Expected number of real zeros for random orthogonal polynomials

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Abstract

We study the expected number of real zeros for random linear combinations of orthogonal polynomials. It is well known that Kac polynomials, spanned by monomials with i.i.d. Gaussian coefficients, have only $(2/\pi + o(1)) \log n$ expected real zeros in terms of the degree n. If the basis is given by the orthonormal polynomials associated with a compactly supported Borel measure on the real line, or associated with a Freud weight, then random linear combinations have $n/\sqrt{3} + o(n)$ expected real zeros. We prove that the same asymptotic relation holds for all random orthogonal polynomials on the real line associated with a large class of weights, and give local results on the expected number of real zeros. We also show that the counting measures of properly scaled zeros of these random polynomials converge weakly to either the Ullman distribution or the arcsine distribution.

Key words: Polynomials, random coefficients, expected number of real zeros, random orthogonal polynomials.

1. Background

Problems on the number of real zeros for polynomials with random coefficients date back to 1930s, and they are considered as some of the most classical in the area of random polynomials. These original contributions dealt with the expected number of real zeros $\mathbb{E}[N_n(\mathbb{R})]$ for polynomials of the form

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$P_n(x) = \sum_{k=0}^{n} c_k x^k$, where $\{c_k\}_{k=0}^{n}$ are independent and identically distributed random variables. Apparently the first paper that initiated the study is due to Bloch and Pólya [4], who gave an upper bound $\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n})$ for polynomials with coefficients selected from the set $\{-1, 0, 1\}$ with equal probabilities. Further results generalizing and improving that estimate were obtained by Littlewood and Offord [26]-[27], Erdős and Offord [11] and others. In particular, Kac [19] established the important asymptotic result

$$\mathbb{E}[N_n(\mathbb{R})] = (2/\pi + o(1)) \log n \quad \text{as } n \to \infty,$$

for polynomials with independent real Gaussian coefficients. More precise forms of this asymptotic were obtained by many authors, including Kac [20], Wang [37], Edelman and Kostlan [10]. It appears that the sharpest known version is given by the asymptotic series of Wilkins [38]. Many additional references and further directions of work on the expected number of real zeros may be found in the books of Bharucha-Reid and Sambandham [1], and of Farahmand [12]. In fact, Kac [19]-[20] found the exact formula for $\mathbb{E}[N_n(\mathbb{R})]$ in the case of standard real Gaussian coefficients:

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{4}{\pi} \int_0^1 \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)}\, dx,$$

where

$$A(x) = \sum_{j=0}^{n} x^{2j}, \quad B(x) = \sum_{j=1}^{n} j x^{2j-1} \quad \text{and} \quad C(x) = \sum_{j=1}^{n} j^2 x^{2j-2}.$$

In the subsequent paper Kac [21], the asymptotic result for the number of real zeros was extended to the case of uniformly distributed coefficients on $[-1, 1]$. Erdős and Offord [11] generalized the Kac asymptotic to Bernoulli distribution (uniform on $\{-1, 1\}$), while Stevens [35] considered a wide class of distributions. Finally, Ibragimov and Maslova [17, 18] extended the result to all mean-zero distributions in the domain of attraction of the normal law.

We state a result on the number of real zeros for the random linear combinations of rather general functions. It originated in the papers of Kac [19]-[21], who used the monomial basis, and was extended to trigonometric polynomials and other bases, see Farahmand [12] and Das [7]-[8]. We are particularly interested in the bases of orthonormal polynomials, which is the case considered by Das [7]. For any set $E \subset \mathbb{C}$, we use the notation $N_n(E)$ for the number of zeros of random functions (1.1) (or random orthogonal polynomials of degree at most n) located in E. The expected number of zeros in E is denoted by $\mathbb{E}[N_n(E)]$, with $\mathbb{E}[N_n(a,b)]$ being the expected number of zeros in $(a,b) \subset \mathbb{R}$.

Proposition 1.1. *Let $[a,b] \subset \mathbb{R}$, and consider real valued functions $g_j(x) \in C^1([a,b])$, $j = 0,\ldots,n$, with $g_0(x)$ being a nonzero constant. Define the random function*

$$G_n(x) = \sum_{j=0}^{n} c_j g_j(x), \tag{1.1}$$

where the coefficients c_j are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0,\sigma^2), \sigma > 0$. If there is $M \in \mathbb{N}$ such that $G_n'(x)$ has at most M zeros in (a,b) for all choices of coefficients, then the expected number of real zeros of $G_n(x)$ in the interval (a,b) is given by

$$\mathbb{E}[N_n(a,b)] = \frac{1}{\pi} \int_a^b \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)}\, dx, \tag{1.2}$$

where

$$A(x) = \sum_{j=0}^{n} g_j^2(x), \quad B(x) = \sum_{j=1}^{n} g_j(x) g_j'(x) \quad \text{and} \quad C(x) = \sum_{j=1}^{n} [g_j'^2. \tag{1.3}$$

Clearly, the original formula of Kac follows from this proposition for $g_j(x) = x^j$, $j = 0,1,\ldots,n$. For a sketch of the proof of Proposition 1.1, see [28]. We note that multiple zeros are counted only once by the standard convention in all of the above results on real zeros. However, the probability of having a multiple zero for a polynomial with Gaussian coefficients is equal to 0, so that we have the same result on the expected number of zeros regardless whether they are counted with or without multiplicities.

2. Random orthogonal polynomials

Let $W = e^{-Q}$, where $Q : \mathbb{R} \longrightarrow [0,\infty)$ is continuous, and all moments

$$\int_{\mathbb{R}} x^j W^2(x)\, dx, \; j = 0,1,2,\cdots,$$

are finite. For $n \geq 0$, let

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + \ldots$$

denote the nth orthonormal polynomial with $\gamma_n > 0$, so that

$$\int p_n p_m W^2 = \delta_{mn}.$$

Using the orthonormal polynomials $\{p_j\}_{j=0}^{\infty}$ as the basis, we consider the ensemble of random polynomials of the form

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \quad n \in \mathbb{N}, \tag{2.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are i.i.d. random variables. Such a family is often called random orthogonal polynomials. If the coefficients have Gaussian distribution, one can apply Proposition 1.1 to study the expected number of real zeros of random orthogonal polynomials. In particular, Das [7] considered random Legendre polynomials, and found that $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$. Wilkins [39] improved the error term in this asymptotic relation by showing that $\mathbb{E}[N_n(-1,1)] = n/\sqrt{3} + o(n^{\varepsilon})$ for any $\varepsilon > 0$. For random Jacobi polynomials, Das and Bhatt [9] concluded that $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$ too. They also stated estimates for the expected number of real zeros of random Hermite and Laguerre polynomials, but those arguments contain significant gaps. The authors recently showed [28] that if the basis is given by orthonormal polynomials associated to a finite Borel measure with compact support on the real line, then random linear combinations have $n/\sqrt{3} + o(n)$ expected real zeros under mild conditions on the weight. The second author and the third author recently also showed [32] that if the basis is given by orthonormal polynomials associated with a Freud weight, then random linear combinations have $n/\sqrt{3} + o(n)$ expected real zeros. The results of this paper provide detailed information on the expected number of real zeros for random polynomials associated with a large class of weights. In particular, they cover the case of random Freud polynomials. Interesting computations and pictures of zero distributions of random orthogonal polynomials may be found on the CHEBFUN web page of Trefethen [36].

For the orthonormal polynomials $\{p_j(x)\}_{j=0}^{\infty}$, define the reproducing kernel by

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x) p_j(y),$$

and the differentiated kernels by

$$K_n^{(k,l)}(x,y) = \sum_{j=0}^{n-1} p_j^{(k)}(x) p_j^{(l)}(y), \quad k,l \in \mathbb{N} \cup \{0\}.$$

The strategy is to apply Proposition 1.1 with $g_j = p_j$, so that

$$A(x) = K_{n+1}(x,x), \quad B(x) = K_{n+1}^{(0,1)}(x,x) \quad \text{and} \quad C(x) = K_{n+1}^{(1,1)}(x,x). \quad (2.2)$$

We use universality limits for the reproducing kernels of orthogonal polynomials (see Levin and Lubinsky [24]-[25]), and asymptotic results on zeros of random polynomials (cf. Pritsker [31]) to give asymptotics for the expected number of real zeros for a class of random orthogonal polynomials associated with weights from the class $\mathcal{F}(C^2)$.

Definition 2.1. *Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0,\infty)$ satisfies the following conditions:*
(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) Q' is non-decreasing in $\mathbb{R}$, and Q'' exists in $\mathbb{R} \setminus \{0\}$.
(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)}, \; t \neq 0,$$

is quasi-increasing in $(0,\infty)$, in the sense that for some $C > 0$,

$$0 < x < y \Rightarrow T(x) \leq CT(y).$$

We assume an analogous restriction for $y < x < 0$. In addition, we assume that for some $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } \mathbb{R} \setminus \{0\}.$$

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \; x \in \mathbb{R} \setminus \{0\}.$$

Then we write $W \in \mathcal{F}(C^2)$.

Theorem 2.2. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty], \tag{2.3}$$

then the expected number of real zeros of random orthogonal polynomials (2.1) *with independent real Gaussian coefficients satisfy*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\sqrt{3}}.$$

Theorem 2.2 is a combination of two results on zeros of random orthogonal polynomials given below. Define the Ullman distribution μ_α for $0 < \alpha < \infty$, by,

$$\mu'_\alpha(x) = \frac{\alpha}{\pi}\int_{|x|}^{1} \frac{t^{\alpha-1}}{\sqrt{t^2 - x^2}}\, dt, \quad x \in [-1, 1],$$

and for $\alpha = \infty$, the arcsine distribution μ_∞ by

$$\mu'_\infty(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad x \in [-1, 1].$$

Also define the contracted version of P_n:

$$P_n^*(s) := P_n(a_n s), \quad n \in \mathbb{N},$$

where a_n is the Mhaskar-Rakhmanov-Saff number associated with the weight W, see [23], [29], [34] and Section 3 below.

For any set $E \subset \mathbb{C}$, we use the notation $N_n^*(E)$ for the number of zeros of random functions $P_n^*(s)$ located in E. The expected number of zeros of $P_n^*(s)$ in E is denoted by $\mathbb{E}[N_n^*(E)]$, with $\mathbb{E}[N_n^*([a,b])]$ being the expected number of zeros in $[a, b] \subset \mathbb{R}$.

Theorem 2.3. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. Assume that the function T in the definition of $\mathcal{F}(C^2)$ satisfies (2.3). If $[a, b] \subset (-1, 1)$ is any closed interval, then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] = \frac{1}{\sqrt{3}}\mu_\alpha([a,b]). \tag{2.4}$$

We will establish a generalization of Theorem 2.3 for non-even weights in Section 3. Define the normalized zero counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$ for the scaled polynomial $P_n^*(s)$ of (??), where $\{z_k\}_{k=1}^n$ are its zeros, and δ_z denotes the unit point mass at z. We can determine the weak limit of τ_n for random polynomials with quite general random coefficients $\{c_j\}_{j=0}^\infty$.

Theorem 2.4. *Let the coefficients $\{c_j\}_{j=0}^\infty$ of random orthogonal polynomials (2.1) be complex i.i.d. random variables such that $\mathbb{E}[|\log|c_0||] < \infty$. If $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even, and the function T in the definition of $\mathcal{F}(C^2)$ satisfies (2.3). Then the normalized zero counting measures τ_n for the scaled polynomials $P_n^*(s)$ converge weakly to μ with probability one.*

Related results on the asymptotic zeros distribution of random orthogonal polynomials with varying weights were proved by Bloom [5] and Bloom and Levenberg [6]. Theorem 2.4 allows to find asymptotics for the expected number of zeros in various sets. In particular, we need the following corollary for the proof of Theorem 2.2.

Corollary 2.5. *Suppose that the assumptions of Theorem 2.4 hold. If $E \subset \mathbb{C}$ is any compact set satisfying $\mu(\partial E) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*(E)\right] = \mu(E), \tag{2.5}$$

where $N_n^*(E)$ is the number of real zeros of $P_n^*(s)$ in E.

It is of interest to relax conditions on random coefficients c_j, e.g., by considering probability distributions from the domain of attraction of normal law as in [17, 18].

3. Proofs

Our proofs require detailed knowledge of potential theory with external fields, see [23] and [34].

Let W be a continuous nonnegative weight function on $\mathbb{R}$ such that W is not identically zero and $\lim_{|x|\to\infty} |x|\, W(x) = 0$. Set $Q(x) := -\log W(x)$. The weighted equilibrium measure μ_W of $\mathbb{R}$ is the unique probability measure with compact support $S_W =$ supp $\mu_W \subset \mathbb{R}$ that minimizes the energy functional

$$I[\nu] = -\iint \log|z-t|\, d\nu(t) d\nu(z) + 2\int Q\, d\nu$$

amongst all probability measures ν with support on $\mathbb{R}$. It satisfies

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) = C, \quad z \in S_W,$$

and

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) \geq C, \quad z \in \mathbb{R},$$

where C is a constant.

For a weight function $W(x) = e^{-Q(x)}$, for which Q is convex on $\mathbb{R}$, the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n} < 0 < a_n$$

are defined for $n \geq 1$ by the relations

$$n = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}\, dx$$

and

$$0 = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}\, dx.$$

We also let

$$\delta_n = \frac{1}{2}\left(a_n + |a_{-n}|\right) \text{ and } \beta_n = \frac{1}{2}\left(a_{-n} + a_n\right).$$

For even Q, $a_{-n} = -a_n$, and we may define a_n by

$$\frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\, dt = n. \tag{3.1}$$

Existence and uniqueness of these numbers is established in the monographs [23], [29], [34], but goes back to earlier work of Mhaskar, Saff, and Rakhmanov. One illustration of their role is the Mhaskar-Saff identity:

$$||PW||_{L^\infty(\mathbb{R})} = ||PW||_{L^\infty([a_{-n},a_n])},$$

which is valid for all polynomials P of degree at most n. We define the Mhaskar-Rakhmanov-Saff interval Δ_n as $\Delta_n := [a_{-n}, a_n]$. The linear transformation

$$L_n(x) = \frac{x-\beta_n}{\delta_n}, \; x \in \mathbb{R},$$

maps Δ_n onto $[-1,1]$. Its inverse is

$$L_n^{[-1]}(s) = \beta_n + \delta_n s, \; s \in \mathbb{R}.$$

For $\varepsilon \in (0,1)$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n].$$

Then the equilibrium density is defined as

$$\sigma_n(x) = \frac{\sqrt{(x-a_{-n})(a_n-x)}}{\pi^2} \int_{a_{-n}}^{a_n} \frac{Q'(s)-Q'(x)}{s-x} \frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}, \; x \in \Delta_n.$$

The equilibrium density satisfies [23, p. 41]:

$$\int_{a_{-n}}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) = C, \; x \in \Delta_n,$$

and

$$\int_{a_{-n}}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) \geq C, \; x \in \mathbb{R}.$$

Note that the measure $\sigma_n(x)\, dx$ has total mass n:

$$\int_{a_{-n}}^{a_n} \sigma_n(x)\, dx = n.$$

We also define the normalized version of σ_n as follows:

$$\sigma_n^*(s) := \frac{\delta_n}{n} \sigma_n(L_n^{-1}(s)), \quad s \in [-1,1].$$

Note that

$$\int_{-1}^{1} \sigma_n^*(s)\, ds = 1.$$

For details on σ_n and σ_n^* one should consult the book [23].

In particular, the Ullman distribution μ'_α is the normalized equilibrium density for the standard Freud weight $w(x) = e^{-\gamma_\alpha |x|^\alpha}$ on $\mathbb{R}$, see Theorem 5.1 of [34, p. 240], where

$$\gamma_\alpha = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})},$$

An alternative formula for the Ullman distribution follows from that for σ_n above, namely,

$$\mu_\alpha'(x) = \frac{2\sqrt{1-x^2}}{\pi^2 B_\alpha} \int_0^1 \frac{t^\alpha - x^\alpha}{t^2 - x^2} \frac{dt}{\sqrt{1-t^2}}, \quad x \in [-1,1], \tag{3.2}$$

where

$$B_\alpha = \frac{2}{\pi} \int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}}\, dt.$$

For $n \geq 1$, we also define the square root factor

$$\rho_n(x) = \sqrt{(x - a_{-n})(a_n - x)}, \quad x \in \Delta_n. \tag{3.3}$$

In the sequel $C, C_1, C_2, \cdots$ denote constants independent of n, x, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. Given sequences $\{c_n\}, \{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that for $n \geq 1$,

$$C_1 \leq c_n / d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

We start with a general result, our only one that allows non-even weights:

Theorem 3.1
Let $W = e^{-Q} \in \mathcal{F}(C^2)$ and $[a,b] \subset (-1,1)$ be any given closed interval. Then as $n \to \infty$,

$$\frac{1}{n} \mathbb{E}\left[N_n^*([a,b])\right] = \frac{1+o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(y)\, dy.$$

New ↓

Proof (Revised)
The strategy is to apply Theorem 1.6 of [24]. It states that for all $r, s \geq 0$ and any $\varepsilon \in (0,1)$, we have uniformly for $x \in J_n(\varepsilon)$ as $n \to \infty$,

$$\frac{W^2(x)\, K_n^{(r,s)}(x,x)}{\sigma_n(x)^{r+s+1}} = \sum_{j=0}^{r} \binom{r}{j} \sum_{k=0}^{s} \binom{s}{k} \tau_{k,s} \pi^{r+s} \left(\frac{Q'(x)}{\sigma_n(x)} \right)^{r+s-j-k} + o(1),$$

where

$$\tau_{r,s} = \begin{cases} 0, & r+s \text{ odd} \\ \frac{(-1)^{(r-s)/2}}{r+s+1}, & r+s \text{ even} \end{cases}.$$

In particular, uniformly for $x \in J_{n+1}(\varepsilon)$,

$$\frac{W^2(x) K_{n+1}^{(0,0)}(x,x)}{\sigma_{n+1}(x)} = 1 + o(1);$$

$$\frac{W^2(x) K_{n+1}^{(0,1)}(x,x)}{(\sigma_{n+1}(x))^2} = \frac{Q'(x)}{\sigma_{n+1}(x)} + o(1);$$

and

$$\frac{W^2(x) K_{n+1}^{(1,1)}(x,x)}{(\sigma_{n+1}(x))^3} = \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{\pi^2}{3} + o(1).$$

Next, from Proposition 1.1, for any closed interval $[\ell, q] \subset J_{n+1}(\varepsilon)$ (where ℓ, q may depend on n),

$$\frac{1}{n}\mathbb{E}[N_n([\ell,q])] = \frac{1}{n\pi}\int_\ell^q \sqrt{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)}\right)^2}\, dx.$$

Substituting the asymptotics above, and cancelling, yields

$$\begin{aligned} &\frac{1}{n}\mathbb{E}[N_n([\ell,q])] \\ = &\frac{1}{n\pi}\int_\ell^q \sigma_{n+1}(x)\sqrt{\frac{\pi^2}{3} + o\left(\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2\right) + o\left(\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2\right) + o(1)dx}. \end{aligned}$$

Next, we note that [24, p. 87, Lemma 5.1(a), (d)] uniformly for $x \in J_{n+1}(\varepsilon)$,

$$\sigma_{n+1}(x) \geq C\frac{n}{\delta_{n+1}}$$

and

$$|Q'(x)| \leq C\frac{n}{\sqrt{(x-a_{-n})(a_n - x)}} \leq C_1\frac{n}{\delta_n},$$

so that

$$\left|\frac{Q'(x)}{\sigma_{n+1}(x)}\right| \leq C_2.$$

Thus, uniformly for all intervals $[\ell, q] \subset J_{n+1}(\varepsilon)$,

$$\begin{aligned}
&\frac{1}{n}\mathbb{E}\left[N_n([\ell, q])\right] \\
&= \frac{1}{n\pi}\int_\ell^q \sigma_{n+1}(x)\sqrt{\frac{\pi^2}{3}+o(1)}dx \\
&= (1+o(1))\frac{1}{n\sqrt{3}}\int_\ell^q \sigma_{n+1}(x)\,dx.
\end{aligned}$$

Note that the number $N_n(E)$ of real zeros of P_n in E equals the number $N_n^*(E)$ of real zeros of P_n^* in $E^* := L_{n+1}(E) = \{L_{n+1}(x) : x \in E\}$, since L_{n+1} is a bijection. Finally, if $[a, b] \subset (-1, 1)$, then by a simple calculation,

$$L_{n+1}^{[-1]}[a, b] = [a_{-n-1} + \delta_{n+1}(1+a), a_{n+1} - \delta_{n+1}(1-b)] \subset J_{n+1}(\varepsilon),$$

if $0 < \varepsilon < \min\{1+a, 1-b\}$. Then

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\left[N_n^*([a, b])\right] &= \frac{1}{n}\mathbb{E}\left[L_{n+1}^{[-1]}[a, b]\right] \\
&= (1+o(1))\frac{1}{n\sqrt{3}}\int_{L_{n+1}^{[-1]}(a)}^{L_{n+1}^{[-1]}(b)} \sigma_{n+1}(x)\,dx \\
&= (1+o(1))\frac{1}{\sqrt{3}}\int_a^b \sigma_{n+1}^*(y)\,dy,
\end{aligned}$$

by the substitution $x = L_{n+1}^{[-1]}(y)$. ■

Lemma 3.1. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. Let $\alpha \in (1, \infty]$. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty],$$

then

$$\lim_{n\to\infty} \sigma_n^*(x) = \mu_\alpha'(x), \quad x \in (-1, 1)\setminus\{0\}.$$

Remark 3.2. *An equivalent form of*

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty)$$

is

$$\lim_{x\to\infty}\frac{Q'(xt)}{Q'(x)} = t^{\alpha-1}, \quad t \in (0, 1). \tag{3.4}$$

Indeed, if this last condition holds, then as $x \to \infty$,

$$T(x)^{-1} = \frac{Q(x)}{xQ'(x)} = \frac{1}{xQ'(x)} \int_0^x Q'(u)\,du$$
$$= \int_0^1 \frac{Q'(xt)}{Q'(x)}\,dt \to \int_0^1 t^{\alpha-1}\,dt = \frac{1}{\alpha}.$$

Here we also used $0 \leq Q'(xt)/Q'(x) \leq 1$ and dominated convergence. In the other direction, as $x \to \infty$,

$$\frac{Q'(xt)}{Q'(x)} = \frac{T(xt)}{T(x)} \frac{Q(xt)}{tQ(x)} = \frac{T(xt)}{tT(x)} \exp\{\log Q(xt) - \log Q(x)\}$$
$$= \frac{T(xt)}{tT(x)} \exp\{-\int_{xt}^x \frac{Q'(u)}{Q(u)}\,du\}$$
$$= \frac{T(xt)}{tT(x)} \exp\{-\int_{xt}^x \frac{T(u)}{u}\,du\}$$
$$= \frac{T(xt)}{tT(x)} \exp\{-\int_{xt}^x \frac{\alpha + o(1)}{u}\,du\}$$
$$= \frac{1+o(1)}{t} \exp\{-(\alpha + o(1)) \log \frac{1}{t}\} = t^{\alpha-1}(1+o(1)).$$

Proof of Lemma 3.2. We prove the case $0 < \alpha < \infty$ first:
From (3.1), as $n \to \infty$,

$$\frac{n}{a_n Q'(a_n)} = \frac{2}{\pi} \int_0^1 \frac{tQ'(a_n t)}{Q'(a_n)\sqrt{1-t^2}}\,dt$$
$$\to \frac{2}{\pi} \int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}}\,dt = B_\alpha. \tag{3.5}$$

Indeed the integrand converges pointwise, and because Q is convex, so $Q'(a_n t)/Q'(a_n) \leq 1$, so we may apply Lebesgue's Dominated Convergence Theorem. In particular, then, for $n \geq 1$, and some $C_1 > 1$ independent of n,

$$C_1^{-1} n \leq a_n Q'(a_n) \leq C_1 n. \tag{3.6}$$

Next, we know that for $x \in (0,1)$,

$$\sigma_n^*(x) = \frac{2\sqrt{1-x^2}}{\pi^2} \int_0^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}}.$$

Here, for $t \in (0,1) \setminus \{x\}$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \\
&= B_\alpha^{-1} \lim_{n\to\infty} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{a_n Q'(a_n)(t^2 - x^2)} \\
&= B_\alpha^{-1} \frac{t^\alpha - x^\alpha}{t^2 - x^2}.
\end{aligned}$$

We need a bound on the integrand so as to apply dominated convergence. First $T(u)$ is bounded above. Next, for some ξ between t and x,

$$\begin{aligned}
&\left| \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \right| \\
&= \left| \frac{\frac{d}{du}(a_n u Q'(a_n u))|_{u=\xi}}{n(t + x)} \right| \\
&\leq \frac{a_n Q'(a_n \xi) + a_n^2 \xi Q''(a_n \xi)}{n(t + x)}.
\end{aligned}$$

Here (3.6) gives (since Q' is increasing)

$$\frac{a_n Q'(a_n \xi)}{n(t + x)} \leq \frac{a_n Q'(a_n)}{n(t + x)} \leq \frac{C}{x}.$$

Next, by definition of $\mathcal{F}(C^2)$ and boundedness of T, if $y > 0$,

$$0 \leq \frac{Q''(y)}{Q'(y)} \leq \frac{CT(y)}{y} \leq \frac{C}{y},$$

so

$$\frac{a_n^2 \xi Q''(a_n \xi)}{n(t + x)} \leq C \frac{a_n^2 \xi Q'(a_n \xi)}{a_n \xi n(t + x)} \leq C \frac{a_n Q'(a_n \xi)}{n(t + x)} \leq \frac{C}{x},$$

as above. Thus, for all $t \in (0,1)$,

$$\left| \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \right| \leq \frac{C}{x}.$$

so we can apply dominated convergence, to deduce that

$$\lim_{n\to\infty} \sigma_n^*(x) = \frac{2\sqrt{1 - x^2}}{\pi^2 B_\alpha} \int_0^1 \frac{t^\alpha - x^\alpha}{t^2 - x^2} \frac{dt}{\sqrt{1 - t^2}} = \mu_\alpha'(x).$$

Next, we deal with the case $\alpha = \infty$:
Let $0 < r < s < 1$. We consider $x \in (0, r]$ and split

$$\sigma_n^*(x) = \frac{2\sqrt{1-x^2}}{\pi^2}\left(\int_0^s + \int_s^1\right) \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}}$$
$$=: I_1 + I_2. \tag{3.7}$$

We shall show that the main contribution to σ_n^* comes from I_2. Now the integrand in the integral defining σ_n^* is nonnegative, so for $x \in (0, r]$,

$$\begin{aligned}
I_2 &= \frac{2\sqrt{1-x^2}}{\pi^2} \int_s^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}} \\
&\le \frac{2\sqrt{1-x^2}}{\pi^2} \int_s^1 \frac{a_n t Q'(a_n t)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}} \\
&\le \frac{2\sqrt{1-x^2}}{\pi^2 (s^2 - x^2) n} \int_s^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}} \\
&\le \frac{\sqrt{1-x^2}}{\pi (s^2 - x^2) n} \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{\sqrt{1-x^2}}{\pi (s^2 - x^2)}.
\end{aligned} \tag{3.8}$$

Next, note that by the lower bound in (3.5) in [23, p. 64], for $t \in [0, r]$,

$$\begin{aligned}
0 &\le \frac{a_n t Q'(a_n t)}{a_n s Q'(a_n s)} \le \frac{a_n r Q'(a_n r)}{a_n s Q'(a_n s)} \le \frac{T(a_n r)}{T(a_n s)} \left(\frac{r}{s}\right)^{\max\{\Lambda, C_2 T(a_n r)\}} \\
&\le C \left(\frac{r}{s}\right)^{C_3 T(a_n r)},
\end{aligned}$$

since T is quasi-increasing. Our hypothesis

$$\lim_{x \to \infty} T(x) = \infty$$

gives

$$\lim_{n \to \infty} \max_{t \in [0, r]} \frac{a_n t Q'(a_n t)}{a_n s Q'(a_n s)} = 0. \tag{3.9}$$

It also then follows easily from (3.1) that for each fixed $\tau \in (0, 1)$,

$$\lim_{n \to \infty} \frac{a_n \tau Q'(a_n \tau)}{n} = 0. \tag{3.10}$$

Now uniformly for $x \in [0, r]$,

$$\begin{aligned}
I_2 &\geq \frac{2\sqrt{1-x^2}}{\pi^2(1-x^2)} \int_s^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n} \frac{dt}{\sqrt{1-t^2}} \\
&\geq \frac{1}{\pi\sqrt{1-x^2}} \frac{2}{\pi n} \int_s^1 a_n t Q'(a_n t)(1+o(1)) \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{1}{\pi\sqrt{1-x^2}} \frac{2}{\pi n} \int_0^1 a_n t Q'(a_n t)(1+o(1)) \frac{dt}{\sqrt{1-t^2}} \\
&= \frac{1}{\pi\sqrt{1-x^2}}(1+o(1)), \qquad (3.11)
\end{aligned}$$

by (3.1) and using (3.9). Now we deal with I_1 - it clearly suffices to show only an upper bound. Let $s < \rho < 1$. Now

$$\begin{aligned}
I_1 &= \frac{2\sqrt{1-x^2}}{\pi^2} \int_0^s \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2-x^2)} \frac{dt}{\sqrt{1-t^2}} \\
&\leq \frac{2\sqrt{1-x^2}}{\pi^2 n x} \max_{u\in[0,s]} \left| \frac{d}{du}(a_n u Q'(a_n u)) \right| \int_0^s \frac{dt}{\sqrt{1-t^2}} \\
&\leq \frac{C}{n}[a_n Q'(a_n s) + \max_{u\in[0,s]} a_n^2 u Q''(a_n u)] \\
&\leq o(1) + \frac{C}{n} \max_{u\in[0,s]} a_n Q'(a_n u) T(a_n u),
\end{aligned}$$

by definition of the class $\mathcal{F}(C^2)$ and (3.10). Using the fact that T is quasi-increasing and the lower bound in (3.5) in [23, p. 64], we continue this as

$$\begin{aligned}
I_1 &\leq o(1) + \frac{C}{n} a_n Q'(a_n s) T(a_n s) \\
&\leq o(1) + \frac{C}{n} a_n Q'(a_n \rho) \frac{T(a_n s)}{T(a_n \rho)} \left(\frac{s}{\rho}\right)^{\max\{\Lambda, C_2 T(a_n s)\}-1} T(a_n s) \\
&\leq o(1) + \frac{C}{n} a_n Q'(a_n \rho) \sup_{y\in[0,\infty)} \left(\frac{s}{\rho}\right)^{\max\{\Lambda, C_2 y\}-1} y = o(1),
\end{aligned}$$

by (3.10) again, and that $s/\rho < 1$. Together with the fact that $I_1 \geq 0$, and using (3.7), (3.8), (3.11), we have shown that for $x \subset (0, r]$,

$$\frac{1}{\pi\sqrt{1-x^2}} \leq \liminf_{n\to\infty} \sigma_n^*(x) \leq \limsup_{n\to\infty} \sigma_n^*(x) \leq \frac{\sqrt{1-x^2}}{\pi(s^2-x^2)}.$$

As s is independent of r, we can let $s \to 1-$ to deduce that for $x \in (0, r]$,

$$\lim_{n\to\infty} \sigma_n^*(x) = \frac{1}{\pi\sqrt{1-x^2}} = \mu_\infty'(x).$$

$\square$

Proof of Theorem 2.3. By Lemma ??,

$$\frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] = \frac{1+o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(y)\, dy.$$

By Lemma 3.1, if $0 < \alpha < \infty$ then

$$\lim_{n\to\infty} \sigma_{n+1}^*(y) = \mu_\alpha'(y), \quad y \in (-1,1) \setminus \{0\};$$

and if $\alpha = \infty$ then

$$\lim_{n\to\infty} \sigma_{n+1}^*(y) = \mu_\infty'(y), \quad y \in (-1,1).$$

Since

$$0 \le \int_a^b \sigma_{n+1}^*(y)\, dy \le \int_{-1}^1 \sigma_{n+1}^*(y)\, dy = 1 < \infty,$$

$$0 \le \int_a^b \mu_\alpha'(y)\, dy \le \int_{-1}^1 \mu_\alpha'(y)\, dy = 1 < \infty$$

and

$$0 \le \int_a^b \mu_\infty'(y)\, dy \le \int_{-1}^1 \mu_\infty'(y)\, dy = 1 < \infty,$$

Lebesgue's Dominated Convergence Theorem gives that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] = \frac{1}{\sqrt{3}} \int_a^b \lim_{n\to\infty} \sigma_{n+1}^*(y)\, dy = \begin{cases} \frac{1}{\sqrt{3}} \int_a^b \mu_\alpha'(y)\, dy & \text{if } 0 < \alpha < \infty \\ \frac{1}{\sqrt{3}} \int_a^b \mu_\infty'(y)\, dy & \text{if } \alpha = \infty \end{cases}$$

$$= \frac{1}{\sqrt{3}} \int_a^b d\mu(y) = \frac{1}{\sqrt{3}} \mu([a,b]).$$

$\square$

Lemma 3.3. *If $W = e^{-Q} \in \mathcal{F}(C^2)$ then*

$$\lim_{n\to\infty} a_n^{1/n} = 1.$$

Proof. Lemma 3.5(c) of [23, p. 72] implies that there is a constant $C > 0$ such that

$$1 \leq \frac{a_n}{a_1} \leq Cn^{1/\Lambda} \text{ for all } n \geq 1,$$

which immediately implies the result. □

Lemma 3.4. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the coefficients $\{c_j\}_{j=0}^{\infty}$ of random orthogonal polynomials (2.1) are complex i.i.d. random variables such that $\mathbb{E}[|\log|c_0||] < \infty$, then*

$$\lim_{n\to\infty} \|P_n W\|_{L^\infty(\mathbb{R})}^{1/n} = 1 \textit{ with probability one.}$$

Proof. Using orthogonality, we obtain for polynomials defined in (2.1) that

$$\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx = \sum_{j=0}^{n} |c_j|^2 \text{ with probability one.}$$

Hence

$$\max_{0\leq j\leq n} |c_j| \leq \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/2} \leq (n+1) \max_{0\leq j\leq n} |c_j|.$$

Lemma 4.2 of [31] (see (4.6) there) implies that

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/(2n)} = \lim_{n\to\infty} \left(\max_{0\leq j\leq n} |c_j|\right)^{1/n} = 1$$

with probability one. That is,

$$\lim_{n\to\infty} \|P_n W\|_{L^2(\mathbb{R})}^{1/n} = 1 \text{ with probability one.} \tag{3.12}$$

Theorem 10.3 of [23, p. 295] states that

$$\|P_n W\|_{L^\infty(\mathbb{R})} \leq C_1 N(\infty, 2, n)\, \|P_n W\|_{L^2(\mathbb{R})},$$

$$\|P_n W\|_{L^2(\mathbb{R})]} \leq C_2 N(2, \infty, n)\, \|P_n W\|_{L^\infty(\mathbb{R})},$$

where $C_1, C_2 > 0$ are constants and for $0 < p, q \leq \infty$,

$$N(p,q,n) := \begin{cases} a_n^{\frac{1}{p}-\frac{1}{q}}, & q > p \\ \left[\frac{n}{\sqrt{a_n}}\sqrt{\max\left(\frac{T(a_n)}{a_n}, \frac{T(-a_n)}{a_n}\right)}\right]^{\frac{1}{q}-\frac{1}{p}} & , q < p. \end{cases}$$

Note that in our case Q is even, so is T. Hence

$$N(\infty, 2, n) = n^{1/2} \frac{1}{\sqrt{a_n}} [T(a_n)]^{1/4} \text{ and } N(2, \infty, n) = \sqrt{a_n}.$$

Lemma 3.7 of [23, p. 76] states that there exists $\epsilon_0 > 0$ and $C > 0$ such that for large n,

$$T(a_n) \leq C n^{2-\epsilon_0}.$$

Since $\lim_{n\to\infty} a_n = \infty$, we have for large n that

$$N(\infty, 2, n) \leq n^{1/2} \frac{1}{\sqrt{a_n}} [C n^{2-\epsilon_0}]^{1/4} \leq C^* n.$$

Hence

$$\frac{1}{C_2} \frac{1}{\sqrt{a_n}} \|P_n W\|_{L^2(\mathbb{R})} \leq \|P_n W\|_{L^\infty(\mathbb{R})} \leq C^* n \|P_n W\|_{L^2(\mathbb{R})}.$$

and the result follows by applying Lemma 3.3 and (3.12). □

Lemma 3.5. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \infty,$$

then

$$\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2,$$

where γ_n is the leading coefficient of the orthonormal polynomial $p_n(x)$ associated with the weight W.

Proof. Theorem 1.22 of [23, p. 25] gives

$$\gamma_n = \frac{1}{\sqrt{2\pi}} \left(\frac{a_n}{2}\right)^{-n-\frac{1}{2}} e^{\frac{1}{\pi}\int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2 - s^2}} ds} (1 + o(1)) \text{ as } n \to \infty,$$

so that

$$\gamma_n^{1/n} a_n = (2\pi)^{-\frac{1}{2n}} a_n^{-\frac{1}{2n}} e^{\frac{1}{n\pi}\int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2 - s^2}} ds} (1 + o(1)) \text{ as } n \to \infty. \tag{3.13}$$

Considering Lemma 3.3, we only need to show that

$$\lim_{n\to\infty} \frac{1}{n\pi} \int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2 - s^2}}\, ds = \lim_{n\to\infty} \int_{-1}^{1} \frac{Q(a_n t)}{n\pi\sqrt{1-t^2}}\, dt = 0.$$

We first prove that

$$\lim_{n\to\infty} \frac{Q(a_n t)}{n} = 0 \text{ uniformly on } t \in [-1, 1]. \tag{3.14}$$

Indeed, Lemma 3.4 of [23, p. 69] says that uniformly for $n > 0$,

$$\frac{Q(a_n)}{n} \sim \frac{1}{\sqrt{T(a_n)}}.$$

Since $\lim_{x\to\infty} T(x) = \infty$ and $\lim_{n\to\infty} a_n = \infty$, we obtain that $\lim_{n\to\infty} Q(a_n)/n = 0$. On the other hand, $Q(a_n t)/n$ is increasing as a function of $t \in (0, 1]$, which implies (3.14). Furthermore,

$$0 \le \int_{-1}^{1} \frac{Q(a_n t)}{n\pi\sqrt{1-t^2}}\, dt \le \frac{Q(a_n)}{n} \int_{-1}^{1} \frac{1}{\pi\sqrt{1-t^2}}\, dt = \frac{Q(a_n)}{n} \to 0 \quad \text{as } n \to \infty.$$

□

Lemma 3.6. *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty),$$

then

$$\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2e^{1/\alpha},$$

where γ_n is the leading coefficient of the orthonormal polynomial $p_n(x)$ associated with the weight W.

Proof (New)

Considering Lemma 3.4 and (3.13), we only need to show that

$$\lim_{n\to\infty} \frac{1}{n\pi} \int_{-a_n}^{a_n} \frac{Q(a_n s)}{\sqrt{a_n^2 - s^2}} ds = \lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} \frac{Q(a_n t)}{\pi\sqrt{1-t^2}} dt = \frac{1}{\alpha}.$$

In terms of the function T, we can recast this as

$$\lim_{n\to\infty}\frac{1}{n}\int_{-1}^{1}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}\frac{1}{T(a_nt)}dt=\frac{1}{\alpha}. \tag{3.15}$$

Using our assumption that $\lim_{t\to\infty}T(s)=\infty$, we have uniformly for $|t|\geq a_n^{-1/2}$, that $T(a_nt)=\alpha(1+o(1))$, so as the integrand is non-negative,

$$\frac{1}{n}\int_{a_n^{-1/2}\leq|t|\leq 1}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}\frac{1}{T(a_nt)}dt=\frac{1+o(1)}{\alpha}\frac{1}{n}\int_{a_n^{-1/2}\leq|t|\leq 1}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}dt. \tag{3.16}$$

Also, the integral over the remaining range is small: for $j=0,1$,

$$\begin{aligned}0 &\leq \frac{1}{n}\int_{|t|\leq a_n^{-1/2}}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}\frac{1}{T(a_nt)^j}dt\\ &\leq \frac{1}{n}\frac{a_n^{1/2}Q'\left(a_n^{1/2}\right)}{\pi\sqrt{1-a_{n-1}}\Lambda^j}2a_n^{-1/2}\leq C\frac{Q'\left(a_n^{1/2}\right)}{n}=o(1),\end{aligned}$$

recall (3.4) and (3.6). Thus (3.16) yields

$$\frac{1}{n}\int_{-1}^{1}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}\frac{1}{T(a_nt)}dt=\frac{1+o(1)}{\alpha}\frac{1}{n}\int_{-1}^{1}\frac{a_ntQ'(a_nt)}{\pi\sqrt{1-t^2}}dt=\frac{1+o(1)}{\alpha}.$$

■

Proof of Theorem 2.4. We first deal with the case

$$\lim_{x\to\infty}T(x)=\infty,$$

and show that the normalized zero counting measures τ_n for the scaled polynomials $P_n^*(s)$ converge weakly to μ with probability one. Theorem 2.1 of [3, p. 310] states that if $\{M_n\}$ is any sequence of monic polynomials of degree $\deg(M_n)=n$ satisfying

$$\limsup_{n\to\infty}\|M_n\|_{[-1,1]}^{1/n}\leq\frac{1}{2}, \tag{3.17}$$

then the normalized zero counting measures τ_n for the polynomials M_n converge weakly to μ_∞. Note that 1/2 in the above equation is the logarithmic capacity of $[-1,1]$, see Corollary 5.2.4 of [33, p. 134], and $\|\cdot\|_{[-1,1]}$ is the supremum norm on $[-1,1]$. We show that the monic polynomials

$$M_n(x) := P_n^*(x)/(c_n\gamma_n a_n^n), \quad n \in \mathbb{N},$$

satisfy (3.17) with probability one, so that the result of Theorem 2.4 follows for $\alpha = \infty$. We know from Lemma 3.4 that

$$\limsup_{n\to\infty} \|P_n W\|_{\mathbb{R}}^{1/n} \le 1 \text{ with probability one.}$$

Using the contracted weight

$$w_n(s) := \sqrt[n]{W(a_n s)} = e^{-\frac{Q(a_n s)}{n}}, \quad s \in \mathbb{R},$$

and the properties of a_n [23, p. 4], we obtain that

$$\|P_n^* w_n^n\|_{[-1,1]} = \|P_n W\|_{[-a_n,a_n]} = \|P_n W\|_{\mathbb{R}}.$$

It follows that

$$\limsup_{n\to\infty} \|P_n^* w_n^n\|_{[-1,1]}^{1/n} \le 1 \text{ with probability one.}$$

Since $\lim_{n\to\infty} Q(a_n)/n = 0$ by (3.14), we have that

$$\limsup_{n\to\infty} \|P_n^*\|_{[-1,1]}^{1/n} \le \limsup_{n\to\infty} \|P_n^* w_n^n\|_{[-1,1]}^{1/n} e^{Q(a_n)/n} \le 1$$

with probability one. We use below that $\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2$ by Lemma 3.5, and that $\lim_{n\to\infty} |c_n|^{1/n} = 1$ with probability one by Lemma 4.2 of [31]. This implies that

$$\limsup_{n\to\infty} \|M_n\|_{[-1,1]}^{1/n} = \limsup_{n\to\infty} \left\| \frac{P_n^*}{c_n\gamma_n a_n^n} \right\|_{[-1,1]}^{1/n} = \limsup_{n\to\infty} \|P_n^*\|_{[-1,1]}^{1/n} \frac{1}{|c_n|^{1/n}} \frac{1}{\gamma_n^{1/n} a_n} \le \frac{1}{2}.$$

with probability one.

Next, we prove the case

$$\lim_{x\to\infty} T(x) = \alpha \in (0,\infty).$$

Recall that the standard Freud weight with index α is given by

$$w(s) = e^{-\gamma_\alpha |s|^\alpha}, \quad s \in \mathbb{R},$$

where

$$\gamma_\alpha = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})} = \int_0^1 \frac{t^{\alpha-1}}{\sqrt{1-t^2}}\,dt$$

see [34, p. 239]. Since $\gamma_{\alpha+1} = B_\alpha \pi/2$, we apply $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(t+1) = t\Gamma(t)$ to obtain that

$$\gamma_\alpha B_\alpha = \gamma_\alpha \frac{2\gamma_{\alpha+1}}{\pi} = \frac{2}{\pi}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})}\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha+1}{2}+\frac{1}{2})} = \frac{1}{\alpha}.$$

Note that by [34, p. 240], $F_w = \log 2 + 1/\alpha$ is the modified Robin constant and $\mu_w = \mu_\alpha$ is the equlibrium measure corresponding to w. Following [34], we call a sequence of monic polynomials $\{M_n\}_{n=1}^\infty$, with $\deg(M_n) = n$, asymptotically extremal with respect to the weight w if it satisfies

$$\lim_{n\to\infty} \|w^n M_n\|_{\mathbb{R}}^{1/n} = e^{-F_w} = e^{-1/\alpha}/2, \tag{3.18}$$

where $\|\cdot\|_{\mathbb{R}}$ is the supremum norm on $\mathbb{R}$. Theorem 4.2 of [34, p. 170] states that asymptotically extremal monic polynomials have their zeros distributed according to the measure μ_w. Namely, the normalized zero counting measures of M_n converge weakly to $\mu_w = \mu_\alpha$. On the other hand, by Corollary 2.6 of [34, p. 157] and Theorem 5.1 of [34, p. 240],

$$\|w^n M_n\|_{\mathbb{R}} = \|w^n M_n\|_{[-1,1]}.$$

Together with Theorem 3.6 of [34, p. 46], (3.18) is equivalent to

$$\limsup_{n\to\infty} \|w^n M_n\|_{[-1,1]}^{1/n} \le e^{-F_w} = e^{-1/\alpha}/2.$$

We show that the monic polynomials

$$M_n(x) := P_n^*(x)/(c_n \gamma_n a_n^n), \quad n \in \mathbb{N},$$

are asymptotically extremal in this sense with probability one, so that the result of Theorem 2.4 follows. Note that

$$\lim_{n\to\infty} \|P_n W\|_{\mathbb{R}}^{1/n} = 1 \text{ with probability one}$$

by Lemma 3.4, and that

$$\|P_n^* w_n^n\|_{[-1,1]} = \|P_n W\|_{[-a_n,a_n]} = \|P_n W\|_{\mathbb{R}}$$

by [23, p. 4]. Hence

$$\limsup_{n\to\infty} \|P_n^* w_n^n\|_{[-1,1]}^{1/n} \le 1 \text{ with probability one.}$$

Since

$$\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2e^{1/\alpha}$$

by Lemma 3.6, and $\lim_{n\to\infty} |c_n|^{1/n} = 1$ with probability one by Lemma 4.2 of [31], it follows that

$$\limsup_{n\to\infty} \|M_n w^n\|_{[-1,1]}^{1/n} = \limsup_{n\to\infty} \|P_n^* w^n\|_{[-1,1]}^{1/n} \frac{1}{c_n^{1/n} \gamma_n^{1/n} a_n} = \frac{1}{2e^{1/\alpha}} \limsup_{n\to\infty} \|P_n^* w^n\|_{[-1,1]}^{1/n}$$
$$= e^{-F_w} \limsup_{n\to\infty} \|P_n^* w^n\|_{[-1,1]}^{1/n}.$$

On the other hand,

$$\limsup_{n\to\infty} \|P_n^* w^n\|_{[-1,1]}^{1/n} \le \limsup_{n\to\infty} \|P_n^* w_n^n\|_{[-1,1]}^{1/n} \|w/w_n\|_{[-1,1]} \le \limsup_{n\to\infty} \|w/w_n\|_{[-1,1]}.$$

Since w_n and w are both even, it remains to show that

$$\limsup_{n\to\infty} \|w/w_n\|_{[0,1]} \le 1.$$

Let $\varepsilon \in (0,1)$. We first show that

$$\lim_{n\to\infty} \frac{a_n Q'(a_n x)}{n} = \frac{x^{\alpha-1}}{B_\alpha}, \quad \text{uniformly for } x \in [\varepsilon, 1]. \tag{3.19}$$

Indeed, (??) and (??) give that

$$\frac{a_n x Q'(a_n x)}{n} = \frac{a_n Q'(a_n)}{n} \frac{T(a_n x)}{T(a_n)} e^{-\int_x^1 T(a_n y)/y\, dy}, \quad x \in [\varepsilon, 1].$$

Since $\lim_{x\to\infty} T(x) = \alpha$, for all $\delta > 0$ there exists $C_\delta > 0$ such that $|T(x) - \alpha| < \delta$ whenever $x \ge C_\delta$. As $\lim_{n\to\infty} a_n \varepsilon = \infty$, there is $N \in \mathbb{N}$ such that $a_n x \ge a_n \varepsilon \ge C_\delta$ for all $n \ge N$. Thus

$$\lim_{n\to\infty} T(a_n x) = \alpha \quad \text{uniformly for } x \in [\varepsilon, 1] \tag{3.20}$$

and

$$\lim_{n\to\infty} e^{-\int_x^1 T(a_n y)/y\, dy} = x^\alpha, \quad \text{uniformly for } x \in [\varepsilon, 1]. \tag{3.21}$$

Recalling that $a_n Q'(a_n)/n \to 1/B_\alpha$ by (3.5), we obtain (3.19) by applying (3.20) and (3.21). It now follows that

$$\frac{Q(a_n)}{n} - \frac{Q(a_n s)}{n} = \int_s^1 \frac{a_n Q'(a_n x)}{n}\, dx \to \int_s^1 \frac{x^{\alpha-1}}{B_\alpha}\, dx = \frac{1}{\alpha B_\alpha} - \frac{s^\alpha}{\alpha B_\alpha} \quad \text{as } n \to \infty,$$

uniformly for $s \in [\varepsilon, 1]$. Hence

$$\lim_{n\to\infty} \frac{Q(a_n s)}{n} = \gamma_\alpha s^\alpha \quad \text{uniformly for } s \in [\varepsilon, 1].$$

This gives that

$$\limsup_{n\to\infty} \|w/w_n\|_{[\varepsilon,1]} = \limsup_{n\to\infty} \max_{s\in[\varepsilon,1]} e^{Q(a_n s)/n - \gamma_\alpha s^\alpha} = 1.$$

Using monotonicity of Q and the above convergence, we also have that

$$\limsup_{n\to\infty} \|w/w_n\|_{[0,\varepsilon]} = \limsup_{n\to\infty} \max_{s\in[0,\varepsilon]} e^{Q(a_n s)/n - \gamma_\alpha s^\alpha} \le \limsup_{n\to\infty} e^{Q(a_n \varepsilon)/n} = e^{\gamma_\alpha \varepsilon^\alpha}.$$

Consequently,

$$\limsup_{n\to\infty} \|w/w_n\|_{[0,1]} \le e^{\gamma_\alpha \varepsilon^\alpha},$$

and we finish the proof by letting $\varepsilon \to 0$. □

Proof of Corollary 2.5. Consider the normalized zero counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$ for the scaled polynomial $P_n^*(s)$ of (??), where $\{z_k\}_{k=1}^n$ are the zeros of that polynomial, and δ_z denotes the unit point mass at z. Theorem 2.4 implies that measures τ_n converge weakly to μ with probability one. Since $\mu(\partial E) = 0$, we obtain that $\tau_n|_E$ converges weakly to $\mu|_E$ with probability one by Theorem 0.5′ of [22] and Theorem 2.1 of [2]. In particular, we have that the random variables $\tau_n(E)$ converge to $\mu(E)$ with probability one. Hence this convergence holds in L^p sense by the Dominated Convergence Theorem, as $\tau_n(E)$ are uniformly bounded by 1, see Chapter 5 of [16]. It follows that

$$\lim_{n\to\infty} \mathbb{E}[|\tau_n(E) - \mu(E)|] = 0$$

for any compact set E such that $\mu(\partial E) = 0$, and

$$|\mathbb{E}[\tau_n(E) - \mu(E)]| \le \mathbb{E}[|\tau_n(E) - \mu(E)|] \to 0 \quad \text{as } n \to \infty.$$

But $\mathbb{E}[\tau_n(E)] = \mathbb{E}[N_n^*(E)]/n$ and $\mathbb{E}[\mu(E)] = \mu(E)$, which immediately gives (2.5). □

Proof of Theorem 2.2. Theorem 2.3 gives that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1}{\sqrt{3}}\mu([a,b])$$

for any interval $[a,b] \subset (-1,1)$. Note that both $\mathbb{E}\left[N_n^*\left(H\right)\right]$ and $\mu(H)$ are additive functions of the set H. Moreover, they both vanish when H is a single point by (2.5) and the absolute continuity of μ with respect to Lebesgue measure on $[-1,1]$. Hence (2.5) gives that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*\left(\mathbb{R}\setminus(-1,1)\right)\right] = \mu(\mathbb{R}\setminus(-1,1)) = 0.$$

It now follows that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n^*(\mathbb{R})] = \frac{1}{\sqrt{3}}\mu((-1,1)) = \frac{1}{\sqrt{3}}.$$

To complete the proof, observe that $N_n^*(\mathbb{R}) = N_n(\mathbb{R})$, so that $\mathbb{E}[N_n^*(\mathbb{R})] = \mathbb{E}[N_n(\mathbb{R})]$, since L_{n+1} is a bijection for each fixed n. Therefore (**??**) is proved. □

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