LOCAL ASYMPTOTICS FOR ORTHONORMAL POLYNOMIALS IN THE INTERIOR OF THE SUPPORT VIA UNIVERSALITY

D. S. LUBINSKY

Abstract. We establish local pointwise asymptotics for orthonormal polynomials inside the support of the measure using universality limits. For example, if a measure μ has compact support, is regular in the sense of Stahl, Totik and Ullmann, and in some subinterval I, μ is absolutely continuous and μ' is positive and continuous, we prove that for y_{jn} in a compact subset of I^o with $p_n'(y_{jn}) = 0$, we have

$$\lim_{n\to\infty} \frac{p_n\left(y_{jn} + \frac{z}{n\omega(y_{jn})}\right)}{p_n(y_{jn})} = \cos \pi z$$

uniformly in y_{jn} and for z in compact subsets of the plane. Here ω is the equilibrium density for the support of μ.

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1. Results

Let μ be a finite positive Borel measure with compact support. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

One of the key limits in random matrix theory, the so-called universality limit [1], [3], [4], [5], [8], [13], [14] involves the reproducing kernel

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y)$$

For x in the interior of supp$[\mu]$ (the "bulk" of the support), at least when $\mu'(x)$ is finite and positive, the universality limit typically takes the form

$$\lim_{n\to\infty} \frac{K_n\left(x + \frac{a}{\mu'(x)K_n(x,x)}, x + \frac{b}{\mu'(x)K_n(x,x)}\right)}{K_n(x, x)} = \mathbb{S}(a - b),$$

uniformly for a, b in compact subsets of $\mathbb{C}$, where $\mathbb{S}$ is the sinc kernel,

$$\mathbb{S}(a) = \frac{\sin \pi a}{\pi a}.$$

One feature of the universality limit is that it holds very generally, far more generally than pointwise asymptotics for orthonormal polynomials, that at one

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stage were used to prove it. It is the purpose of this paper to show that we can at least partially go in the other direction. We establish local asymptotics for orthonormal polynomials inside the support of the measure. The main ideas we use were introduced and first applied in the context of universality at the endpoints of the interval of orthogonality in [6].

We shall need more notation. Let supp$[\mu]$ denote the compact support of the measure μ. We say that μ is *regular* (in the sense of Stahl, Totik, and Ullmann) if for every sequence of polynomials $\{P_n\}$ with degree P_n at most n,

$$\limsup_{n\to\infty}\left(\frac{|P_n(x)|}{\left(\int |P_n|^2\,d\mu\right)^{1/2}}\right)^{1/n}\leq 1$$

for quasi-every $x\in$supp$[\mu]$ (that is except in a set of logarithmic capacity 0). If the support consists of finitely many intervals, and $\mu'>0$ a.e. in each subinterval, then μ is regular, though much less is required [9]. An equivalent formulation involves the leading coefficients $\{\gamma_n\}$ of the orthonormal polynomials for μ :

$$\lim_{n\to\infty}\gamma_n^{1/n}=\frac{1}{cap\,(\text{supp}\,[\mu])},$$

where cap denotes logarithmic capacity.

Recall that the equilibrium measure for the compact set supp$[\mu]$ is the probability measure that minimizes the energy integral

$$\int\int \log\frac{1}{|x-y|}\,d\nu\,(x)\,d\nu\,(y)$$

amongst all probability measures ν supported on supp$[\mu]$. If I is an interval contained in supp$[\mu]$, then the equilibrium measure is absolutely continuous in I, and moreover its density, which we denote throughout by ω, is continuous in the interior I^o of I [7, p.216, Thm. IV.2.5].

The zeros x_{jn} of p_n are listed in decreasing order:

$$x_{1n}>x_{2n}>...>x_{n-1,n}>x_{nn}.$$

They interlace the zeros y_{jn} of p_n' :

$$p_n'\,(y_{jn})=0 \text{ and } y_{jn}\in(x_{j+1,n},x_{jn})\,,\ 1\leq j\leq n-1.$$

We prove:

Theorem 1.1
Assume that μ is a regular measure with compact support. Let I be a closed subinterval of the support in which μ is absolutely continuous, and μ' is positive and continuous. Let J be a compact subset of the interior I^o of I. Then

$$\lim_{n\to\infty}\frac{p_n\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)}{p_n\,(y_{jn})}=\cos\pi z$$

uniformly for $y_{jn}\in J$ and z in compact subsets of $\mathbb{C}$. Here ω is the equilibrium density for the support of μ.

Corollary 1.2

$$\lim_{n\to\infty}\frac{\left(p_n\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)\right)^2+\left(\frac{1}{n\pi\omega(y_{jn})}\right)^2\left(p_n'\left(y_{jn}+\frac{z}{n\omega(y_{jn})}\right)\right)^2}{p_n\left(y_{jn}\right)^2}=1,$$

uniformly for z in compact subsets of the plane.

We note that Bernstein-Szegő inequalities involving expressions of the form $P\left(x\right)^2+\left(\frac{1}{n\pi\omega(x)}\right)^2P'\left(x\right)^2$, where P is a polynomial of degree at most n, have been established by Totik [11, Thm. 3.2].

We shall deduce Theorem 1.1 from a general proposition for a sequence of measures $\{\mu_n\}$. The corresponding orthonormal polynomials and reproducing kernels are denoted respectively by $p_n\left(\mu_n,x\right)$ and $K_n\left(\mu_n,x,x\right)$. The zeros of $p_n\left(\mu_n,x\right)$ are denoted by

$$x_{nn,n}<x_{n-1n,n}<...<x_{2n,n}<x_{1n,n}.$$

Theorem 1.3
Assume that for $n\geq 1$ we have a measure μ_n supported on the real line with infinitely many points in its support. Let $\{\xi_n\}$ be a bounded sequence of real numbers, and $\{\tau_n\}$ be a sequence of positive numbers that is bounded above and below by positive constants, such that uniformly for a,b in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty}\frac{K_n\left(\mu_n,\xi_n+\frac{a\tau_n}{n},\xi_n+\frac{b\tau_n}{n}\right)}{K_n\left(\mu_n,\xi_n,\xi_n\right)}=\mathbb{S}\left(a-b\right). \tag{1.1}$$

Let us be given some infinite sequence of integers $\mathcal{T}$. The following are equivalent:
(I)

$$\sup_{n\in\mathcal{T}}\frac{1}{n}\left|\sum_{j=1}^{n}\frac{1}{\xi_n-x_{jn,n}}\right|<\infty \text{ and } \sup_{n\in\mathcal{T}}\frac{1}{n^2}\sum_{j=1}^{n}\frac{1}{\left(\xi_n-x_{jn,n}\right)^2}<\infty. \tag{1.2}$$

(II) For each $R>0$, there exists C_R such that

$$\sup_{n\in\mathcal{T}}\sup_{|z|\leq R}\left|\frac{p_n\left(\mu_n,\xi_n+\frac{\tau_n z}{n}\right)}{p_n\left(\mu_n,\xi_n\right)}\right|\leq C_R. \tag{1.3}$$

(III) From every subsequence of $\mathcal{T}$, there is a further subsequence $\mathcal{S}$ such that

$$\lim_{n\to\infty,n\in\mathcal{S}}\frac{p_n\left(\mu_n,\xi_n+\frac{z\tau_n}{n}\right)}{p_n\left(\mu_n,\xi_n\right)}=\cos\left(\pi z\right)+\frac{\alpha}{\pi}\sin\pi z, \tag{1.4}$$

uniformly for z in compact subsets of $\mathbb{C}$, where

$$\alpha=\lim_{n\to\infty,n\in\mathcal{S}}\frac{\tau_n}{n}\frac{p_n'\left(\mu_n,\xi_n\right)}{p_n\left(\mu_n,\xi_n\right)} \tag{1.5}$$

and α is bounded independently of $\mathcal{S}$.

This paper is organised as follows. In the next section, we prove Theorem 1.3. In Section 3, we prove Theorem 1.1 and Corollary 1.2. In the sequel $C,C_1,C_2,...$ denote constants independent of n,x,θ. The same symbol does not necessarily denote the same constant in different occurences.

2. Proof of Theorem 1.3

In this section only, we abbreviate $p_n(\mu_n, z)$ as $p_n(z)$ and $K_n(\mu_n, z, w)$ as $K_n(z, w)$.

Lemma 2.1
Assume (1.1) and that through the subsequence $\mathcal{S}$, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{\mathcal{S}} \frac{p_n\left(\xi_n + \frac{z\tau_n}{n}\right)}{p_n(\xi_n)} = f(z). \tag{2.1}$$

(a) Then for $u, z, w \in \mathbb{C}$,

$$f(u)\sin\pi(w-z) = f(w)\sin\pi(u-z) + f(z)\sin\pi(w-u). \tag{2.2}$$

(b)

$$f(z) = \cos\pi z + \frac{1}{\pi} f'(0)\sin\pi z. \tag{2.3}$$

Here

$$f'(0) = \lim_{\mathcal{S}} \frac{\tau_n}{n}\frac{p_n'(\xi_n)}{p_n(\xi_n)}. \tag{2.4}$$

Proof
(a) From

$$\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(w) = \left[\frac{p_{n-1}}{p_n}(z) - \frac{p_{n-1}}{p_n}(u)\right] + \left[\frac{p_{n-1}}{p_n}(u) - \frac{p_{n-1}}{p_n}(w)\right]$$

and the Christoffel-Darboux formula, we deduce that

$$\frac{K_n(z,w)}{p_n(z)\,p_n(w)}(w-z) = \frac{K_n(u,z)}{p_n(z)\,p_n(u)}(u-z) + \frac{K_n(w,u)}{p_n(u)\,p_n(w)}(w-u).$$

Replace z, w, u respectively by $\xi_n + \frac{z\tau_n}{n}$, $\xi_n + \frac{w\tau_n}{n}, \xi_n + \frac{u\tau_n}{n}$. Divide each denominator by $p_n(\xi_n)^2$ and each numerator by $K_n(\xi_n, \xi_n)$. Take limits through the subsequence $\mathcal{S}$. If $f(u)\,f(w) f(z) \neq 0$, we obtain from (1.1) and (2.1),

$$\frac{\mathbb{S}(z-w)}{f(z)\,f(w)}(w-z) = \frac{\mathbb{S}(u-z)}{f(z)\,f(u)}(u-z) + \frac{\mathbb{S}(w-u)}{f(u)\,f(w)}(w-u)$$

and hence

$$\frac{\sin\pi(w-z)}{f(z)\,f(w)} = \frac{\sin\pi(u-z)}{f(z)\,f(u)} + \frac{\sin\pi(w-u)}{f(u)\,f(w)}$$

yielding (2.2).
(b) The double angle formula for trigomometric functions yields the elementary identity

$$\cos\pi u \sin\pi(w-z) = \cos\pi w \sin\pi(u-z) + \cos\pi z \sin\pi(w-u).$$

Then we can recast (2.2) as
(2.5)

$$[f(u) - \cos\pi u]\sin\pi(w-z) = [f(w) - \cos\pi w]\sin\pi(u-z) + [f(z) - \cos\pi z]\sin\pi(w-u).$$

Setting $u = 0$ and using $f(0) = 1$ gives

$$0 = -[f(w) - \cos\pi w]\sin\pi z + [f(z) - \cos\pi z]\sin\pi w$$

so if $(\sin \pi z)(\sin \pi w) \neq 0$, we have

$$\frac{f(z) - \cos \pi z}{\sin \pi z} = \frac{f(w) - \cos \pi w}{\sin \pi w}.$$

So both sides are necessarily constant. Fix any such w, and call the right-hand side c. We have at first for all non-integer z, and then for all z,

$$f(z) - \cos \pi z = c \sin \pi z.$$

We see that

$$f'(0) = c\pi,$$

so

$$f(z) = \cos \pi z + \frac{1}{\pi} f'(0) \sin \pi z.$$

Finally, because of the uniform convergence, we can differentiate the asymptotic relation, and deduce (2.4). ■

Proof of Theorem 1.3
(I) ⇒ (II)

$$\begin{aligned}
\log \left| \frac{p_n\left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n(\xi_n)} \right| &= \sum_{j=1}^{n} \log \left| 1 + \frac{\tau_n z}{n(\xi_n - x_{jn,n})} \right| \\
&= \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{2\tau_n \operatorname{Re}(z)}{n(\xi_n - x_{jn,n})} + \frac{\tau_n^2 |z|^2}{(n(\xi_n - x_{jn,n}))^2} \right) \\
&\leq \frac{\tau_n \operatorname{Re}(z)}{n} \sum_{j=1}^{n} \frac{1}{\xi_n - x_{jn,n}} + \frac{\tau_n^2 |z|^2}{2n^2} \sum_{j=1}^{n} \frac{1}{(\xi_n - x_{jn,n})^2}.
\end{aligned}$$

(2.6)

Then our hypotheses give the uniform boundedness.
(II)⇒(I)
Suppose we have the uniform boundedness (1.3). Then by normality from every subsequence, we can choose another subsequence $\mathcal{S}$ such that

$$\lim_{\mathcal{S}} \frac{p_n\left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n(\xi_n)} = f(z),$$

where f is an entire function. Then also from (1.3), with $R = 1$,

$$\sup_{|z| \leq 1} |f(z)| \leq C_1.$$

Because of the uniform convergence for z in compact sets, the differentiated sequence also converges, so

$$\lim_{n \to \infty} \left| \frac{\tau_n}{n} \frac{p_n'(\xi_n)}{p_n(\xi_n)} \right| = |f'(0)|.$$

Since $\{\tau_n\}$ is bounded above and below, and by Cauchy's inequalities, $|f'(0)|$ is bounded above independently of the subsequence $\mathcal{S}$,

$$\sup_{n \in \mathcal{T}} \left| \sum_{j=1}^{n} \frac{1}{n(\xi_n - x_{jn,n})} \right| = \sup_{n \in \mathcal{T}} \frac{1}{n} \left| \frac{p_n'(\xi_n)}{p_n(\xi_n)} \right| < \infty.$$

Next, setting $z = iy$, we have for $y \in [-R, R]$,

$$C_R \geq \log \left| \frac{p_n \left(\xi_n + \frac{i\tau_n y}{n}\right)}{p_n (\xi_n)} \right| = \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{\tau_n^2 y^2}{(n (\xi_n - x_{jn}))^2} \right).$$

Let us assume that $\tau_n \geq d$ for all n. Then also for each j,

$$\begin{aligned} C_1 &\geq \log \left(1 + \frac{d^2}{(n (\xi_n - x_{jn,n}))^2} \right) \\ &\Rightarrow e^{C_1} \geq 1 + \frac{d^2}{(n (\xi_n - x_{jn,n}))^2} \\ &\Rightarrow C_1 := e^{C_1} - 1 \geq \frac{d^2}{(n (\xi_n - x_{jn,n}))^2}. \end{aligned}$$

Now there exists C_2 depending only on C_1 such that

$$\log (1 + t) \geq C_2 t \text{ for } t \in [0, C_1].$$

Then

$$\begin{aligned} C_1 &\geq \log \left| \frac{p_n \left(\xi_n + \frac{i\tau_n}{n}\right)}{p_n (\xi_n)} \right| \\ &= \frac{1}{2} \sum_{j=1}^{n} \log \left(1 + \frac{\tau_n^2}{(n (\xi_n - x_{jn,n}))^2} \right) \\ &\geq \frac{C_2}{2} d^2 \sum_{j=1}^{n} \frac{1}{(n (\xi_n - x_{jn,n}))^2}. \end{aligned}$$

Here C, C_2, d are independent of n, so we have also

$$\sup_{n \in \mathcal{T}} \sum_{j=1}^{n} \frac{1}{(n (\xi_n - x_{jn,n}))^2} < \infty.$$

(II)⇒(III)
Because of the uniform convergence, we can extract a subsequence $\mathcal{S}$ of $\mathcal{T}$ such that

$$\lim_{n \in \mathcal{S}} \frac{p_n \left(\xi_n + \frac{\tau_n z}{n}\right)}{p_n (\xi_n)} = f(z)$$

uniformly for z in compact subsets of $\mathbb{C}$. Then Lemma 2.1 shows that f has the form (1.4-5).
(III)⇒(II)
Since α is bounded independently of the subsequence, we obtain the uniform boundedness in (1.3). ■

3. Proof of Theorem 1.1

Our analysis depends heavily on results established by Vili Totik [13], [14]. We list the results we need in the following lemma:

Lemma 3.1
Assume that μ is a regular measure with compact support. Assume that in some closed subinterval I of the support, $\log \mu' \in L_1(I)$. *Assume that $x \in I$ is a Lebesgue*

point of both μ' and $\log \mu'$.
(a)

$$\lim_{n\to\infty} \frac{K_n\left(x+\frac{a}{\mu'(x)K_n(x,x)}, x+\frac{b}{\mu'(x)K_n(x,x)}\right)}{K_n(x,x)} = \mathbb{S}(a-b)$$

uniformly for a, b in compact subsets of $\mathbb{C}$.
(b) Given any $L > 0$, we have for $|x_{kn} - x| \le L/n$,

$$\lim_{n\to\infty} n\left(x_{kn} - x_{k+1,n}\right)\omega(x) = 1.$$

(c) Given any $A > 0$, we have uniformly for $a \in [-A, A]$

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(x+\frac{a}{n}, x+\frac{a}{n}\right)\mu'(x) = \omega(x).$$

When μ' is continuous and positive in I all the above results hold uniformly for $x \in J$, where J is any closed subinterval of I^0.

Proof

(a), (b), See Theorems 2.1 and 2.2 in [13] (see also Theorems 1-3 in [14] for the statement without uniformity).
(c) See Theorem 3.1 in [13]. ■

Recall that $y_{jn} \in (x_{j+1,n}, x_{jn})$ has $p_n'(y_{jn}) = 0$. Let

$$z_{jn} = \frac{1}{2}\left(x_{j+1,n} + x_{jn}\right),\ 1 \le j \le n-1.$$

Lemma 3.2

Assume the hypotheses of Theorem 1.1. Let J be a compact subinterval of the interior of I.
(a) For $n \ge 1$, $j \ge 1$,

$$\sum_{k=1}^{n} \frac{1}{y_{jn} - x_{kn}} = 0. \tag{3.1}$$

(b) There exists C such that for $x_{jn} \in J$ and $k = j, j+1$,

$$|y_{jn} - x_{kn}| \ge \frac{C}{n}. \tag{3.2}$$

(c) The bounds (1.2) hold with all $\mu_n = \mu$ and $\xi_n = y_{jn}$ for the full sequence $\mathcal{T} = \{1, 2, 3, ...\}$.

Proof

(a)

$$0 = \frac{p_n'(y_{jn})}{p_n(y_{jn})} = \sum_{k=1}^{n} \frac{1}{y_{jn} - x_{kn}}. \tag{3.3}$$

(b) Now from the Christoffel-Darboux formula,

$$K_n(x_{kn}, z_{jn}) = \frac{\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn})\, p_n(z_{jn})}{x_{kn} - z_{jn}}$$

so as z_{jn} is the midpoint of $[x_{j+1,n}, x_{jn}]$,

$$\frac{\gamma_{n-1}}{\gamma_n} p_{n-1}(x_{kn})\, p_n(z_{jn}) = \frac{1}{2}\left(x_{jn} - x_{j+1,n}\right) K_n(x_{kn}, z_{jn}).$$

From Lemma 3.1(b) and continuity of ω in I [7, p. 216, Thm. IV.2.5],

$$\lim_{n\to\infty}\left(x_{jn}-x_{j+1,n}\right)n\omega\left(x_{jn}\right)=1,$$

so for $k=j,j+1$,

$$\begin{aligned}
&\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(x_{kn}\right)p_n\left(z_{jn}\right)\mu'\left(z_{jn}\right)\\
=\;&\frac{1}{2}\frac{1+o\left(1\right)}{n\omega\left(x_{jn}\right)}K_n\left(x_{kn},x_{kn}-\frac{1}{2}\frac{1}{n\omega\left(x_{jn}\right)}\left(1+o\left(1\right)\right)\right)\mu'\left(z_{jn}\right)\\
=\;&\frac{1}{2}\mathbb{S}\left(\frac{1}{2}\right)+o\left(1\right)=\frac{1}{\pi}+o\left(1\right),
\end{aligned}$$

by Lemma 3.1(a), (c) and uniformity. Thus uniformly for $x_{jn}\in J$ and $k=j,j+1$,

$$\lim_{n\to\infty}\frac{\gamma_{n-1}}{\gamma_n}\left|p_{n-1}\left(x_{kn}\right)p_n\left(z_{jn}\right)\right|\mu'\left(z_{jn}\right)=\frac{1}{\pi}.$$

As $\left|p_n\left(y_{jn}\right)\right|=\max_{[x_{j+1,n},x_{jn}]}\left|p_n\right|\geq\left|p_n\left(z_{jn}\right)\right|$, then uniformly for $x_{jn}\in J$ and $k=j,j+1$,

$$\liminf_{n\to\infty}\frac{\gamma_{n-1}}{\gamma_n}\left|p_{n-1}\left(x_{kn}\right)p_n\left(y_{jn}\right)\right|\mu'\left(y_{jn}\right)\geq\frac{1}{\pi}. \tag{3.4}$$

Here we also have used continuity of μ'. Suppose there does not exist C satisfying (3.2). Then for infinitely many n and j, we have

$$y_{jn}-x_{kn}=\frac{\varepsilon_n}{n\omega\left(x_{jn}\right)}$$

where $\varepsilon_n\to 0$ as $n\to\infty$. Then from the Christoffel-Darboux formula,

$$\begin{aligned}
&\frac{\gamma_{n-1}}{\gamma_n}p_{n-1}\left(x_{kn}\right)p_n\left(y_{jn}\right)\mu'\left(y_{jn}\right)\\
=\;&\frac{\varepsilon_n}{n\omega\left(x_{jn}\right)}K_n\left(y_{jn},x_{kn}\right)\mu'\left(y_{jn}\right)\\
=\;&\frac{\varepsilon_n}{n\omega\left(x_{jn}\right)}K_n\left(x_{kn}\pm\varepsilon_n\frac{1}{n\omega\left(x_{jn}\right)}\left(1+o\left(1\right)\right),x_{kn}\right)\mu'\left(y_{jn}\right)\\
=\;&\varepsilon_n\mathbb{S}\left(\varepsilon_n\right)+o\left(1\right)=o\left(1\right),
\end{aligned}$$

by Lemma 3.1(a), (c), contradicting (3.4). Thus we have (3.2).
(c) Let J be a subinterval of I^o. From Lemma 3.1(c), we have uniformly for $n\geq 1,x\in J$,

$$\lambda_n\left(\mu,x\right)=\frac{1}{K_n\left(x,x\right)}\sim\frac{1}{n}.$$

By the Markov-Stieltjes inequalities [2, p. 33, (I.5.10)] for $x_{kn}\in J$, and (for example) $j\leq k-2$,

$$\int_{x_{kn}}^{x_{jn}}\mu'\geq\sum_{i=j+1}^{k-1}\lambda_n\left(\mu,x_{in}\right)\geq C\frac{k-j}{n}.$$

It follows that for $|j-k|\geq 2$, such that $x_{jn}\in I$,

$$\left|y_{jn}-x_{kn}\right|\geq C\frac{|j-k|}{n}$$

and then

$$\frac{1}{n^2}\left(\sum_{k=1}^{j-2}+\sum_{k=j+2}^{n}\right)\frac{1}{\left(y_{jn}-x_{kn}\right)^2}\leq C\sum_{k:|j-k|\geq 2}\frac{1}{\left(j-k\right)^2}\leq C_1.$$

Here we are also using that $\text{supp}[\mu]\setminus I$ is a positive distance from J. Together with (b), this shows that

$$\sup_{n\geq 1}\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{\left(y_{jn}-x_{kn}\right)^2}\leq C.$$

Together with (a), this shows that (1.2) is satisfied for the full sequence $\mathcal{T}=\{1,2,3,...\}$. ■

Proof of Theorem 1.1

We have shown (1.2) holds, so from Theorem 1.3 with $\tau_n=\frac{1}{\omega_n(y_{jn})}$ that for appropriate subsequences $\mathcal{S}$,

$$\lim_{n\to\infty,n\in\mathcal{S}}\frac{p_n\left(y_{jn}+z\frac{1}{n\omega(y_{jn})}\right)}{p_n\left(y_{jn}\right)}=\cos\pi z+\alpha\sin\pi z,$$

where

$$\alpha=\lim_{n\to\infty,n\in\mathcal{S}}\frac{1}{n\omega\left(y_{jn}\right)}\frac{p_n'\left(y_{jn}\right)}{p_n\left(y_{jn}\right)}=0.$$

Then the result follows, for full sequences, as the limit is independent of the subsequence. ■

Proof of Corollary 1.2

We can differentiate the asymptotics in Theorem 1.1 to obtain

$$\lim_{n\to\infty}\frac{p_n'\left(y_{jn}+z\frac{1}{n\omega(y_{jn})}\right)}{n\omega\left(y_{jn}\right)p_n\left(y_{jn}\right)}=-\pi\sin\pi z,$$

so that Corollary 1.2 follows. ■

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School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu