in(0,\pi)$ is bounded away from 0 and π. A similar estimate holds for the sin sum. We deduce that

$$\begin{aligned}&\frac{w\left(\mu,\nu,\xi\right)}{2n}K_n^{\mu,\nu}\left(x_n,y_n\right)\\ =\ &\frac{1}{2}\left\{\begin{array}{c}\cos\left(\rho_\mu\left(\xi\right)-\rho_\nu\left(\xi\right)\right)S\left(\frac{b-a}{2}\pi\right)\cos\left(\frac{b-a}{2}\pi\right)\\ -\sin\left(\rho_\mu\left(\xi\right)-\rho_\nu\left(\xi\right)\right)S\left(\frac{b-a}{2}\pi\right)\sin\left(\frac{b-a}{2}\pi\right)\end{array}\right\}+o\left(1\right)\\ =\ &\frac{1}{2}S\left(\frac{b-a}{2}\pi\right)\cos\left(\frac{b-a}{2}\pi+\rho_\mu\left(\xi\right)-\rho_\nu\left(\xi\right)\right)+o\left(1\right),\end{aligned}$$

uniformly for $\xi\in J$ and a,b in compact subsets of the real line. ■

Next, we extend this to complex a,b. For a given ξ, let

$$f_n\left(a,b\right)=\frac{w\left(\mu,\nu,\xi\right)}{n}K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right).$$

Lemma 2.1
Assume the extra hypothesis in Proposition 1.1 *that* μ' *and* $\nu' \geq C$ *in* I. *Then for each* $\xi \in J$, $\{f_n\}$ *are a normal family, that is,* $\{f_n(a,b)\}$ *are uniformly bounded for* a,b *in compact subsets of* $\mathbb{C}$.
Proof
We sketch the proof - full details in a very similar situation are given in [11, pp. 383-385]. Since $\sqrt{1-\xi^2}$ is bounded below for $\xi \in I$, as are μ', ν', our assumed asymptotics for p_n^μ, p_n^ν give

$$\sup_{t\in I} |p_n^\mu(t)| \leq C_1; \sup_{t\in I} |p_n^\nu(t)| \leq C_1.$$

Then also

$$\sup_{u,v\in I} |K_n^\mu(u,v)| \leq C_1 n.$$

Next, recall Bernstein's inequality

$$|P(z)| \leq \left| z + \sqrt{z^2-1} \right|^n \|P\|_{L_\infty[-1,1]},$$

valid for polynomials P of degree $\leq n$, and $z \in \mathbb{C}$. Let $r > 0$. Using this on subintervals J of the interior of I, separately in u and v, yields for $u,v \in J$, complex $|a|, |b| \leq r$, and $n \geq n_0(r)$

$$\left| \frac{1}{n} K_n^\mu \left(u + \frac{a}{n}, v + \frac{b}{n} \right) \right| \leq C_1 e^{C_2(|a|+|b|)}.$$

Here C_1 and C_2 are independent of u,v,n,r,a,b. A similar inequality holds for K_n^ν, and then Cauchy-Schwarz yields for $u \in J$, $|a|, |b| \leq r$, and $n \geq n_0(r)$,

$$\begin{aligned} & \left| \frac{1}{n} K_n^{\mu,\nu} \left(u + \frac{a}{n}, u + \frac{b}{n} \right) \right| \\ \leq & \left(\frac{1}{n} K_n^\mu \left(u + \frac{a}{n}, u + \frac{\bar{a}}{n} \right) \right)^{1/2} \left(\frac{1}{n} K_n^\nu \left(u + \frac{b}{n}, u + \frac{\bar{b}}{n} \right) \right)^{1/2} \\ \leq & \ C_1 e^{C_2(|a|+|b|)} \leq C_1 e^{C_2 r}. \end{aligned}$$

It then follows easily that $\{f_n(a,b)\}$ is uniformly bounded for a,b in compact subsets of the plane. ■

Proof of Proposition 1.1 for complex a,b
We already have the result for real a,b. The uniform boundedness of the sequence $\{f_n\}$ ensures that they are a normal family, while both sides of (1.4) are entire in a,b. Convergence continuation theorems give the result in general. ■

Remark Let us cast the three term recurrence relations for p_n^μ and p_n^ν in the following form:

$$\begin{aligned} xp_n^\mu(x) &= \alpha_{n+1}^\mu p_{n+1}^\mu(x) + \beta_n^\mu p_n^\mu(x) + \alpha_n^\mu p_{n-1}^\mu(x); \\ xp_n^\nu(x) &= \alpha_{n+1}^\nu p_{n+1}^\nu(x) + \beta_n^\nu p_n^\nu(x) + \alpha_n^\nu p_{n-1}^x(x), \end{aligned} \tag{2.2}$$

with $p_{-1}^{\mu}=p_{-1}^{\nu}=0$. We also define a "remainder term"

$$\begin{aligned}\Delta_j(x,y) &= p_j^{\mu}(x)\,p_j^{\nu}(y)\left[\left(1-\frac{\alpha_{j+1}^{\mu}}{\alpha_{j+1}^{\nu}}\right)y+\alpha_{j+1}^{\mu}\left(\frac{\beta_j^{\mu}}{\alpha_{j+1}^{\mu}}-\frac{\beta_j^{\nu}}{\alpha_{j+1}^{\nu}}\right)\right]\\ &\quad +p_j^{\mu}(x)\,p_{j-1}^{\nu}(y)\,\alpha_{j+1}^{\mu}\left(\frac{\alpha_j^{\nu}}{\alpha_{j+1}^{\nu}}-\frac{\alpha_j^{\mu}}{\alpha_{j+1}^{\mu}}\right). \qquad (2.3)\end{aligned}$$

Then the proof of the Christoffel-Darboux formula via the recurrence relation also shows that

$$\begin{aligned}&(x-y)\,K_n^{\mu,\nu}(x,y)\\ &= \alpha_n^{\mu}\left(p_n^{\mu}(x)\,p_{n-1}^{\nu}(y)-p_{n-1}^{\mu}(x)\,p_n^{\nu}(y)\right)-\sum_{j=0}^{n-1}\Delta_j(x,y). \qquad (2.4)\end{aligned}$$

One can use this to prove scaling limits for $K_n^{\mu,\nu}(x,y)$, at least when $\sum_{j=0}^{\infty}\Delta_j(x,y)$ converges uniformly for x,y in a neighborhood of ξ. However, the method above is more general.

3. A Basic Comparison Estimate

Our basic tool is contained in the following theorem. For any positive measure ω, and $\delta\geq 0$, define

$$\mathcal{L}(\omega,\delta)=\{x:\omega'(x)>\delta\}. \qquad (3.1)$$

Recall that meas denotes linear Lebesgue measure.

Theorem 3.1
Let $\mu,\hat{\mu},\omega$ be positive measures on the real line with compact support. Let $n\geq 1$ and

$$\eta_n=1+\frac{1}{n}\int K_n^{\hat{\mu}}(x,x)\,d\mu(x)-\frac{2}{n}\sum_{j=0}^{n-1}\frac{\gamma_j^{\hat{\mu}}}{\gamma_j^{\mu}}. \qquad (3.2)$$

Let $\varepsilon,R>0$. *Then there exist sets $\mathcal{E}_n$ and $\mathcal{F}_n$, satisfying*

$$\text{meas}(\mathcal{E}_n)<\eta_n^{1/4}+\varepsilon;\text{meas}(\mathcal{F}_n)<\varepsilon, \qquad (3.3)$$

and such that for $x\in\mathcal{L}\left(\mu,\eta_n^{1/4}\right)\backslash\mathcal{E}_n$ and $y\in\mathcal{L}(\omega,0)\backslash\mathcal{F}_n$, and all complex a,b, with $|a|,|b|\leq R$,

$$\frac{1}{n}\left|K_n^{\mu,\omega}-K_n^{\hat{\mu},\omega}\right|\left(x+\frac{a}{n},y+\frac{b}{n}\right)\leq\hat{C}\eta_n^{1/4}. \qquad (3.4)$$

Here $\hat{C}$ is independent of $n,x,y,a,b,\mu,\hat{\mu},\eta_n$, but depends on ω,ε and R.

The dependence of $\hat{C}$ on R is explicitly given in the proof of Theorem 3.1, but will not be used in the sequel.

We use in an essential way, ideas from [13]. We begin with:

Lemma 3.2
Let $n\geq 1,\varepsilon>0$, and $P(u,v)$ be a continuous function of two variables $u,v\in\mathbb{C}$. Assume that $V\subset\mathbb{C}$ is bounded, and that for each $v\in V$, $P(u,v)$ is a polynomial

of degree $\leq n$ *in* u*. Let* $\mathcal{K}\subset\mathbb{R}$ *be bounded and have positive Lebesgue measure, such that*

$$|P(u,v)|\leq 1 \text{ for } u\in\mathcal{K} \text{ and } v\in V. \tag{3.5}$$

Then there is a set $\mathcal{H}_{n,\varepsilon}$ *with* $\operatorname{meas}(\mathcal{H}_{n,\varepsilon})<\varepsilon$*, and*

$$|P(x+z,v)|\leq 2e^{nC_1|z|/\varepsilon} \textit{ for all } x\in\mathcal{K}\backslash\mathcal{H}_{n,\varepsilon}, \textit{ all } z\in\mathbb{C}, \textit{ and all } v\in V. \tag{3.6}$$

The constant C_1 *is independent of* $\mathcal{K}, V, n, x, z, v, P, \varepsilon$*. The set* $\mathcal{H}_{n,\varepsilon}$ *depends on* $\mathcal{K}$*,*P*,*V *but not on* x,z,v*.*

Remark

The essential feature is that the same exceptional set $\mathcal{H}_{n,\varepsilon}$ works for all $v\in V$.

Proof

Step 1: Some classical tools

First recall the notion of the equilibrium measure $\nu_{\hat{\mathcal{K}}}$ of a compact set $\hat{\mathcal{K}}$. This is a probability measure supported on $\hat{\mathcal{K}}$ that minimizes

$$\int\int\log|z-t|^{-1}\,d\rho(z)\,d\rho(t)$$

amongst all probability measures ρ supported on $\hat{\mathcal{K}}$. The Green's function $g_{\mathbb{C}\backslash\hat{\mathcal{K}}}(z)$ with pole at ∞ is a function harmonic on $\mathbb{C}\backslash\hat{\mathcal{K}}$ with boundary values 0 (suitably interpreted) on $\hat{\mathcal{K}}$, that behaves like $\log|z|+O(1)$ as $z\to\infty$. Readers lacking the potential theory background can refer to [15], [17]. Next, the maximal function of a measure such as $\nu_{\hat{\mathcal{K}}}$ is

$$\mathcal{M}\left[d\nu_{\hat{\mathcal{K}}}\right](x)=\sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h}d\nu_{\hat{\mathcal{K}}}.$$

Finally, $\mathcal{H}^*$ denotes the maximal Hilbert transform, defined by

$$\mathcal{H}^*\left[d\nu_{\hat{\mathcal{K}}}\right](x)=\sup_{\varepsilon>0}\left|\int_{|t-x|\geq\varepsilon}\frac{1}{t-x}d\nu_{\hat{\mathcal{K}}}(t)\right|.$$

Step 2: Replace $\mathcal{K}$ by a set $\hat{\mathcal{K}}$ consisting of finitely many intervals

In order to apply classical bounds on $\mathcal{H}^*$, we want $\mathcal{K}$ to be compact and $\nu_{\mathcal{K}}$ to be absolutely continuous. This is true if $\mathcal{K}$ consisted of finitely many intervals (in such a case $\nu'_{\mathcal{K}}$ is even analytic in the interior of $\mathcal{K}$ [17, p. 412]). So we replace $\mathcal{K}$ by a larger set $\hat{\mathcal{K}}$ consisting of finitely many intervals. First, recall that P is uniformly continuous in compact subsets of $\mathbb{C}^2$, and $\mathcal{K}$,V are bounded. Choose an interval I containing $\mathcal{K}$. Then we can find finitely many balls $B_1,B_2,...,B_m$ that cover V and have centers in V, and such that

$$v_1,v_2\in B_j\Rightarrow\sup_{t\in I}|P(t,v_1)-P(t,v_2)|\leq 1. \tag{3.7}$$

Suppose a_j is the center of B_j for each j. Now as $P(t,a_j)$ is a polynomial of degree $\leq n$ in t, the set

$$\mathcal{K}_j=\{t\in\mathbb{R}:|P(t,a_j)|\leq 1\}$$

consists of at most $2n$ intervals. Also, by our hypothesis (3.5), $\mathcal{K}\subset\mathcal{K}_j$. Let

$$\hat{K}=I\cap\bigcap_{j=1}^{m}\mathcal{K}_j.$$

This set consists of finitely many intervals, some of which may reduce to a point. As $\mathcal{K} \subset \hat{\mathcal{K}}$, and $\mathcal{K}$ has positive measure, some of the intervals in $\hat{\mathcal{K}}$ have positive length. Then $\nu_{\hat{\mathcal{K}}}$ is absolutely continuous. Moreover, for $t \in \hat{\mathcal{K}}$ and $v \in B_j$, we have that $t \in I \cap \mathcal{K}_j$, so by (3.7),

$$|P(t,v)| \leq |P(t,a_j)| + 1 \leq 2,$$

so

$$|P(t,v)| \leq 2 \text{ for } t \in \hat{\mathcal{K}} \text{ and } v \in V.$$

Step 3: Apply Classical Potential Theory Estimates
By the Bernstein-Walsh inequality [15, p. 156, Theorem 5.5.7], we have for any x, for $z \in \mathbb{C}$ and $v \in V$

$$|P(x+z,v)| \leq 2e^{n g_{\mathbb{C}\setminus\hat{\mathcal{K}}}(x+z)}. \tag{3.8}$$

Next, by Lemma 4.1 in [13, p. 231, Lemma 4.1], if x is a regular point of the set $\hat{\mathcal{K}}$, in the sense of potential theory, we have the bound

$$g_{\mathbb{C}\setminus\hat{\mathcal{K}}}(x+z) \leq 26\,|z|\,\mathcal{M}\left[d\nu_{\hat{\mathcal{K}}}\right](x) + |\mathrm{Re}\, z|\,\mathcal{H}^*\left[d\nu_{\hat{\mathcal{K}}}\right](x),$$

for all complex z. Inasmuch as $\hat{\mathcal{K}}$ consists of finitely many intervals, at most finitely many points are not regular points. Moreover, $\hat{\mathcal{K}}$ and $\mathrm{supp}\left[\nu_{\hat{\mathcal{K}}}\right]$ are identical except perhaps for finitely many isolated points in $\hat{\mathcal{K}}$. Next, we use the fact that both the maximal function and the maximal Hilbert transform are weak type (1,1). That is, for $\lambda > 0$, [16, p. 137, Thm. 7.4]

$$meas\left\{x : \mathcal{M}\left[\nu_{\hat{\mathcal{K}}}\right](x) > \lambda\right\} \leq \frac{3}{\lambda}\int d\nu_{\hat{\mathcal{K}}} = \frac{3}{\lambda};$$

and as $\nu_{\mathcal{K}}$ is absolutely continuous, [3, p. 130, Propn. 4.6; p. 134, Thm. 4.7], [7, p. 128 ff.]

$$meas\left\{x : \mathcal{H}^*\left[d\nu_{\hat{\mathcal{K}}}\right](x) > \lambda\right\} \leq \frac{C_0}{\lambda}\int d\nu_{\hat{\mathcal{K}}} = \frac{C_0}{\lambda}.$$

Here C_0 is independent of $\nu_{\hat{\mathcal{K}}}$ and λ, and this is the only place we need $\nu_{\hat{\mathcal{K}}}$ to be absolutely continuous. Choosing $\lambda = \frac{2}{\varepsilon}\max\{3, C_0\}$, we obtain a set $\mathcal{H}_{n,\varepsilon}$ of measure $\leq \varepsilon$ such that for $x \in \mathrm{supp}\left[\nu_{\hat{\mathcal{K}}}\right] \setminus \mathcal{H}_{n,\varepsilon}$, all complex u, and all $v \in V$,

$$|P(x+z,v)| \leq 2e^{60n|z|\max\{3,C_0\}/\varepsilon}.$$

As $\mathcal{K} \subset \mathrm{supp}\left[\nu_{\hat{\mathcal{K}}}\right]$, (except possibly for a set of measure 0), we are done. ∎

Proof of Theorem 3.1
Now by orthogonality,

$$\begin{aligned}
&\int \left[\int \left(K_n^{\mu,\omega}(x,t) - K_n^{\hat{\mu},\omega}(x,t)\right)^2 d\omega(t)\right] d\mu(x) \\
= &\int \left[K_n^{\mu}(x,x) - 2K_n^{\mu,\hat{\mu}}(x,x) + K_n^{\hat{\mu}}(x,x)\right] d\mu(x) \\
= &\; n - 2\sum_{j=0}^{n-1} \frac{\gamma_j^{\hat{\mu}}}{\gamma_j^{\mu}} + \int K_n^{\hat{\mu}}(x,x)\, d\mu(x) = n\eta_n.
\end{aligned}$$

Let $\delta=\eta_n^{1/4}$ in the sequel. Also, let $\mathcal{E}_{n,1}$ denote the set of $x\in\mathcal{L}(\mu,\delta)=\mathcal{L}\left(\mu,\eta_n^{1/4}\right)$ for which

$$\int\left(K_n^{\mu,\omega}(x,t)-K_n^{\hat{\mu},\omega}(x,t)\right)^2 d\omega(t)\geq n\sqrt{\eta_n}. \tag{3.9}$$

By our lower bound of δ for μ' in $\mathcal{L}(\mu,\delta)$, we have

$$\begin{aligned}&n\sqrt{\eta_n}\delta\text{meas}(\mathcal{E}_{n,1})\\ \leq\ &\int_{\mathcal{E}_{n,1}}\left[\int\left(K_n^{\mu,\omega}(x,t)-K_n^{\hat{\mu},\omega}(x,t)\right)^2 d\omega(t)\right]d\mu(x)\leq n\eta_n,\end{aligned}$$

Thus

$$\text{meas}(\mathcal{E}_{n,1})\leq\frac{\sqrt{\eta_n}}{\delta}=\eta_n^{1/4}. \tag{3.10}$$

Next, for polynomials P of degree $\leq n-1$, and all complex y, we have the Christofel function inequality

$$|P(y)|^2\leq K_n^{\omega}(y,\bar{y})\int P^2 d\omega.$$

Applying this to $P(t)=K_n^{\mu,\omega}(x,t)-K_n^{\hat{\mu},\omega}(x,t)$, we obtain for $x\in\mathcal{L}(\mu,\delta)\backslash\mathcal{E}_{n,1}$, and all complex y,

$$\begin{aligned}&\left|K_n^{\mu,\omega}(x,y)-K_n^{\hat{\mu},\omega}(x,y)\right|^2\\ \leq\ &K_n^{\omega}(y,\bar{y})\int\left(K_n^{\mu,\omega}(x,t)-K_n^{\hat{\mu},\omega}(x,t)\right)^2 d\omega(t)\\ \leq\ &K_n^{\omega}(y,\bar{y})\,n\sqrt{\eta_n}.\end{aligned} \tag{3.11}$$

Now we apply Corollary 4.3 in [13, p. 232]: there is a set $\mathcal{F}_n$ of linear Lebesgue measure less than ε, depending on n and ω, such that for $\xi\in\mathcal{L}(\omega,0)\backslash\mathcal{F}_n$ and all complex b with $|b|\leq R$,

$$\left|K_n^{\omega}\left(\xi+\frac{b}{n},\xi+\frac{\bar{b}}{n}\right)\right|\leq C_1 n e^{C_2|b|/\varepsilon}\leq C_1 n e^{C_2 R/\varepsilon}.$$

Here C_1 and C_2 are independent of $n,b,\xi,R,\mu,\hat{\mu}$ and C_2 is independent of ε, but C_1 does depend on ε. Thus for $x\in\mathcal{L}(\mu,\delta)\backslash\mathcal{E}_{n,1}$, and for $\xi\in\mathcal{L}(\omega,0)\backslash\mathcal{F}_n$ and all $|b|\leq R$,

$$\frac{1}{n}\left|K_n^{\mu,\omega}\left(x,\xi+\frac{b}{n}\right)-K_n^{\hat{\mu},\omega}\left(x,\xi+\frac{b}{n}\right)\right|\leq C_1 e^{C_2R/\varepsilon}\eta_n^{1/4}. \tag{3.12}$$

Next we apply Lemma 3.2 to the polynomial

$$P(t,v)=\frac{1}{n}\left(K_n^{\mu,\omega}(t,v)-K_n^{\hat{\mu},\omega}(t,v)\right)/\left(C_1 e^{C_2R/\varepsilon}\eta_n^{1/4}\right).$$

As our set $\mathcal{K}$, we take $\mathcal{L}(\mu,\delta)\backslash\mathcal{E}_{n,1}$, and as our set V, we take $\left\{\xi+\frac{b}{n}:\xi\in\mathcal{L}(\omega,0)\backslash\mathcal{F}_n \text{ and } |b|\leq R\right\}$. Inasmuch as $\mathcal{K}$ and V are bounded sets, and P is continuous as a function of two variables on $\mathbb{C}^2$, the hypotheses of Lemma 3.2 are satisfied. Then we obtain a set $\mathcal{H}_{n,\varepsilon}$ with meas $(\mathcal{H}_{n,\varepsilon})<\varepsilon$, and

$$\frac{1}{n}\left|K_n^{\mu,\omega}-K_n^{\hat{\mu},\omega}\right|\left(x+z,\xi+\frac{b}{n}\right)\leq 2C_1 e^{(CR+nC_0|z|)/\varepsilon}\eta_n^{1/4}$$

for all $x \in \mathcal{L}(\mu,\delta) \setminus (\mathcal{E}_{n,1} \cup \mathcal{H}_\varepsilon)$, all $z \in \mathbb{C}$, and all $v = \xi + \frac{b}{n} \in V$. In particular, for $|a| \leq R$,

$$\frac{1}{n}\left|K_n^{\mu,\omega} - K_n^{\hat{\mu},\omega}\right|\left(x + \frac{a}{n}, \xi + \frac{b}{n}\right) \leq 2C_1 e^{C_3 R/\varepsilon} \eta_n^{1/4}.$$

Replacing ξ by y, setting $\hat{C} = 2C_1 e^{C_3 R/\varepsilon}$ and $\mathcal{E}_n = \mathcal{E}_{n,1} \cup \mathcal{H}_{n,\varepsilon}$ gives the result. ■

4. Proof of Theorem 1.4

We shall approximate the measures μ, ν in Theorem 1.4, by measures $\hat{\mu}, \hat{\nu}$ to which Proposition 1.1. can be applied. We begin with

Lemma 4.1
Let $R > 0, \varepsilon \in \left(0, \frac{1}{2}\right)$ and μ be as in Theorem 1.4, with the additional restriction that μ is absolutely continuous. Let ω be a measure satisfying $\omega' > 0$ a.e. in $(-1,1)$. There is a measure $\hat{\mu}$ with support $(-1,1)$, that satisfies the Szegő condition as in (1.5), that has $\hat{\mu}'$ analytic in $(-1+\varepsilon, 1-\varepsilon)$, and in addition satisfies the following: there exists n_0 and for $n \geq n_0$, a set $\tilde{\mathcal{E}}_n$ such that for $x \in (-1,1) \setminus \tilde{\mathcal{E}}_n$, and all complex a, b with $|a|, |b| \leq R$,

$$\frac{1}{n}\left|K_n^{\mu,\omega} - K_n^{\hat{\mu},\omega}\right|\left(x + \frac{a}{n}, x + \frac{b}{n}\right) \leq \varepsilon, \tag{4.1}$$

while

$$\operatorname{meas}\left(\tilde{\mathcal{E}}_n\right) \leq 3\varepsilon. \tag{4.2}$$

Proof
We shall use Theorem 3.1, and break the proof into several steps. We shall fix an auxiliary small parameter $\varepsilon_1 \in (0, \varepsilon)$. Later on, we shall choose this small enough depending on the given R and ε.
Step 1: Limits that indicate how to choose $\hat{\mu}$
We first note that if our approximating measure $\hat{\mu}$ is absolutely continuous, and satisfies for some $A > 1$,

$$\frac{1}{A\pi\sqrt{1-x^2}} \leq \hat{\mu}'(x) \leq \frac{A}{\pi\sqrt{1-x^2}}, x \in (-1,1), \tag{4.3}$$

then by the monotonicity of Christoffel functions/ reproducing kernels, and bounds for the Christoffel functions of the Chebyshev weight, we have [6, pp. 103-5, Lemma 3.2, Theorem 3.4]

$$C_1 A^{-1} n \leq K_n^{\hat{\mu}}(x,x) \leq C_2 A n, x \in [-1,1],$$

where C_1 and C_2 are independent of $\hat{\mu}, n, x, A$. Moreover, we have by results of Maté, Nevai, and Totik [14, p. 449, Theorem 8],

$$\lim_{n\to\infty} \frac{1}{n} K_n^{\hat{\mu}}(x,x) = \left(\pi\sqrt{1-x^2}\hat{\mu}'(x)\right)^{-1}, \text{ a.e. } x \in (-1,1).$$

Hence, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{1}{n} \int K_n^{\hat{\mu}}(x,x)\, d\mu(x) = \int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} \frac{\mu'(x)}{\hat{\mu}'(x)} dx.$$

(It is only here that we need μ to be absolutely continuous). Next, Szegő asymptotics for leading coefficients of orthogonal polynomials [19, p. 309, eqn. (12.7.2)], [6, p. 204, p. 245] yield

$$\lim_{n\to\infty}\frac{\gamma_n^{\hat{\mu}}}{\gamma_n^{\mu}}=\exp\left(-\frac{1}{2\pi}\int_{-1}^{1}\log\frac{\hat{\mu}'(x)}{\mu'(x)}\frac{dx}{\sqrt{1-x^2}}\right).$$

Then the average also has the same limit:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\frac{\gamma_j^{\hat{\mu}}}{\gamma_j^{\mu}}=\exp\left(-\frac{1}{2\pi}\int_{-1}^{1}\log\frac{\hat{\mu}'(x)}{\mu'(x)}\frac{dx}{\sqrt{1-x^2}}\right),$$

so

(4.4)

$$\lim_{n\to\infty}\eta_n=\eta_\infty:=1+\int_{-1}^{1}\frac{1}{\pi\sqrt{1-x^2}}\frac{\mu'(x)}{\hat{\mu}'(x)}dx-2\exp\left(-\frac{1}{2\pi}\int_{-1}^{1}\log\frac{\hat{\mu}'(x)}{\mu'(x)}\frac{dx}{\sqrt{1-x^2}}\right).$$

Step 2: Choosing $\hat{\mu}$

We want η_∞ to be small. Let $\delta_1\in(0,\varepsilon_1)$ be so small that all of

$$\int_{1\geq|x|\geq1-\delta_1}\left(\mu'(x)+\frac{1}{\sqrt{1-x^2}}\right)dx<\frac{\varepsilon_1}{8}; \tag{4.5}$$

$$\int_{1\geq|x|\geq1-\delta_1}|\log\mu'(x)|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8}; \tag{4.6}$$

$$\int_{1\geq|x|\geq1-\delta_1}\left|\log\sqrt{1-x^2}\right|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8}. \tag{4.7}$$

Next let $M>0$ be a number so large that if $\mathcal{S}=\{x\in[-1+\delta_1,1-\delta_1]:|\log\mu'(x)|\geq M\}$,

$$\int_{\mathcal{S}}|\log\mu'(x)|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8};\int_{\mathcal{S}}(\mu'(x)+1)\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8}. \tag{4.8}$$

For $|x|\leq1-\delta_1$, define

$$g(x)=\begin{cases}-M, & \text{if } \log\mu'(x)<-M;\\ \log\mu'(x), & \text{if } |\log\mu'(x)|\leq M;\\ M, & \text{if } \log\mu'(x)>M.\end{cases}$$

Then $|g|\leq M$, and we can find a polynomial P such that $|P|\leq 2M$ in $[-1+\delta_1,1-\delta_1]$, and

$$e^{3M}\int_{-1+\delta_1}^{1-\delta_1}|g(x)-P(x)|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8}. \tag{4.9}$$

We set

$$\hat{\mu}'(x)=\frac{1}{\sqrt{1-x^2}},1-\delta_1\leq|x|<1$$

and

$$\hat{\mu}'(x)=e^{P(x)},|x|<1-\delta_1.$$

Note that $\hat{\mu}'$ satisfies (4.3) for some $A>0$, so (4.4) holds. Also, $\hat{\mu}'$ is analytic in $(-1+\varepsilon,1-\varepsilon)$, recall that $\delta_1<\varepsilon_1<\varepsilon$.

Step 3: Estimates involving $\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|$
Now,

$$\begin{aligned}&\int_{-1+\delta_1}^{1-\delta_1}\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|\frac{dx}{\sqrt{1-x^2}}\\ \le&\int_{[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}}\left|g(x)-P(x)\right|\frac{dx}{\sqrt{1-x^2}}\\ &+\int_{\mathcal{S}}\left(\left|\log\mu'(x)\right|+2M\right)\frac{dx}{\sqrt{1-x^2}}\\ <&\frac{\varepsilon_1}{8}+3\int_{\mathcal{S}}\left|\log\mu'(x)\right|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{2}.\end{aligned}$$

Here we have used (4.9), (4.8) and that $|\log\mu'|\ge M$ in $\mathcal{S}$. Also, by (4.6), (4.7),

$$\begin{aligned}&\int_{1\ge|x|\ge1-\delta_1}\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|\frac{dx}{\sqrt{1-x^2}}\\ \le&\frac{\varepsilon_1}{8}+\int_{1\ge|x|\ge1-\delta_1}\left|\log\sqrt{1-x^2}\right|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{4}.\end{aligned}$$

Thus

$$\int_{-1}^{1}\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|\frac{dx}{\sqrt{1-x^2}}<\frac{3}{4}\varepsilon_1. \tag{4.10}$$

Using the inequality $|e^t-1|\le2|t|$ for $|t|\le\frac{1}{2}$, we obtain

$$1-\exp\left(-\frac{1}{2\pi}\int_{-1}^{1}\log\frac{\hat{\mu}'(x)}{\mu'(x)}\frac{dx}{\sqrt{1-x^2}}\right)<\frac{2}{2\pi}\frac{3}{4}\varepsilon_1<\frac{\varepsilon_1}{4}. \tag{4.11}$$

Step 4: Estimates involving $\left|\frac{\mu'}{\hat{\mu}'}-1\right|$
Next, for $x\in[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}$, $\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|\le3M$, so the inequality $|e^t-1|\le|t|\,e^{|t|}, t\in\mathbb{R}$, gives

$$\begin{aligned}&\int_{[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}}\frac{1}{\sqrt{1-x^2}}\left|\frac{\mu'(x)}{\hat{\mu}'(x)}-1\right|dx\\ =&\int_{[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}}\frac{1}{\sqrt{1-x^2}}\left|e^{\log\mu'(x)-\log\hat{\mu}'(x)}-1\right|dx\\ \le&e^{3M}\int_{[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}}\left|\log\mu'(x)-\log\hat{\mu}'(x)\right|\frac{dx}{\sqrt{1-x^2}}\\ =&e^{3M}\int_{[-1+\delta_1,1-\delta_1]\backslash\mathcal{S}}\left|g(x)-P(x)\right|\frac{dx}{\sqrt{1-x^2}}<\frac{\varepsilon_1}{8},\end{aligned}$$

by (4.9). Also, if $S_+=\{x\in[-1+\delta_1,1-\delta_1]:\log\mu'(x)\ge M\}$

$$\begin{aligned}&\int_{\mathcal{S}_+}\frac{1}{\sqrt{1-x^2}}\left|\frac{\mu'(x)}{\hat{\mu}'(x)}-1\right|dx=\int_{\mathcal{S}_+}\frac{1}{\sqrt{1-x^2}}\left|\frac{\mu'(x)}{e^M}-1\right|dx\\ \le&2\int_{\mathcal{S}_+}\frac{\mu'(x)}{\sqrt{1-x^2}}dx<\frac{\varepsilon_1}{4},\end{aligned}$$

by (4.8), while if $S_- = \{x \in [-1+\delta_1, 1-\delta_1] : \log \mu'(x) \leq -M\}$

$$\int_{\mathcal{S}_-} \frac{1}{\sqrt{1-x^2}} \left| \frac{\mu'(x)}{\hat{\mu}'(x)} - 1 \right| dx = \int_{\mathcal{S}_-} \frac{1}{\sqrt{1-x^2}} \left| \frac{\mu'(x)}{e^{-M}} - 1 \right| dx$$

$$\leq \quad 2 \int_{\mathcal{S}-} \frac{1}{\sqrt{1-x^2}} dx < \frac{\varepsilon_1}{4},$$

by (4.8) again. Finally,

$$\int_{1 \geq |x| \geq 1-\delta_1} \frac{1}{\sqrt{1-x^2}} \left| \frac{\mu'(x)}{\hat{\mu}'(x)} - 1 \right| dx = \int_{1 \geq |x| \geq 1-\delta_1} \frac{1}{\sqrt{1-x^2}} \left| \mu'(x) \sqrt{1-x^2} - 1 \right| dx$$

$$\leq \quad \int_{1 \geq |x| \geq 1-\delta_1} \left(\mu'(x) + \frac{1}{\sqrt{1-x^2}} \right) dx \leq \frac{\varepsilon_1}{4},$$

by (4.5). Combining the above inequalities gives

$$\begin{aligned} & \int_{-1}^{1} \frac{1}{\pi\sqrt{1-x^2}} \left| \frac{\mu'(x)}{\hat{\mu}'(x)} - 1 \right| dx \\ & < \frac{\varepsilon_1}{8} + 2\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4} < \varepsilon_1. \end{aligned} \tag{4.12}$$

Together with (4.11), and recalling (4.4), this gives

$$\eta_\infty < \varepsilon_1 + \frac{\varepsilon_1}{2} = \frac{3}{2}\varepsilon_1.$$

Step 5: Completion of the Proof
Choose n_0 such that for $n \geq n_0$,

$$\eta_n < 2\varepsilon_1.$$

Then using Theorem 3.1, for the given $\varepsilon, R > 0$, there exist for $n > n_0$, sets $\mathcal{E}_n$ and $\mathcal{F}_n$, such that for $x \in \left(\mathcal{L}\left(\mu, \eta_n^{1/4}\right) \backslash \mathcal{E}_n \right) \cap ((-1,1) \backslash \mathcal{F}_n)$, and all complex a, b, with $|a|, |b| \leq R$,

$$\frac{1}{n} \left| K_n^{\mu,\omega} - K_n^{\hat{\mu},\omega} \right| \left(x + \frac{a}{n}, x + \frac{b}{n} \right) \leq \hat{C} \eta_n^{1/4} \leq \hat{C} (2\varepsilon_1)^{1/4}.$$

Here, $\hat{C}$ depends on R, ε, but is independent of $n, x, a, b, \mu, \hat{\mu}$. Crucially, also, ε_1 is independent of R and ε. Set

$$\tilde{\mathcal{E}}_n = \mathcal{E}_n \cup \mathcal{F}_n \cup \left((-1,1) \backslash \mathcal{L}\left(\mu, \varepsilon_1^{1/4}\right) \right).$$

By (3.3),

$$\text{meas}\left(\tilde{\mathcal{E}}_n\right) \leq (2\varepsilon_1)^{1/4} + 2\varepsilon + \text{meas}\left((-1,1) \backslash \mathcal{L}\left(\mu, (2\varepsilon_1)^{1/4}\right) \right).$$

We now choose ε_1 so small that this last right-hand side is less than 3ε, while also $\hat{C}(2\varepsilon_1)^{1/4} < \varepsilon$. ■.

Proof of Theorem 1.4 for absolutely continuous μ and ν
We assume that μ and ν are absolutely continuous. Let $\varepsilon, R > 0$. Choose a measure $\hat{\mu}$ approximating μ as in Lemma 4.1 with corresponding exceptional sets $\left\{\tilde{\mathcal{E}}_n\right\}$ as in (4.2). Similarly choose a measure $\hat{\nu}$ approximating ν and corresponding exceptional sets $\left\{\tilde{\mathcal{F}}_n\right\}$. We shall also use the estimate (4.10) and its analogue for ν, as well as the number $\delta_1 < \varepsilon$ from the proof of Lemma 4.1. Since the pair $(\hat{\mu}, \hat{\nu})$ satisfies

the requirements of Proposition 1.1, and Example 1.2, for ξ in compact subsets of $(-1+\varepsilon, 1-\varepsilon)$, and uniformly for a, b in compact subsets of the plane, we have

$$\lim_{n\to\infty} \frac{w(\hat{\mu},\hat{\nu},\xi)}{n} K_n^{\hat{\mu},\hat{\nu}}\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{\bar{b}\pi\sqrt{1-\xi^2}}{n}\right)$$
$$= S\left(\frac{b-a}{2}\right)\cos\left(\frac{\pi(b-a)}{2} + \rho_{\hat{\mu}}(\xi) - \rho_{\hat{\nu}}(\xi)\right), \tag{4.13}$$

where

$$\hat{B}(\xi) := \rho_{\hat{\mu}}(\xi) - \rho_{\hat{\nu}}(\xi) = \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \log \frac{\hat{\mu}'(\cos t)}{\hat{\nu}'(\cos t)} \cot \frac{\theta - t}{2} dt. \tag{4.14}$$

Next, uniformly for ξ in $[-1+2\varepsilon, 1-2\varepsilon]$, and complex a, b with $|a|, |b| \le R$,

$$\left|\frac{w(\hat{\mu},\hat{\nu},\xi)}{n} K_n^{\mu,\nu}\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n}\right) - S\left(\frac{a-b}{2}\right)\cos\left(\frac{\pi(a-b)}{2} + B(\xi)\right)\right|$$
$$\le \left|\frac{w(\hat{\mu},\hat{\nu},\xi)}{n}\left[K_n^{\mu,\nu} - K_n^{\hat{\mu},\nu}\right]\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{\bar{b}\pi\sqrt{1-\xi^2}}{n}\right)\right|$$
$$+\left|\frac{w(\hat{\mu},\hat{\nu},\xi)}{n}\left[K_n^{\hat{\mu},\nu} - K_n^{\hat{\mu},\hat{\nu}}\right]\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{\bar{b}\pi\sqrt{1-\xi^2}}{n}\right)\right|$$
$$+\left|\frac{w(\hat{\mu},\hat{\nu},\xi)}{n} K_n^{\hat{\mu},\hat{\nu}}\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{\bar{b}\pi\sqrt{1-\xi^2}}{n}\right) - S\left(\frac{a-b}{2}\right)\cos\left(\frac{\pi(a-b)}{2} + \hat{B}(\xi)\right)\right|$$
$$+\left|S\left(\frac{a-b}{2}\right)\left[\cos\left(\frac{\pi(a-b)}{2} + \hat{B}(\xi)\right) - \cos\left(\frac{\pi(a-b)}{2} + B(\xi)\right)\right]\right|$$
$$=: w(\hat{\mu},\hat{\nu},\xi)(T_1 + T_2) + o(1) + T_3, \tag{4.15}$$

where we have used (4.13). Here, by Lemma 4.1 applied to $(\mu, \hat{\mu}, \nu)$, we have for $|a|, |b| \le R$, and $\xi \in (-1,1)\setminus\tilde{\mathcal{E}}_n$,

$$|T_1| = \frac{1}{n}\left|K_n^{\mu,\nu} - K_n^{\hat{\mu},\nu}\right|\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n}\right) \le \varepsilon, \tag{4.16}$$

and by that lemma applied to $(\mu, \hat{\mu}, \hat{\nu})$, we have for $|a|, |b| \le R$, and $\xi \in (-1,1)\setminus\hat{\mathcal{F}}_n$,

$$|T_2| = \frac{1}{n}\left|K_n^{\hat{\mu},\nu} - K_n^{\hat{\mu},\hat{\nu}}\right|\left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{\bar{b}\pi\sqrt{1-\xi^2}}{n}\right) \le \varepsilon. \tag{4.17}$$

Next,

$$T_3 = \left|S\left(\frac{a-b}{2}\right)\left[\cos\left(\frac{\pi(a-b)}{2} + \hat{B}(\xi)\right) - \cos\left(\frac{\pi(a-b)}{2} + B(\xi)\right)\right]\right|$$
$$\le 2\left|\sin\left(\frac{\hat{B}(\xi) - B(\xi)}{2}\right)\right| \le \left|\hat{B}(\xi) - B(\xi)\right|. \tag{4.18}$$

For $\xi = \cos\theta$,

$$\begin{aligned} & \hat{B}(\xi) - B(\xi) \\ = & \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \left[\log \frac{\hat{\mu}'(\cos t)}{\hat{\nu}'(\cos t)} - \log \frac{\mu'(\cos t)}{\nu'(\cos t)} \right] \cot \frac{\theta - t}{2} dt \\ = & \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \left[\log \frac{\hat{\mu}'(\cos t)}{\mu'(\cos t)} - \log \frac{\hat{\nu}'(\cos t)}{\nu'(\cos t)} \right] \cot \frac{\theta - t}{2} dt. \end{aligned}$$

Next, we use the fact that the conjugate function operator is weak type $(1,1)$ [3, p. 160, Thm. 6.8], so that for $\lambda > 0$,

$$\begin{aligned} & \text{meas} \left\{ \theta \in (0,\pi) : \left| \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \log \frac{\hat{\mu}'(\cos t)}{\mu'(\cos t)} \cot \frac{\theta - t}{2} dt \right| > \lambda \right\} \\ \leq & \frac{C_0}{\lambda} \int_{-\pi}^{\pi} \left| \log \frac{\hat{\mu}'(\cos t)}{\mu'(\cos t)} \right| dt \leq \frac{C_0}{\lambda} \varepsilon_1 \leq \frac{C_0}{\lambda} \varepsilon, \end{aligned}$$

and similarly

$$\text{meas} \left\{ \theta \in (0,\pi) : \left| \frac{1}{4\pi} PV \int_{-\pi}^{\pi} \log \frac{\hat{\nu}'(\cos t)}{\nu'(\cos t)} \cot \frac{\theta - t}{2} dt \right| > \lambda \right\} \leq \frac{C_0}{\lambda} \varepsilon.$$

Here C_0 is an absolute constant, and we are using (4.10) for μ, ν. We choose $\lambda = \sqrt{\varepsilon}$, and let

$$\mathcal{G} = \left\{ \xi \in (-1,1) : \left| \hat{B}(\xi) - B(\xi) \right| > \varepsilon^{1/2} \right\},$$

so that by the above inequalities, $\text{meas}(\mathcal{G}) \leq 2C_0 \varepsilon^{1/2}$. Now let

$$\mathcal{H}_n = \tilde{\mathcal{E}}_n \cup \tilde{\mathcal{F}}_n \cup \mathcal{G} \cup [-1 + 2\varepsilon, 1] \cup [1 - 2\varepsilon, 1].$$

We see that for large enough n, we have

$$\text{meas}(\mathcal{H}_n) \leq 2\varepsilon + 2C_0 \varepsilon^{1/2} + 4\varepsilon \leq (6 + 2C_0) \varepsilon^{1/2}.$$

Moreover, for $n \geq n_0$, $\xi \in (-1,1) \backslash \mathcal{H}_n$, and $|a|, |b| \leq R$, (4.15)-(4.18) give

$$\begin{aligned} & \left| \frac{w(\hat{\mu}, \hat{\nu}, \xi)}{n} K_n^{\mu,\nu} \left(\xi + \frac{a\pi\sqrt{1-\xi^2}}{n}, \xi + \frac{b\pi\sqrt{1-\xi^2}}{n} \right) - S\left(\frac{a-b}{2} \right) \cos\left(\frac{\pi(a-b)}{2} + B(\xi) \right) \right| \\ \leq & \; w(\hat{\mu}, \hat{\nu}, \xi) \, 2\varepsilon + \varepsilon + \varepsilon^{1/2}. \end{aligned}$$

Finally, $w(\hat{\mu}, \hat{\nu}, \xi)$ is close to $w(\mu, \nu, \xi)$ except on a set of small measure. Indeed, (4.12) shows that

$$\text{meas} \left\{ x \in (-1,1) : \left| \frac{\mu'(x)}{\hat{\mu}'(x)} - 1 \right| > \varepsilon_1^{1/2} \right\} < \pi \varepsilon_1^{1/2}.$$

A similar inequality holds for ν and $\hat{\nu}$. Since $\varepsilon_1 < \varepsilon$, we are finished. ■

Proof of Theorem 1.4 for the case where μ, ν may have singular parts
Let μ_{ac} and ν_{ac} denote the absolutely continuous parts of μ and ν respectively. The conclusion of Theorem 1.4 holds for $K_n^{\mu_{ac},\nu_{ac}}$. We write

$$\begin{aligned} & K_n^{\mu,\nu} - K_n^{\mu_{ac},\nu_{ac}} \\ = & \left[K_n^{\mu,\nu} - K_n^{\mu,\nu_{ac}} \right] + \left[K_n^{\mu,\nu_{ac}} - K_n^{\mu_{ac},\nu_{ac}} \right] \end{aligned}$$

and use Theorem 3.1 to show that each of the terms in [] is small outside a set of small measure. Let us illustrate on the first term. Now since $\nu_{ac} \leq \nu$, we have

$$K_n^{\nu_{ac}}(x,x) \geq K_n^{\nu}(x,x) \text{ for } x \in \mathbb{R},$$

by the variational property of reproducing kernels along the diagonal. Then

$$\frac{1}{n}\int K_n^{\nu}(x,x)\,d\nu_{ac}(x) \leq \frac{1}{n}\int K_n^{\nu_{ac}}(x,x)\,d\nu_{ac}(x) = 1.$$

We apply Theorem 3.1 to the measures (ν, ν_{ac}, μ). Then η_n of (3.2) becomes

$$\begin{aligned} \eta_n &= 1 + \frac{1}{n}\int K_n^{\nu}(x,x)\,d\nu_{ac}(x) - \frac{2}{n}\sum_{j=0}^{n-1}\frac{\gamma_j^{\nu}}{\gamma_j^{\nu_{ac}}} \\ &\leq 2\left(1 - \frac{1}{n}\sum_{j=0}^{n-1}\frac{\gamma_j^{\nu}}{\gamma_j^{\nu_{ac}}}\right). \end{aligned}$$

Now Szego's theorem for the leading coefficients gives, as above,

$$\lim_{j\to\infty}\frac{\gamma_j^{\nu}}{\gamma_j^{\nu_{ac}}} = 1.$$

Hence

$$\lim_{n\to\infty}\eta_n = 0.$$

By Theorem 3.1, given $R, \varepsilon > 0$,

$$\begin{aligned} &\sup_{|a|,|b|\leq R}\left|K_n^{\mu,\nu} - K_n^{\mu,\nu_{ac}}\right|\left(x+\frac{a}{n}, x+\frac{b}{n}\right) \\ &= \sup_{|a|,|b|\leq R}\left|K_n^{\nu,\mu} - K_n^{\nu_{ac},\mu}\right|\left(x+\frac{a}{n}, x+\frac{b}{n}\right) < \varepsilon, \end{aligned}$$

outside a set of small measure. A similar estimate holds for

$$\sup_{|a|,|b|\leq R}\left|K_n^{\mu,\nu_{ac}} - K_n^{\mu_{ac},\nu_{ac}}\right|\left(x+\frac{a}{n}, x+\frac{b}{n}\right).$$

■

5. Proof of Corollaries 1.5-1.7

Proof of Corollary 1.5

Our hypothesis (1.10) on μ' and ν' ensures that uniformly for x in I, we have

$$K_n^{\mu}(x,x) \leq C_1 n \text{ and } K_n^{\nu}(x,x) \leq C_1 n.$$

Cauchy-Schwarz then shows that

$$\sup_{x,y\in I, n\geq 1}\frac{1}{n}\left|K_n^{\mu,\nu}(x,y)\right| \leq C_1.$$

Also $w(\mu,\nu,\xi)$ is bounded above for $\xi \in I$. Combining the convergence in measure in Theorem 1.4 and this last bound easily yields the result. ■

Proof of Corollary 1.6

The main idea is to use Cauchy's estimates for Taylor series of analytic functions. First we expand the right-hand side of (1.4) as a double Taylor series in a, b. Now

$$\begin{aligned}\Psi(a,b) &= S\left(\frac{\pi(a-b)}{2}\right)\cos\left(\frac{\pi(a-b)}{2}+B(\xi)\right)\\ &= S(\pi(a-b))\cos B(\xi) - \frac{\sin^2\left(\frac{\pi(a-b)}{2}\right)}{\frac{\pi(a-b)}{2}}\sin B(\xi).\end{aligned}$$

Here using Euler's formula $\sin u = \frac{1}{2i}\left(e^{iu}-e^{-iu}\right)$, the binomial expansion, and some elementary manipulation, we see that

$$S(\pi(a-b)) = \sum_{\ell,m=0}^{\infty}\frac{a^\ell}{\ell!}\frac{b^m}{m!}\pi^{\ell+m}\tau_{\ell,m};$$

$$\frac{\sin^2\left(\frac{\pi(a-b)}{2}\right)}{\frac{\pi(a-b)}{2}} = \sum_{\ell,m=0}^{\infty}\frac{a^\ell}{\ell!}\frac{b^m}{m!}\pi^{\ell+m}\rho_{\ell,m},$$

where $\{\tau_{\ell,m}\}, \{\rho_{\ell,m}\}$ are defined respectively by (1.13) and (1.14). Now let,

$$\Delta_n(a,b) = \frac{w(\mu,\nu,\xi)}{n}K_n^{\mu,\nu}\left(\xi+\frac{a\pi\sqrt{1-\xi^2}}{n},\xi+\frac{b\pi\sqrt{1-\xi^2}}{n}\right)-\Psi(a,b).$$

Using a double Taylor series expansion on $K_n^{\mu,\nu}$, and those above, we see that

$$\begin{aligned}&\Delta_n(a,b)\\ =& \sum_{\ell,m=0}^{\infty}\frac{a^\ell}{\ell!}\frac{b^m}{m!}\pi^{\ell+m}\left[\begin{array}{c}\frac{w(\mu,\nu,\xi)}{n}\left(\frac{\sqrt{1-\xi^2}}{n}\right)^{\ell+m}K_n^{\mu,\nu(\ell,m)}(\xi,\xi)\\ -\left(\tau_{\ell.m}\cos B(\xi)-\rho_{\ell,m}\sin B(\xi)\right)\end{array}\right].\end{aligned}$$

Finally, Cauchy's inequalities show that for a given ℓ, m,

$$\begin{aligned}&\pi^{\ell+m}\left|\frac{w(\mu,\nu,\xi)}{n}\left(\frac{\sqrt{1-\xi^2}}{n}\right)^{\ell+m}K_n^{\mu,\nu(\ell,m)}(\xi,\xi)-\left(\tau_{\ell.m}\cos B(\xi)-\rho_{\ell,m}\sin B(\xi)\right)\right|\\ \leq& \sup_{|a|,|b|\leq r}|\Delta_n(a,b)|/r^{\ell+m}.\end{aligned}$$

Now we integrate for ξ over I, and use Corollary 1.5. For pointwise convergence, Cauchy's estimates directly give the result. ■

Proof of Corollary 1.7

This follows from Hurwitz's theorem, as $S\left(\frac{\pi(b-a)}{2}\right)=0$ when, and only when $b = a+2j$, while $\cos\left(\frac{\pi(b-a)}{2}+B(\xi)\right)=0$, when and only when $b = a+2j+1-\frac{2}{\pi}B(\xi)$. ■

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