LOCAL ASYMPTOTICS FOR ORTHONORMAL POLYNOMIALS ON THE UNIT CIRCLE VIA UNIVERSALITY

D. S. LUBINSKY

Abstract. Let μ be a positive measure on the unit circle that is regular in the sense of Stahl, Totik, and Ullmann. Assume that in some subarc J, μ is absolutely continuous, while μ' is positive and continuous. Let $\{\varphi_n\}$ be the orthonormal polynomials for μ. We show that for appropriate $\zeta_n \in J$, $\left\{\frac{\varphi_n\left(\zeta_n\left(1+\frac{z}{n}\right)\right)}{\varphi_n(\zeta_n)}\right\}_{n\geq 1}$ is a normal family in compact subsets of $\mathbb{C}$. Using universality limits, we show that limits of subsequences have the form $e^z + C\left(e^z - 1\right)$ for some constant C. Under additional conditions, we can set $C = 0$.

Dedicated to L. Zalcman

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1. Results

Let μ be a finite positive Borel measure on $[-\pi, \pi)$ (or equivalently on the unit circle) with infinitely many points in its support. Then we may define orthonormal polynomials

$$\varphi_n(z) = \kappa_n z^n + ..., \kappa_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_n(z)\,\overline{\varphi_m(z)}d\mu(\theta) = \delta_{mn},$$

where $z = e^{i\theta}$. We shall often assume that μ is *regular* in the sense of Stahl, Totik and Ullmann [14], so that

$$\lim_{n\to\infty} \kappa_n^{1/n} = 1.$$

This is true if for example $\mu' > 0$ a.e. in $[-\pi, \pi)$, but there are pure jump and pure singularly continuous measures that are regular. We denote the zeros of φ_n by $\{z_{jn}\}_{j=1}^{n}$. They lie inside the unit circle, and may not be distinct.

The nth reproducing kernel for μ is

$$K_n(z,u) = \sum_{j=0}^{n-1} \varphi_j(z)\,\overline{\varphi_j(u)}.$$

One of the key limits in random matrix theory, the so-called universality limit [4], [5], [7], [8], [13], [16], [17] can be cast in the following form for measures on the unit circle [6, Thm. 6.3, p. 559]:

Theorem A

Let μ be a finite positive Borel measure on the unit circle that is regular. Let

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$J\subset(-\pi,\pi)$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume moreover, that μ' is positive and continuous at each point of J. Then uniformly for $\theta\in J, z=e^{i\theta}$ and a,b in compact subsets of the complex plane, we have

$$\lim_{n\to\infty}\frac{K_n\left(z\left(1+\frac{i2\pi a}{n}\right),z\left(1+\frac{i2\pi\bar{b}}{n}\right)\right)}{K_n(z,z)}=e^{i\pi(a-b)}\mathbb{S}(a-b),\tag{1.1}$$

where $\mathbb{S}(t)=\frac{\sin\pi t}{\pi t}$.

There are several refinements and generalizations of this result, see for example, [11], [13], [16], [17].

In this paper, we shall use the universality limit to establish "local" asymptotics for the ratio $\varphi_n\left(z\left(1+\frac{u}{n}\right)\right)/\varphi_n(z)$ with u as our variable. Analogous results for orthogonal polynomials associated with measures on compact subsets of the real line were established in [9], [10]. In [9], we showed that if μ is a regular measure on $[-1,1]$ for which $\mu'(x)(1-x)^{-\alpha}$ has a finite positive limit as $x\to 1-$, then the orthonormal polynomials $\{p_n\}$ for μ satisfy, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{p_n\left(1-\frac{z^2}{2n^2}\right)}{p_n(1)}=\frac{J_\alpha^*(z)}{J_\alpha^*(0)},$$

where $J_\alpha^*(z)=J_\alpha(z)/z^\alpha$ is the normalized Bessel function of order α. In [10], we showed that if μ is a regular measure with compact support in the real line, and in some closed subinterval J of the support, μ is absolutely continuous, while μ' is continuous, then for points y_{jn} in a compact subset of J^o with $p_n'(y_{jn})=0$, we have

$$\lim_{n\to\infty}\frac{p_n\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)}{p_n(y_{jn})}=\cos\pi z$$

uniformly in y_{jn} and for z in compact subsets of the plane. Here ω is the density of the equilibrium measure of the support.

The case of the unit circle turns out to be more difficult, because there is no obvious analogue of the point 1 at the endpoint of $[-1,1]$, or the local maximum point y_{jn} of $|p_n|$ inside the support. The derivative φ_n' of the orthonormal polynomial φ_n has all its zeros inside the unit circle. Moreover, $\left|\varphi_n\left(e^{i\theta}\right)\right|$ might have only a few local maxima for $\theta\in[-\pi,\pi]$. One could consider points where paraorthogonal polynomials assume local maximal absolute values and indeed we shall do this in Lemma 4.4. However, for the most part we shall consider points and rotations of them through small angles. We prove:

Theorem 1.1

Let μ be a positive measure on the unit circle that is regular in the sense of Stahl, Totik and Ullmann. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. Let J_1 be a subarc of the (relative) interior of J. Let $\{z_n\}_{n\geq 1}$ be a sequence in J_1. For $n\geq 1$, we can choose at least one of $\zeta_n=z_n$ or $\zeta_n=z_ne^{i\pi/n}$ such that from any infinite sequence of positive integers, we can extract a further subsequence $\mathcal{S}$ such that uniformly for

u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty,n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)} = e^u + C(e^u-1) \tag{1.2}$$

where

$$C = \lim_{n\to\infty,n\in\mathcal{S}} \left(\frac{\zeta_n}{n}\frac{\varphi_n'\left(\zeta_n\right)}{\varphi_n\left(\zeta_n\right)} - 1\right). \tag{1.3}$$

Moreover, $|C| \leq 1$.

Next, we consider when we may take $C = 0$:

Theorem 1.2
Let μ be a positive measure on the unit circle that is regular in the sense of Stahl, Totik and Ullmann. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. Let J_1 be a subarc of the interior of J. Let $\{\zeta_n\} \subset J_1$ and $\mathcal{S}$ be an infinite sequence of positive integers. The following are equivalent:
(I) Uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty,n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)} = e^u. \tag{1.4}$$

(II)

$$\lim_{n\to\infty,n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)}{\varphi_n\left(\zeta_n\right)} = -1. \tag{1.5}$$

(III) Both

$$\lim_{n\to\infty,n\in\mathcal{S}} \left|\varphi_n\left(\zeta_n\right)\right|^2 \mu'\left(\zeta_n\right) = 1 \tag{1.6}$$

and

$$\lim_{n\to\infty,n\in\mathcal{S}} \operatorname{Im}\left(\frac{\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)}{\varphi_n\left(\zeta_n\right)}\right) = 0. \tag{1.7}$$

(IV) If $\{z_{jn}\}_{j=1}^n$ are the zeros of φ_n, both

$$\lim_{n\to\infty,n\in\mathcal{S}} \frac{1}{n}\sum_{j=1}^{n} \frac{1-\left|z_{jn}\right|^2}{\left|\zeta_n - z_{jn}\right|^2} = 1 \tag{1.8}$$

and (1.7) holds.

Remarks
(a) If (1.4) holds uniformly in $\zeta_n \in J_1$ and for u in compact sets, our proof shows that (1.6) holds uniformly in J_1, and in particular $\left\{\left|\varphi_n\right|^2\right\}$ is uniformly bounded in J_1. The latter requires far more of μ than the initial hypotheses stated. Indeed, there are measures μ satisfying the initial hypotheses of Theorem 1.2 for which $\left\{\left|\varphi_n\right|^2\right\}$ is not uniformly bounded [1], [2].
(b) Note that (II) is essentially the special case of (I) with $u = \pm i\pi$.
(c) What sort of explicit assumptions on μ guarantee that (1.4) holds? If we have uniform pointwise asymptotics of $\{\varphi_n\}$ on J_1, then we have (1.5) and hence (1.4).

The most general conditions that guarantee these are due to Badkov [3]. Let μ satisfy Szegő's condition $\int_{-\pi}^{\pi} \log \mu'(\theta)\, d\theta > -\infty$. Assume in addition that in J, μ is absolutely continuous, and the local modulus of continuity $\omega(\cdot)$ of μ' satisfies the Dini-Lipschitz condition

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

Then Badkov proved uniform pointwise asymptotics that imply (1.5). We note that the conclusion of Theorem 1.2 is new even for the measures considered by Badkov. Indeed, the standard asymptotics for Szegő measures outside the unit circle, hold only at a positive distance to the unit circle, while Badkov's asymptotics hold only on the unit circle.

Theorem 1.1 is a consequence of a more general result:

Theorem 1.3

Let μ be a finite positive Borel measure on the unit circle with infinitely many points in its support. Assume that $\{\zeta_n\}$ is a sequence of numbers on the unit circle, and that uniformly for a, b in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} \frac{K_n\left(\zeta_n\left(1+\frac{i2\pi a}{n}\right), \zeta_n\left(1+\frac{i2\pi \bar{b}}{n}\right)\right)}{K_n(\zeta_n, \zeta_n)} = e^{i\pi(a-b)}\mathbb{S}(a-b). \tag{1.9}$$

The following are equivalent:
(a)

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty; \ \sup_{n\geq 1} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{|\zeta_n - z_{jn}|^2} < \infty. \tag{1.10}$$

(b) From every infinite sequence of positive integers, we can choose an infinite subsequence $\mathcal{S}$ such that uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)} = e^u + C(e^u - 1), \tag{1.11}$$

where

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{\zeta_n}{n}\frac{\varphi_n'(\zeta_n)}{\varphi_n(\zeta_n)} - 1\right), \tag{1.12}$$

and C is bounded independently of the subsequence $\mathcal{S}$.

Remarks

(a) An equivalent formulation of (1.11) is

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n e^{2\pi i a/n}\right)}{\varphi_n(\zeta_n)} = e^{2\pi i a} + C_1 e^{\pi i a}\sin(\pi a), \tag{1.13}$$

uniformly for a in compact subsets of $\mathbb{C}$, for some constant C_1.

(b) We note that one can dispense with the first condition in (1.10) provided we assume that u is restricted to the half plane in which

$$\operatorname{Re}\left(u\sum_{j=1}^{n} \frac{\zeta_n}{\zeta_n - z_{jn}}\right) \leq 0,$$

which in turn is true provided u lies in the quadrant where

$$\operatorname{Re} u \leq 0 \text{ and } (\operatorname{Im} u)\left(\operatorname{Im}\sum_{j=1}^{n}\frac{\zeta_n}{\zeta_n - z_{jn}}\right) \geq 0.$$

The second condition in (1.10) is somewhat easier to satisfy than the first, see Lemma 4.3 below. This yields normality of $\left\{\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)}\right\}_{n\in\mathcal{S}}$ at least for u in a suitable quadrant, and subsequences $\mathcal{S}$ of integers.

(c) It is possible to formulate a version of Theorem 1.3 where μ is replaced at the nth stage by a measure μ_n so that we are handling varying measures, as was done in [9], [10] for measures on the real line. Moreover, we could consider an infinite subsequence of integers rather than the full sequence of positive integers.

It is instructive to consider converse results, where we assume only the limit (1.2):

Theorem 1.4

Let μ be a positive measure on the unit circle with infinitely many points in its support. Let $\mathcal{S}$ be an infinite sequence of positive integers and assume that for $n \in \mathcal{S}$, we are given ζ_n on the unit circle. Assume that uniformly for u in compact subsets of $\mathbb{C}$, we have the limit (1.2), for some constant C. Then uniformly for a, b in compact subsets of $\mathbb{C}$,

$$\begin{aligned}&\lim_{n\to\infty, n\in\mathcal{S}}\frac{1}{n}K_n\left(\zeta_n\left(1+\frac{2\pi i a}{n}\right),\zeta_n\left(1+\frac{2\pi i\bar{b}}{n}\right)\right)/\left|\varphi_n(\zeta_n)\right|^2\\ (1.14)\qquad &= \left(2\operatorname{Re}C+1\right)e^{i\pi(a-b)}\mathbb{S}(a-b).\end{aligned}$$

In particular, if $\mu'(\zeta_n)$ exists and is finite and positive for $n \in \mathcal{S}$, then we have the usual universality limit

(1.15)

$$\lim_{n\to\infty, n\in\mathcal{S}}\frac{1}{n}K_n\left(\zeta_n\left(1+\frac{2\pi i a}{n}\right),\zeta_n\left(1+\frac{2\pi i\bar{b}}{n}\right)\right)\mu'(\zeta_n) = e^{i\pi(a-b)}\mathbb{S}(a-b)$$

iff

$$\lim_{n\to\infty, n\in\mathcal{S}}\frac{1}{\left|\varphi_n(\zeta_n)\right|^2\mu'(\zeta_n)} = 2\operatorname{Re}C+1. \tag{1.16}$$

Our proofs very heavily use the fact that there is a Christoffel-Darboux formula for orthogonal polynomials on the unit circle. Since such a formula is lacking for more general contours, it will be a significant challenge to extend the results of this paper to such a setting.

Normal families of analytic functions play an important role in the research of Larry Zalcman. They also play a role in our proofs, so this paper is appropriate for dedication to Larry Zalcman.

This paper is organised as follows: Theorem 1.3 is proved in Section 2. Theorem 1.4 is proved in Section 3. Theorems 1.1 and 1.2 are proved in Section 4.

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2. Proof of Theorem 1.3

We shall use the Christoffel-Darboux formula [12, p. 125], [15, p. 293]

$$K_n(z,t) = \sum_{k=0}^{n-1} \varphi_k(z)\overline{\varphi_k(t)} = \frac{\overline{\varphi_n^*(t)}\varphi_n^*(z) - \overline{\varphi_n(t)}\varphi_n(z)}{1-\bar{t}z}, \tag{2.1}$$

where

$$\varphi_n^*(z) = z^n \overline{\varphi_n\left(\frac{1}{\bar{z}}\right)}$$

is the reversed polynomial. (Note that Simon [12, p. 120] sums to n in the definition of K_n.) Let

$$H_n(z,t) = \frac{\varphi_n^*(z)}{\varphi_n(z)} - \frac{\varphi_n^*(t)}{\varphi_n(t)}. \tag{2.2}$$

Lemma 2.1
(a)

$$H_n(z,t) = \frac{t^n K_n\left(z,\frac{1}{\bar{t}}\right)\left(1-\frac{z}{t}\right)}{\varphi_n(t)\,\varphi_n(z)}. \tag{2.3}$$

(b)

$$H_n(z,t) = H_n(z,u) + H_n(u,t). \tag{2.4}$$

Proof
(a) From the Christoffel-Darboux formula,

$$\begin{aligned} & t^n K_n\left(z,\frac{1}{\bar{t}}\right)\left(1-\frac{z}{t}\right) \\ = \; & t^n \left[\overline{\varphi_n^*\left(\frac{1}{\bar{t}}\right)}\varphi_n^*(z) - \overline{\varphi_n\left(\frac{1}{\bar{t}}\right)}\varphi_n(z)\right] \\ = \; & \varphi_n(t)\,\varphi_n^*(z) - \varphi_n^*(t)\,\varphi_n(z), \end{aligned}$$

so (2.3) follows from the definition of $H_n(z,t)$.
(b) This is immediate from the definition of H_n. ■

Lemma 2.2
Let $\{\zeta_n\}$ be a sequence on the unit circle. The following are equivalent:
(a)

$$\sup_{n\geq 1} \frac{1}{n}\left|\sum_{j=1}^{n} \frac{1}{\zeta_n - z_{jn}}\right| < \infty; \sup_{n\geq 1} \frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{\left|\zeta_n - z_{jn}\right|^2} < \infty. \tag{2.5}$$

(b) The functions $\left\{\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)}\right\}_{n\geq 1}$ are uniformly bounded for u in compact subsets of $\mathbb{C}$.
Proof

(a)⇒(b)
Now

$$
\begin{aligned}
&\log \left| \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)} \right| \\
(2.6)\qquad &= \frac{1}{2}\sum_{j=1}^{n} \log\left(1+2\operatorname{Re}\left(\frac{u\zeta_n}{n\left(\zeta_n - z_{jn}\right)}\right) + \left|\frac{u\zeta_n}{n\left(\zeta_n - z_{jn}\right)}\right|^2\right) \\
&\leq \operatorname{Re}\left(\frac{u\zeta_n}{n}\sum_{j=1}^{n}\frac{1}{\zeta_n - z_{jn}}\right) + \frac{|u|^2}{2n^2}\sum_{j=1}^{n}\frac{1}{\left|\zeta_n - z_{jn}\right|^2}.
\end{aligned}
$$

Then given $R > 0$, we obtain from (2.5),

$$\sup_{n\geq 1}\sup_{|u|\leq R}\left|\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)}\right| < \infty.$$

(b)⇒(a)
Let

$$A = \sup_{n\geq 1}\sup_{|u|\leq 1}\log\left|\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n\left(\zeta_n\right)}\right|.$$

We use the fact that for each j,

$$\operatorname{Re}\left(\frac{\zeta_n}{\zeta_n - z_{jn}}\right) = \frac{1-\operatorname{Re}\left(\zeta_n\overline{z_{jn}}\right)}{\left|\zeta_n - z_{jn}\right|^2} \geq 0,$$

so that

$$2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n - z_{jn}\right)}\right) + \left|\frac{1}{n\left(\zeta_n - z_{jn}\right)}\right|^2 \geq 0.$$

Then also, for each j, we have from the identity (2.6) above,

$$e^{2A} - 1 \geq 2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n - z_{jn}\right)}\right) + \left|\frac{1}{n\left(\zeta_n - z_{jn}\right)}\right|^2 \geq 0.$$

Choose $C_1 > 0$ such that

$$\log\left(1+t\right) \geq C_1 t \text{ for } t \in [0, e^{2A} - 1].$$

Then from (2.6),

$$
\begin{aligned}
A &\geq \frac{C_1}{2}\sum_{j=1}^{n}\left(2\operatorname{Re}\left(\frac{\zeta_n}{n\left(\zeta_n - z_{jn}\right)}\right) + \left|\frac{1}{n\left(\zeta_n - z_{jn}\right)}\right|^2\right) \\
&= \frac{C_1}{n}\operatorname{Re}\left(\zeta_n\sum_{j=1}^{n}\frac{1}{\zeta_n - z_{jn}}\right) + \frac{C_1}{2n^2}\sum_{j=1}^{n}\frac{1}{\left|\zeta_n - z_{jn}\right|^2}.
\end{aligned}
$$

As both terms are nonnegative, we obtain

$$\sup_{n\geq 1}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left|\zeta_n - z_{jn}\right|^2} < \infty\ .$$

Next, we apply Cauchy's inequalities for derivatives to $f_n(u) = \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)}$. We obtain

$$\left|\frac{\zeta_n}{n}\sum_{j=1}^{n}\frac{1}{\zeta_n - z_{jn}}\right| = |f_n'(0)| \leq \sup_{|u|\leq 1}|f_n(u)| \leq e^A.$$

■

Proof of Theorem 1.3
(a)⇒(b)
By Lemma 2.2, the functions $\{f_n(u)\}_{n\geq 1} = \left\{\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)}\right\}_{n\geq 1}$ form a normal family in $\mathbb{C}$. Assume that $\mathcal{S}$ is an infinite subsequence of integers such that

$$\lim_{n\in\mathcal{S}} f_n(u) = G(u),$$

uniformly for u in compact subsets of the plane, where G is an entire function. Note too that $G(0) = 1$. Let

$$\Delta_n = \frac{n\varphi_n(\zeta_n)^2}{\zeta_n^n K_n(\zeta_n, \zeta_n)}.$$

Then uniformly for u, v in compact sets, and u, v with $G(u), G(v)$ non-zero, Lemma 2.1(a) gives

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right), \zeta_n\left(1+\frac{v}{n}\right)\right) \\
= \quad & \frac{\left(\zeta_n\left(1+\frac{v}{n}\right)\right)^n \frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)} n\left(1-\frac{\zeta_n\left(1+\frac{u}{n}\right)}{\zeta_n\left(1+\frac{v}{n}\right)}\right)}{\zeta_n^n\left[\frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)}\frac{\varphi_n\left(\zeta_n\left(1+\frac{v}{n}\right)\right)}{\varphi_n(\zeta_n)}\right]} \\
= \quad & \frac{e^v}{G(u)G(v)}\frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)}(v-u)(1+o(1)).
\end{aligned}$$

Write $u = 2\pi i a$, $-\bar{v} = 2\pi i\bar{b}$ so that $v = 2\pi i b$. Here by the uniform convergence in (1.9),

$$\begin{aligned}
\frac{K_n\left(\zeta_n\left(1+\frac{u}{n}\right), \frac{1}{\overline{\zeta_n\left(1+\frac{v}{n}\right)}}\right)}{K_n(\zeta_n,\zeta_n)} &= \frac{K_n\left(\zeta_n\left(1+\frac{i2\pi a}{n}\right), \zeta_n\left(1+\frac{i2\pi\bar{b}}{n}+O\left(\frac{1}{n^2}\right)\right)\right)}{K_n(\zeta_n,\zeta_n)} \\
&= e^{i\pi(a-b)}\mathbb{S}(a-b)+o(1) \\
&= e^{(u-v)/2}\mathbb{S}\left(\frac{u-v}{2\pi i}\right)+o(1),
\end{aligned}$$

so

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right),\zeta_n\left(1+\frac{v}{n}\right)\right)\\
=\;& \frac{e^v}{G(u)\,G(v)}e^{(u-v)/2}\mathbb{S}\left(\frac{u-v}{2\pi i}\right)(v-u)+o(1)\\
=\;& 2i\frac{e^{(u+v)/2}}{G(u)\,G(v)}\sin\left(\frac{v-u}{2i}\right)+o(1).
\end{aligned}$$

Now we use this in (2.4). We have for $u,v,w\in\mathbb{C}$,

$$\begin{aligned}
&\Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right),\zeta_n\left(1+\frac{v}{n}\right)\right)\\
=\;& \Delta_n H_n\left(\zeta_n\left(1+\frac{u}{n}\right),\zeta_n\left(1+\frac{w}{n}\right)\right)+\Delta_n H_n\left(\zeta_n\left(1+\frac{w}{n}\right),\zeta_n\left(1+\frac{v}{n}\right)\right)
\end{aligned}$$

and hence for u,v,w with $G(u)\,G(v)\,G(w)\neq 0$,

$$\begin{aligned}
&\frac{e^{(u+v)/2}}{G(u)\,G(v)}\sin\left(\frac{v-u}{2i}\right)\\
=\;& \frac{e^{(u+w)/2}}{G(u)\,G(w)}\sin\left(\frac{w-u}{2i}\right)+\frac{e^{(w+v)/2}}{G(w)\,G(v)}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

Then

$$\begin{aligned}
&G(w)\,e^{(u+v)/2}\sin\left(\frac{v-u}{2i}\right)\\
=\;& G(v)\,e^{(u+w)/2}\sin\left(\frac{w-u}{2i}\right)+G(u)\,e^{(w+v)/2}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

By analytic continuation, this holds for all u,v,w. Next, we note the elementary identity

$$\begin{aligned}
&e^w e^{(u+v)/2}\sin\left(\frac{v-u}{2i}\right)\\
=\;& e^v e^{(u+w)/2}\sin\left(\frac{w-u}{2i}\right)+e^u e^{(w+v)/2}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

(This can be verified directly, or by simply applying the identity above to the case of normalized Lebesgue measure on the unit circle, where we know that $\varphi_n(z)=z^n$ and $G(u)=e^u$). Subtracting the two, we have

$$\begin{aligned}
&[G(w)-e^w]\,e^{(u+v)/2}\sin\left(\frac{v-u}{2i}\right)\\
=\;& [G(v)-e^v]\,e^{(u+w)/2}\sin\left(\frac{w-u}{2i}\right)+[G(u)-e^u]\,e^{(w+v)/2}\sin\left(\frac{v-w}{2i}\right).
\end{aligned}$$

Now we set $w=0$ and use $G(0)=1:$

$$0=-[G(v)-e^v]\,e^{u/2}\sin\left(\frac{u}{2i}\right)+[G(u)-e^u]\,e^{v/2}\sin\left(\frac{v}{2i}\right)$$

so that

$$\frac{G(v)-e^v}{e^{v/2}\sin\left(\frac{v}{2i}\right)}=\frac{G(u)-e^u}{e^{u/2}\sin\left(\frac{u}{2i}\right)}.$$

Then both sides are constant, so calling the right-hand side C_1,

$$\begin{aligned} G(v) &= e^v + C_1 e^{v/2}\sin\left(\frac{v}{2i}\right) \\ &= e^v - \frac{1}{2}iC_1\left(e^v - 1\right). \end{aligned}$$

To determine C_1, we note that

$$G'(0) = 1 - \frac{1}{2}iC_1$$

and also

$$G'(0) = \lim_{n\in\mathcal{S}} f_n'(0) = \lim_{n\in\mathcal{S}} \frac{\zeta_n}{n}\frac{\varphi_n'(\zeta_n)}{\varphi_n(\zeta_n)}.$$

Thus

$$-\frac{1}{2}iC_1 = \lim_{n\in\mathcal{S}}\left(\frac{\zeta_n}{n}\frac{\varphi_n'(\zeta_n)}{\varphi_n(\zeta_n)} - 1\right).$$

Now set $C = -\frac{1}{2}iC_1$ to obtain (1.11). Since $\{f_n\}_{n\geq 1}$ are uniformly bounded, it also follows that C is bounded independent of the subsequence.

(b)⇒(a)

Since C is bounded independently of the subsequence $\mathcal{S}$, the uniform convergence we are assuming gives that $\{f_n\}$ is uniformly bounded in compact subsets of the plane. Lemma 2.2 gives (1.10). ■

3. Proof of Theorem 1.4

Proof of Theorem 1.4

We assume that uniformly for u in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)} = e^u + C(e^u - 1).$$

Equivalently, uniformly for a in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n e^{2\pi i a/n}\right)}{\varphi_n(\zeta_n)} = e^{2\pi i a} + C(e^{2\pi i a} - 1).$$

Then, uniformly for a in compact subsets of $\mathbb{C}$,

$$\begin{aligned} \lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n^*\left(\zeta_n e^{2\pi i a/n}\right)}{\varphi_n^*(\zeta_n)} &= \lim_{n\to\infty, n\in\mathcal{S}} e^{2\pi i a}\overline{\left[\frac{\varphi_n\left(\zeta_n e^{2\pi i\bar{a}/n}\right)}{\varphi_n(\zeta_n)}\right]} \\ &= e^{2\pi i a}\overline{\left[e^{2\pi i\bar{a}} + C(e^{2\pi i\bar{a}} - 1\right]} \\ &= 1 + \bar{C}\left(1 - e^{2\pi i a}\right). \end{aligned}$$

Then assuming $a \neq b$,

$$\begin{aligned} g_n(a,b) \quad : &= \frac{1}{n}K_n\left(\zeta_n e^{2\pi i a/n}, \zeta_n e^{2\pi i\bar{b}/n}\right) / \left|\varphi_n(\zeta_n)\right|^2 \\ &= \frac{1}{n\left(1 - e^{2\pi i(a-b)/n}\right)}\left\{\begin{array}{c} \overline{\left[\frac{\varphi_n^*\left(\zeta_n e^{2\pi i\bar{b}/n}\right)}{\varphi_n^*(\zeta_n)}\right]}\left[\frac{\varphi_n^*\left(\zeta_n e^{2\pi i a/n}\right)}{\varphi_n^*(\zeta_n)}\right] \\ -\overline{\left[\frac{\varphi_n\left(\zeta_n e^{2\pi i\bar{b}/n}\right)}{\varphi_n(\zeta_n)}\right]}\left[\frac{\varphi_n\left(\zeta_n e^{2\pi i a/n}\right)}{\varphi_n(\zeta_n)}\right] \end{array}\right\} \\ &= \frac{1+o(1)}{-2\pi i(a-b)}\left\{\begin{array}{c} \overline{\left[1+\bar{C}\left(1-e^{2\pi i\bar{b}}\right)\right]}\left[1+\bar{C}\left(1-e^{2\pi i a}\right)\right] \\ -\overline{\left[e^{2\pi i\bar{b}} + C(e^{2\pi i\bar{b}} - 1)\right]}\left[e^{2\pi i a} + C(e^{2\pi i a} - 1)\right] \end{array}\right\}. \end{aligned}$$

This holds uniformly for (a,b) in compact subsets of $\mathbb{C}^2$ for which $a \neq b$. We continue this as

$$\begin{aligned} &= \frac{1+o(1)}{-2\pi i(a-b)} \left\{ \begin{array}{c} 1+|C|^2\left(1-e^{-2\pi ib}\right)\left(1-e^{2\pi ia}\right) \\ +C\left(1-e^{-2\pi ib}\right)+\bar{C}\left(1-e^{2\pi ia}\right) \\ -|C|^2\left(e^{-2\pi ib}-1\right)\left(e^{2\pi ia}-1\right)-e^{2\pi i(a-b)} \\ -C\left(e^{2\pi ia}-1\right)e^{-2\pi ib}-\bar{C}\left(e^{-2\pi ib}-1\right)e^{2\pi ia} \end{array} \right\} \\ &= \frac{1+o(1)}{-2\pi i(a-b)} e^{i\pi(a-b)} \left[e^{-i\pi(a-b)}-e^{i\pi(a-b)}\right]\left(1+C+\bar{C}\right) \\ &= (1+o(1))(1+2\operatorname{Re} C)\, e^{\pi i(a-b)}\mathbb{S}(a-b). \end{aligned} \tag{3.1}$$

Thus we have (1.14). Next, we remove the restriction that $a \neq b$. Let $r > 0$. We have (3.1) uniformly for $|a| \leq r$ and $|b| = r+1$. But then the maximum modulus principle and convergence continuation shows that we have (3.1) for all $|a|, |b| \leq r$. Finally, if we have the usual universality limit (1.15), then (1.16) follows from (1.14). ■

4. Proof of Theorems 1.1 and 1.2

As we have noted, it is not trivial to verify the conditions (1.10) in the case of the unit circle. Recall that the zeros of φ_n are denoted by $\{z_{jn}\}_{j=1}^{n}$. In the sequel, we use the notation

$$R_n(z) = \sum_{j=1}^{n} \frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}. \tag{4.1}$$

Lemma 4.1
Let $|z| = 1$. Then
(a)

$$\operatorname{Re}\left[\frac{z}{n}\frac{\varphi_n'(z)}{\varphi_n(z)} - 1\right] = \frac{1}{2}\left[\frac{1}{n}R_n(z) - 1\right]. \tag{4.2}$$

(b)

$$z\frac{\varphi_n^{*\prime}(z)}{\varphi_n^*(z)} - z\frac{\varphi_n'(z)}{\varphi_n(z)} = -R_n(z). \tag{4.3}$$

Proof
(a) Now

$$\operatorname{Re}\left[z\frac{\varphi_n'(z)}{\varphi_n(z)}\right] = \sum_{j=1}^{n} \frac{1-\operatorname{Re}\left(z\overline{z_{jn}}\right)}{|z-z_{jn}|^2}.$$

Substituting the identity

$$-\operatorname{Re}\left(z\overline{z_{jn}}\right) = \frac{1}{2}\left(|z-z_{jn}|^2 - 1 - |z_{jn}|^2\right)$$

in the sum and rearranging yields (4.2).
(b)

$$\begin{aligned} & \frac{\varphi_n^{*\prime}(z)}{\varphi_n^*(z)} - \frac{\varphi_n'(z)}{\varphi_n(z)} \\ = & \sum_{j=1}^{n} \frac{-\overline{z_{jn}}}{1-\overline{z_{jn}}z} - \sum_{j=1}^{n} \frac{1}{z-z_{jn}} \\ = & -\frac{1}{z}\sum_{j=1}^{n} \frac{1-|z_{jn}|^2}{|z-z_{jn}|^2}. \end{aligned}$$

■

Next we turn to quantitative estimates.

Lemma 4.2
Let μ be a positive measure on the unit circle that is regular. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. Let J_1 be a subarc of the (relative) interior of J.
(a) As $n \to \infty$,

$$n \inf \left\{ 1 - |z_{jn}| : z_{jn} \neq 0 \text{ and } \frac{z_{jn}}{|z_{jn}|} \in J_1 \right\} \to \infty. \tag{4.4}$$

(b) Uniformly for a in compact subsets of $\mathbb{C}$, for $z \in J_1$, and $\zeta = \zeta(a,n) = ze^{2\pi i a/n}$,

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) - \overline{\varphi_n(z)}\varphi_n(\zeta) \right] \mu'(z) = -2ie^{i\pi a}\sin \pi a, \tag{4.5}$$

and

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(\zeta) - \overline{\varphi_n(z)}\varphi_n'(\zeta) \right] \frac{\zeta}{n}\mu'(z) = -ie^{i\pi a}\sin \pi a - e^{\iota\pi a}\cos \pi a. \tag{4.6}$$

(c) Uniformly for $z \in J_1$,

$$\lim_{n\to\infty} \operatorname{Re}\left[\varphi_n(z) \overline{\varphi_n\left(ze^{i\pi/n}\right)} \right] \mu'(z) = -1. \tag{4.7}$$

(d) Uniformly for $z \in J_1$,

$$\lim_{n\to\infty} \frac{1}{n} R_n(z) |\varphi_n(z)|^2 \mu'(z) = 1. \tag{4.8}$$

(e) Uniformly for $z \in J_1$,

$$\operatorname{Re}\left(\frac{z\varphi_n'(z)}{n\varphi_n(z)} - 1 \right) = \frac{1}{2}\left(\frac{1}{|\varphi_n(z)|^2 \mu'(z)(1+o(1))} - 1 \right). \tag{4.9}$$

Proof
(a) Suppose for infinitely many j, with $\frac{z_{jn}}{|z_{jn}|} \in J_1$, we have

$$1 - |z_{jn}| \leq C/n.$$

Write

$$z_{jn} = \zeta_n \left(1 + \frac{i2\pi a_n}{n} \right),$$

where $\arg(z_{jn}) = \arg(\zeta_n)$ and $|\zeta_n| = 1$. We can assume that in a subsequence $a_n \to a$. Let

$$t_{jn} = 1/\overline{z_{jn}}.$$

Then

$$\begin{aligned} t_{jn} &= \zeta_n \left(1 + \frac{\overline{i2\pi a_n}}{n}\right)^{-1} \\ &= \zeta_n \left(1 + \frac{i2\pi \overline{a_n}}{n} + O\left(\frac{1}{n^2}\right)\right). \end{aligned}$$

Now

$$K_n(z_{jn}, t_{jn}) = \frac{\overline{\varphi_n^*(t_{jn})}\varphi_n^*(z_{jn}) - \overline{\varphi_n(t_{jn})}\varphi_n(z_{jn})}{1 - \bar{t}_{jn} z_{jn}} = 0$$

but from the universality (1.1), and the uniform convergence,

$$\lim_{n\to\infty} \frac{K_n(z_{jn}, t_{jn})}{K_n(\zeta_n, \zeta_n)} = \lim_{n\to\infty} \frac{K_n\left(\zeta_n\left(1 + \frac{i2\pi a_n}{n}\right), \zeta_n\left(1 + \frac{i2\pi \overline{a_n}}{n}\right)\right)}{K_n(\zeta_n, \zeta_n)} = 1,$$

a contradiction.
(b) First we note the classical limit for Christoffel functions [13, p. 123, Thm. 2.16.1]: uniformly for z in J_1,

$$\lim_{n\to\infty} \frac{1}{n} K_n(z, z) = \mu'(z)^{-1}. \tag{4.10}$$

Next we recast a special case of the universality limit (1.1) in the form

$$\lim_{n\to\infty} \frac{K_n\left(ze^{2\pi i a/n}, z\right)}{K_n(z, z)} = e^{i\pi a}\mathbb{S}(a), \tag{4.11}$$

uniformly for $z \in J_1$ and a in compact subsets of $\mathbb{C}$. Then by the Christoffel-Darboux formula,

$$\lim_{n\to\infty} \frac{\overline{\varphi_n^*(z)}\varphi_n^*\left(ze^{2\pi i a/n}\right) - \overline{\varphi_n(z)}\varphi_n\left(ze^{2\pi i a/n}\right)}{\left[1 - \bar{z}\left(ze^{2\pi i a/n}\right)\right] K_n(z, z)} = e^{i\pi a}\mathbb{S}(a).$$

Here by (4.10),

$$\lim_{n\to\infty} \left[1 - \bar{z}\left(ze^{2\pi i a/n}\right)\right] K_n(z, z) = -2\pi i a \mu'(z)^{-1}.$$

Then (4.5) follows. Because of the uniformity, we can differentiate (4.5) with respect to a :

$$\begin{aligned} &\lim_{n\to\infty} \left[\overline{\varphi_n^*(z)}\varphi_n^{*\prime}\left(ze^{2\pi i a/n}\right) - \overline{\varphi_n(z)}\varphi_n'\left(ze^{2\pi i a/n}\right)\right] \frac{2\pi i z e^{2\pi i a/n}}{n} \mu'(z) \\ = \;& 2\pi e^{i\pi a} \sin \pi a - 2i\pi e^{\iota\pi a} \cos \pi a. \end{aligned}$$

Dividing by $2\pi i$ yields (4.6).
(c) Taking $a = \frac{1}{2}$ in (4.5) gives $\zeta = ze^{i\pi/n}$ and

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(z)}\varphi_n^*(\zeta) - \overline{\varphi_n(z)}\varphi_n(\zeta)\right] \mu'(z) = 2,$$

so that

$$\lim_{n\to\infty} \left[(\bar{z}\zeta)^n \varphi_n(z)\overline{\varphi_n(\zeta)} - \overline{\varphi_n(z)}\varphi_n(\zeta)\right] \mu'(z) = 2.$$

Since $(\bar{z}\zeta)^n = -1$, we obtain (4.7).
(d) Taking $a = 0$ in (4.6) gives,

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(z)}\varphi_n^{*\prime}(z) - \overline{\varphi_n(z)}\varphi_n'(z)\right] \frac{z}{n}\mu'(z) = -1$$

so as $|\varphi_n^*(z)| = |\varphi_n(z)|$,

$$\lim_{n\to\infty} \left[\frac{z\varphi_n^{*\prime}(z)}{\varphi_n^*(z)} - \frac{z\varphi_n'(z)}{\varphi_n(z)}\right] \frac{|\varphi_n(z)|^2}{n}\mu'(z) = -1.$$

Now apply (4.3).
(e) This follows directly from (d) and Lemma 4.1(a). ■

Lemma 4.3
Let μ be a positive measure on the unit circle that is regular. Assume that J is a closed subarc of the unit circle such that μ is absolutely continuous and μ' is positive and continuous in J. Let J_1 be a subarc of the (relative) interior of J.
(a) Uniformly for $z \in J_1$,

$$\frac{z\varphi_n'(z)}{n\varphi_n(z)} - 1 = -\frac{1}{2}\left\{1 + (1 + o(1))\frac{\varphi_n\left(ze^{\pm i\pi/n}\right)}{\varphi_n(z)} + o(1)\right\}. \tag{4.12}$$

(b) For $z \in J_1$, if $|\varphi_n(z)| \geq \left|\varphi_n\left(ze^{i\pi/n}\right)\right|$, *then*

$$\left|\frac{\varphi_n'(z)}{n\varphi_n(z)}\right| \leq 1 + o(1) \tag{4.13}$$

and

$$|\varphi_n(z)|^2 \mu'(z) \geq 1 + o(1). \tag{4.14}$$

(c) In the contrary case where $|\varphi_n(z)| < \left|\varphi_n\left(ze^{i\pi/n}\right)\right|$, both (4.13) and (4.14) hold with z replaced by $ze^{i\pi/n}$.
(d) For $z \in J_1$ and at least one of $\zeta_n = z, \zeta_n = ze^{i\pi/n}$, for which $|\varphi_n(\zeta_n)|^2 \mu'(\zeta_n) \geq 1 + o(1)$,

$$\frac{1}{n^2}\sum_{j=1}^{n} \frac{1}{|\zeta_n - z_{jn}|^2} = o(1). \tag{4.15}$$

Proof
(a) Let $\sigma = \pm 1$. Because of the uniformity in z in (4.6), we can apply it with z replaced by $ze^{\sigma i\pi/n}$ and $a = -\frac{\sigma}{2}$, so that $\zeta = \left(ze^{\sigma i\pi/n}\right)e^{2\pi i a/n} = z$, so

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*\left(ze^{\sigma i\pi/n}\right)}\varphi_n^{*\prime}(z) - \overline{\varphi_n\left(ze^{\sigma i\pi/n}\right)}\varphi_n'(z)\right] \frac{z}{n}\mu'\left(ze^{\sigma i\pi/n}\right) = 1.$$

Let us set $\xi = \xi(n, z) = ze^{\sigma i\pi/n}$, so that the last limit becomes

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(\xi)}\varphi_n^{*\prime}(z) - \overline{\varphi_n(\xi)}\varphi_n'(z)\right] \frac{z}{n}\mu'(\xi) = 1. \tag{4.16}$$

Next from (4.3),

$$\varphi_n^{*\prime}(z) = \varphi_n^*(z)\left[\frac{\varphi_n'(z)}{\varphi_n(z)} - \frac{R_n(z)}{z}\right],$$

so substituting in (4.16), we obtain

$$\lim_{n\to\infty} \left[\overline{\varphi_n^*(\xi)}\varphi_n^*(z)\left[\frac{\varphi_n'(z)}{\varphi_n(z)} - \frac{R_n(z)}{z}\right] - \overline{\varphi_n(\xi)}\varphi_n'(z)\right] \frac{z}{n}\mu'(\xi) = 1$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\varphi_n'(z)}{\varphi_n(z)}\left\{\overline{\varphi_n^*(\xi)}\varphi_n^*(z)-\overline{\varphi_n(\xi)}\varphi_n(z)\right\}-\frac{R_n(z)}{z}\overline{\varphi_n^*(\xi)}\varphi_n^*(z)\right]\frac{z}{n}\mu'(\xi)=1.$$

(4.17)

Here from the Christoffel-Darboux formula, and then the universality (1.1),

$$\begin{aligned}&\lim_{n\to\infty}\left\{\overline{\varphi_n^*(\xi)}\varphi_n^*(z)-\overline{\varphi_n(\xi)}\varphi_n(z)\right\}\mu'(z)\\ =\;&\lim_{n\to\infty}\left(1-\bar{\xi}z\right)K_n(z,\xi)\,\mu'(z)\\ =\;&\lim_{n\to\infty}\frac{\sigma i\pi}{n}K_n\left(z,ze^{\sigma i\pi/n}\right)\mu'(z)=2,\end{aligned}$$

recalling $\sigma=\pm1$. Moreover,

$$\overline{\varphi_n^*(\xi)}\varphi_n^*(z)=\left(\bar{\xi}z\right)^n\varphi_n(\xi)\,\overline{\varphi_n(z)}=-\varphi_n(\xi)\,\overline{\varphi_n(z)}=-\left|\varphi_n(z)\right|^2\frac{\varphi_n(\xi)}{\varphi_n(z)},$$

so from (4.8),

$$-\frac{1}{n}R_n(z)\,\overline{\varphi_n^*(\xi)}\varphi_n^*(z)\,\mu'(z)=(1+o(1))\,\frac{\varphi_n(\xi)}{\varphi_n(z)}.$$

Substituting all this in (4.17) yields

$$\frac{z\varphi_n'(z)}{n\varphi_n(z)}\left(2+o(1)\right)+\left(1+o(1)\right)\frac{\varphi_n(\xi)}{\varphi_n(z)}=1+o(1)\,.$$

We obtain (4.12) after dividing by $2\left(1+o(1)\right)$ and subtracting 1.
(b) Our hypothesis $\left|\varphi_n(z)\right|\geq\left|\varphi_n\left(ze^{i\pi/n}\right)\right|$ and (4.12) give (4.13). Next Lemma 4.2(c) gives

$$\left|\varphi_n(z)\right|^2\mu'(z)\geq\left|\varphi_n(z)\,\varphi_n\left(ze^{i\pi/n}\right)\right|\mu'(z)\geq1+o(1)\,.$$

(c) Replacing z by $ze^{i\pi/n}$ in (4.12), and choosing the $-$ sign so that $ze^{-i\pi/n}$ becomes $\left(ze^{i\pi/n}\right)e^{-i\pi/n}=z$,

$$\frac{ze^{i\pi/n}\varphi_n'\left(ze^{i\pi/n}\right)}{n\varphi_n\left(ze^{i\pi/n}\right)}-1=-\frac{1}{2}\left\{1+\frac{\varphi_n(z)}{\varphi_n\left(ze^{i\pi/n}\right)}\left(1+o(1)\right)+o(1)\right\}.$$

Since $\left|\varphi_n(z)\right|<\left|\varphi_n\left(ze^{i\pi/n}\right)\right|$, we then obtain

$$\left|\frac{ze^{i\pi/n}\varphi_n'\left(ze^{i\pi/n}\right)}{n\varphi_n\left(ze^{i\pi/n}\right)}\right|\leq1+o(1)$$

and then (4.13) follows with z replaced by $ze^{i\pi/n}$, while (4.14) follows from our hypothesis, as in (b).
(d) Since $\left|\varphi_n(\zeta_n)\right|^2\mu'(\zeta_n)\geq1+o(1)$, we have from Lemma 4.2(d) ,

$$\limsup_{n\to\infty}\frac{1}{n}R_n(\zeta_n)\leq1. \tag{4.18}$$

Choose an arc J_2 contained in the interior of J but such that the interior of J_2 contains J_1. For $\zeta_n\in J_1$, we have from Lemma 4.2(a) (applied to J_2),

$$\sum_{z_{jn}\in J_2}\frac{1}{\left|\zeta_n-z_{jn}\right|^2}=o(n)\sum_{z_{jn}\in J_2}\frac{1-\left|z_{jn}\right|^2}{\left|\zeta_n-z_{jn}\right|^2}=o\left(n^2\right),$$

by (4.18). Next, for $\zeta_n \in J_1$, and $z_{jn} \notin J_2$, we have $\left|\zeta_n - z_{jn}\right|^2 \geq C$ so

$$\sum_{z_{jn} \notin J_2} \frac{1}{\left|\zeta_n - z_{jn}\right|^2} \leq C^{-1} n.$$

Combining the last two estimates gives (4.15). ■

Proof of Theorem 1.1

It follows from the previous lemma that the conditions (1.10) of Theorem 1.3 are satisfied for $n \geq 1$ and for at least one of $\zeta_n = z_n$ or $\zeta_n = z_n e^{i\pi/n}$. By Theorem 1.3, from any subsequence of integers, we can extract another subsequence $\mathcal{S}$ for which (1.11) holds. Moreover, from Lemma 4.3(a),

$$|C| = \lim_{n\to\infty, n\in\mathcal{S}} \left| \frac{\zeta_n \varphi_n'(\zeta_n)}{n\varphi_n(\zeta_n)} - 1 \right| \leq 1,$$

recall that above we had $\left|\varphi_n\left(ze^{\pm i\pi/n}\right)/\varphi_n(z)\right| \leq 1$ in the right-hand side of (4.12) with appropriate z. ■

Proof of Theorem 1.2

(I)⇒(II)

If we have (1.4), then necessarily $C = 0$ in Theorem 1.3, so that

$$C = \lim_{n\to\infty, n\in\mathcal{S}} \left(\frac{\zeta_n \varphi_n'(\zeta_n)}{n\varphi_n(\zeta_n)} - 1 \right) = 0. \tag{4.19}$$

Lemma 4.3(a) gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{\varphi_n\left(\zeta_n e^{\pm i\pi/n}\right)}{\varphi_n(\zeta_n)} = -1.$$

(II)⇒(III)

Lemma 4.3(a) gives (4.19) and then Lemma 4.2(e) gives

$$\lim_{n\to\infty, n\in\mathcal{S}} \left|\varphi_n(\zeta_n)\right|^2 \mu'(\zeta_n) = 1, \tag{4.20}$$

while we have also assumed (1.7).

(III)⇒(IV)

Lemma 4.2(d) and (1.6) give

$$\lim_{n\to\infty, n\in\mathcal{S}} \frac{1}{n} R_n(z) = 1$$

while we also assumed (1.7).

(IV)⇒(I)

From Lemma 4.2(d), we have (4.20), so from Lemma 4.2(e),

$$\lim_{n\to\infty, n\in\mathcal{S}} \operatorname{Re}\left(\frac{\zeta_n \varphi_n'(\zeta_n)}{n\varphi_n(\zeta_n)} - 1 \right) = 0.$$

We also have (1.7), so that (4.19) follows. Next, Lemma 4.3(d) and (4.19) show that the conditions (1.10) of Theorem 1.3 are fulfilled. By Theorem 1.3, from every subsequence of $\mathcal{S}$, we can extract a further subsequence $\mathcal{S}_1$, for which

$$\lim_{n\to\infty, n\in\mathcal{S}_1} \frac{\varphi_n\left(\zeta_n\left(1+\frac{u}{n}\right)\right)}{\varphi_n(\zeta_n)} = e^u, \tag{4.21}$$

recall that C given by (4.19) is 0. As the limit is independent of the subsequence $\mathcal{S}_1$ of $\mathcal{S}$, we obtain (1.4). ■

We end by showing that we can find sequences $\{\zeta_n\}$ close to other given sequences with $\lim_{n\to\infty} |\varphi_n(\zeta_n)|^2 \mu'(\zeta_n) = 1$. Of course this on its own is not enough to give the local limit, as we still need something such as (1.7). Recall [13] that given $|\beta| = 1$, we can define the paraorthogonal polynomial

$$\varphi_{n+1}(t;\beta) = t\varphi_n(t) - \bar{\beta}\varphi_n^*(t)$$

and the related finite Blaschke product

$$B_n(z) = \frac{z\varphi_n(z)}{\varphi_n^*(z)}.$$

There are $n+1$ simple distinct zeros of $\varphi_{n+1}(\cdot;\beta)$ on the unit circle. Moreover, they interlace for different β [13, p. 113 ff.].

Lemma 4.4
Assume the hypotheses of Theorem 1.1 on μ, J, J_1. For $n \geq 1$, let $w_n \in J_1$. There exists a sequence $\{\varepsilon_n\}$ with limit 0 and for large enough n, $\zeta_n = w_n e^{2\pi i d_n/n}$, where $d_n \in [0, 1+\varepsilon_n]$, $n \geq 1$, with

$$\lim_{n\to\infty} |\varphi_n(\zeta_n)|^2 \mu'(\zeta_n) = 1. \tag{4.22}$$

Proof
For $n \geq 1$, let

$$\beta_n = \overline{B_n(w_n)} = \frac{\overline{w_n\varphi_n(w_n)}}{\overline{\varphi_n^*(w_n)}}.$$

Then

$$\begin{aligned}
\varphi_{n+1}(t;\beta_n) &= t\varphi_n(t) - \bar{\beta}_n\varphi_n^*(t) \\
&= \frac{\bar{\beta}_n}{\overline{\varphi_n^*(w_n)}}\left[t\overline{w_n\varphi_n(w_n)}\varphi_n(t) - \varphi_n^*(t)\overline{\varphi_n^*(w_n)}\right] \\
&= -\frac{\bar{\beta}_n}{\overline{\varphi_n^*(w_n)}}(1 - t\bar{w}_n)K_{n+1}(t, w_n),
\end{aligned}$$

by an alternative form of the Christoffel-Darboux formula [12, p. 954]. Note that this also essentially appears as (2.4) in [18] and that our K_{n+1} is Simon's K_n. Then

$$\varphi_{n+1}\left(w_n e^{2\pi i a/n};\beta_n\right) = -\frac{\bar{\beta}_n}{\overline{\varphi_n^*(w_n)}}\left(1 - e^{2\pi i a/n}\right)K_{n+1}\left(w_n e^{2\pi i a/n}, w_n\right)$$

so

$$\begin{aligned}
&\left|\varphi_{n+1}\left(w_n e^{2\pi i a/n};\beta_n\right)\right| |\varphi_n(w_n)| \\
&= \frac{2\pi|a|\left(1 + O\left(\frac{1}{n}\right)\right)}{n}K_{n+1}(w_n, w_n)\left|\frac{K_{n+1}\left(w_n e^{2\pi i a/n}, w_n\right)}{K_{n+1}(w_n, w_n)}\right| \\
&= 2\mu'(w_n)^{-1}|\sin\pi a|(1 + o(1)),
\end{aligned}$$

(4.23)

uniformly for a in compact subsets of the real line, by (1.1). Next $\varphi_{n+1}(w_n;\beta_n) = 0$ and from the last formula, there exists $a_n = 1+o(1)$ such that $\varphi_{n+1}\left(w_n e^{2\pi i a_n/n};\beta_n\right) = 0$, while $\varphi_{n+1}\left(w_n e^{2\pi i a/n};\beta_n\right) \neq 0$ for all other $a \in (0, a_n)$. Equivalently,

$$B_n(w_n) = \bar{\beta}_n \text{ and } B_n\left(w_n e^{2\pi i a_n/n}\right) = \bar{\beta}_n$$

and there are no other zeros of $B_n - \bar{\beta}_n$ in the closed minor arc I_n of the unit circle joining w_n and $w_n e^{2\pi i a_n/n}$. We claim that as t traverses I_n, B_n traverses the unit circle exactly once. If not, as B_n is not constant, it would have to assume some value $\bar{\Delta} \neq \bar{\beta}_n$ twice. Then the paraorthogonal polynomial $\varphi_{n+1}(\cdot;\Delta)$ would have two zeros in the arc I_n, contradicting that its zeros interlace those of of $\varphi_{n+1}(\cdot;\beta_n)$ [13, p. 116, Thm. 2.14.4]. It follows that we can find $\zeta_n \in I_n$ such that

$$B_n(\zeta_n) = -\bar{\beta}_n,$$

or equivalently

$$\varphi_{n+1}(\zeta_n;\beta_n) = -2\bar{\beta}_n \varphi_n^*(\zeta_n).$$

Write $\zeta_n = w_n e^{2\pi i d_n/n}$, where for large enough n, $d_n \in (0, a_n)$. By (4.23),

$$\begin{aligned} & 2\left|\varphi_n(\zeta_n)\right|\left|\varphi_n(w_n)\right| \\ = & \left|\varphi_{n+1}\left(w_n e^{2\pi i d_n/n};\beta_n\right)\right|\left|\varphi_n(w_n)\right| \\ = & 2\mu'(w_n)^{-1}\left|\sin \pi d_n\right|(1+o(1)) \leq 2\mu'(w_n)^{-1}(1+o(1)). \end{aligned}$$

Hence, using also that μ' is continuous,

$$\min_{t\in I_n}\left|\varphi_n(t)\right|^2 \mu'(t) \leq 1+o(1). \tag{4.24}$$

In the other direction, from the definition of $\varphi_{n+1}(\cdot;\beta_n)$, and from (4.7),

$$\left|\varphi_n\left(w_n e^{\pi i/n}\right)\right|\left|\varphi_n(w_n)\right|\mu'(w_n) \geq \left|\mathrm{Re}\left[\varphi_n(w_n)\overline{\varphi_n\left(w_n e^{\pi i/n}\right)}\right]\right|\mu'(w_n) = 1+o(1)$$

so that

$$\max_{t\in I_n}\left|\varphi_n(t)\right|^2 \mu'(t) \geq 1+o(1).$$

Combining this and (4.24) and the continuity of $\left|\varphi_n(\cdot)\right|^2 \mu'(\cdot)$, we see that there must exist for large enough n, $\zeta_n \in I_n$ with the property (4.22). ■

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu