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QUADRATURE IDENTITIES FOR INTERLACING AND ORTHOGONAL POLYNOMIALS

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Abstract. Let S be a real polynomial of degree n with real simple zeros $\{x_j\}_{j=1}^n$. Let R be a real polynomial of degree $n-1$, whose zeros interlace those of S. We prove the quadrature identity

$$\int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\frac{R}{S}(t)\right) dt = \left(\int_{-\infty}^{\infty} h(t)\,dt\right) \sum_{j=1}^{n} \frac{P(x_j)}{(RS')(x_j)}$$

valid for all polynomials P of degree $\leq 2n-2$ and any $h \in L_1(\mathbb{R})$. We deduce identities involving orthogonal polynomials, and weak convergence results involving orthogonal polynomials.

1. Introduction

Let μ be a positive measure on the real line with infinitely many points in its support, and $\int x^j d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \gamma_n > 0,$$

satisfying

$$\int_{-\infty}^{\infty} p_n p_m d\mu = \delta_{mn}.$$

Barry Simon [15, Theorem 2.1, p. 5], proved that for polynomials P of degree $\leq 2n-2$,

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{P(t)}{\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 p_n^2(t) + p_{n-1}^2(t)} dt = \int P(t)\ d\mu(t).$$

Simon calls this a *Carmona type formula* because of its analogy to identities of Carmona in the theory of Schrodinger operators [3]. He also refers to earlier work of Krutikov and Remling [8].

Without being aware of this formula, we used complex analytic methods to prove more general formulae in [11], [12]: if $\mathrm{Im}(z) \neq 0$, then for polynomials P of degree $\leq 2n-2$,

$$\frac{1}{\pi}|\mathrm{Im} z| \int_{-\infty}^{\infty} \frac{P(t)}{|zp_n(t) - p_{n-1}(t)|^2} dt = \frac{\gamma_{n-1}}{\gamma_n}\int P(t)\ d\mu(t), \tag{1.1}$$

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and via Poisson integrals

$$\int_{-\infty}^{\infty} \frac{P(t)}{p_n(t)^2} h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt = \frac{\gamma_{n-1}}{\gamma_n}\left(\int_{-\infty}^{\infty} h(t)\, dt\right)\left(\int P(t)\ d\mu(t)\right). \tag{1.2}$$

for any $h \in L_1(\mathbb{R})$. In discussions with Adhemar Bultheel, the author learned that (1.1) can be recovered from identities for orthogonal rational functions [2, Thm. 6.3.2, p. 136; Thm. 6.4.3, p. 145], and there are analogues within systems theory [4]. However, even (1.1) was evidently new to researchers in orthogonal polynomials.

In this paper, we generalize these, and provide substantially shorter proofs, using Wendroff's theorem on interlacing and orthogonal polynomials and Gauss type quadrature. We begin with:

Theorem 1.1. *Let R, S be real polynomials of respective degrees $n-1$ and n, with positive leading coefficients, and real simple zeros that interlace. Denote the zeros of S by $\{x_j\}_{j=1}^n$. Let $h \in L_1(\mathbb{R})$ and P be a polynomial of degree $\leq 2n-2$. Then*

$$\int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\frac{R(t)}{S(t)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \sum_{j=1}^{n} \frac{P(x_j)}{(S'R)(x_j)}. \tag{1.3}$$

Note that the integral on the left is an ordinary Lebesgue integral, because in intervals where S has a zero, convergence follows from the integrability of h. Note too that all $(S'R)(x_j) > 0$. We can also replace $h(t)\,dt$ by a measure $d\nu(t) = \nu'(t)\,dt + d\nu_s(t)$, at least when the singular part ν_s has compact support:

Theorem 1.2. *Assume the hypotheses of Theorem 1.1 on P, R, S. Let ν be a finite signed measure on the real line, whose singular part ν_s has compact support. Then*

$$\int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} d\nu\left(\frac{R(t)}{S(t)}\right) = \left(\int_{-\infty}^{\infty} d\nu(t)\right) \sum_{j=1}^{n} \frac{P(x_j)}{(S'R)(x_j)}. \tag{1.4}$$

The definition of the integral on the left is discussed in more detail in the proof of Theorem 1.2.

One interesting case is where R is a multiple of S':

Corollary 1.3. *Let S be a real polynomial of degree n, with real simple zeros $\{x_j\}_{j=1}^n$. Let $\alpha > 0, h \in L_1(\mathbb{R})$ and P be a polynomial of degree $\leq 2n-2$. Then*

$$\int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\alpha\frac{S'(t)}{S(t)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \sum_{j=1}^{n} \frac{P(x_j)}{\alpha S'(x_j)^2}. \tag{1.5}$$

Theorem 1.1 also gives us the orthonormal polynomial of degree $n-1$ for a special weight:

Corollary 1.4. *Let R and S be as in Theorem 1.1. Let $h \in L_1(\mathbb{R})$ with $\int_{-\infty}^{\infty} h \neq 0$. Let*

$$\Delta = \sum_{j=1}^{n} \frac{R}{S'}(x_j). \tag{1.6}$$

Then $\Delta > 0$ *and* $\frac{1}{\sqrt{\Delta}}R$ *is the orthonormal polynomial of degree* $\leq n-1$ *for the possibly signed weight*

$$W(t) = \frac{1}{S^2(t)\left(\int_{-\infty}^{\infty} h\right)} h\left(\frac{R(t)}{S(t)}\right), \quad t \in \mathbb{R}. \tag{1.7}$$

That is,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\Delta}} R(t) P(t) W(t)\, dt = \begin{cases} 0, & \deg(P) < n-1 \\ 1, & P = \frac{1}{\sqrt{\Delta}} R \end{cases}.$$

Theorem 1.1 leads to a substantial generalization of identities involving orthogonal polynomials from [12]:

Corollary 1.5. *Let* μ *be a positive measure on the real line with at least* $n+1$ *points in its support, with finite power moments* $\int x^j d\mu(x)$ *for* $j = 0, 1, 2, \ldots, 2n$, *and orthonormal polynomials* $\{p_j\}_{j=0}^{n}$. *Let* $\tau \in \mathbb{R}$ *not be a zero of* p_{n-1}, *and*

$$\psi_n(t, \tau) = p_n(t)\, p_{n-1}(\tau) - p_{n-1}(t)\, p_n(\tau). \tag{1.8}$$

Let $\beta \in \mathbb{R}$ *have the same sign as* $p_{n-1}(\tau)$, *and let* $h \in L_1(\mathbb{R})$. *Then for polynomials* P *of degree* $\leq 2n-2$,

(1.9)
$$\int_{-\infty}^{\infty} P(t) \frac{p_{n-1}(\tau)}{\beta \psi_n(t,\tau)^2} h\left(\frac{p_{n-1}(t)}{\beta \psi_n(t,\tau)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \frac{\gamma_{n-1}}{\gamma_n} \int P(t)\, d\mu(t).$$

In the special case where $p_n(\tau) = 0$, and $\beta = 1/p_{n-1}(\tau)$, we see that $\psi_n(t) = p_n(t)$, and this reduces to (1.2) above. We note that this circle of ideas also leads to explicit formulae for orthogonal polynomials associated with a reciprocal polynomial weight [9].

We can deduce weak convergence results that substantially generalize those in [12]. Recall that a determinate measure is one that is the unique solution to its moment problem. In particular such a measure has all finite power moments, and infinitely many points in its support.

Theorem 1.6. *Let* μ *be a determinate positive measure on the real line, and let* $\{p_n\}$ *denote the orthonormal polynomials for* μ. *For* $n \geq 1$, *let* $a_n > 0, b_n \geq 0$, *and*

$$q_n(t) = a_n p_n(t) - b_n p_{n-1}(t). \tag{1.10}$$

Let $h \in L_1(\mathbb{R})$ *with* $\int_{-\infty}^{\infty} h \neq 0$, *and let* ν_n *be the measure given by*

$$d\nu_n(t) = \left(\frac{\gamma_{n-1}}{\gamma_n} \int_{-\infty}^{\infty} h\right)^{-1} \frac{a_n}{q_n(t)^2} h\left(\frac{p_{n-1}(t)}{q_n(t)}\right) dt. \tag{1.11}$$

Then as $n \to \infty$,

$$d\nu_n \to d\mu \tag{1.12}$$

weakly, in the following sense: for all functions $f : \mathbb{R} \to \mathbb{R}$ *having polynomial growth at* ∞, *and that are Riemann-Stieltjes integrable with respect to* μ, *we have*

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(t)\, d\nu_n(t) = \left(\int f(t)\ d\mu(t)\right). \tag{1.13}$$

We note two special cases of this last result, with $\{a_n\}, \{b_n\}, \{q_n\}$ as above:

Example 1.7. For $n \geq 1$, let $a_n > 0$, $b_n \geq 0$, $\rho \geq 0$, and

$$d\nu_n(t) = \left(\frac{\gamma_{n-1}}{\gamma_n} \frac{\sqrt{\pi}\Gamma\left(\rho + \frac{1}{2}\right)}{\Gamma(\rho + 1)} \right)^{-1} \frac{a_n}{q_n^2(t) + p_{n-1}^2(t)} \left(\frac{q_n(t)^2}{q_n^2(t) + p_{n-1}^2(t)} \right)^{\rho} dt. \tag{1.14}$$

Then $\{d\nu_n\}$ satisfies the weak convergence in Theorem 1.6.

Note that Simon used his Carmona type formula to prove a special case of this result where $a_n = \frac{\gamma_{n-1}}{\gamma_n}$; $b_n = 0$, $\rho = 0$. The case of general a_n, but $b_n = 0$, $\rho = 0$ appears in [12].

Example 1.8. For $n \geq 1$, let

$$d\nu_n(t) = 2\left(\frac{\gamma_{n-1}}{\gamma_n} \pi^2 \right)^{-1} \frac{\log|q_n(t)| - \log|p_{n-1}(t)|}{q_n^2(t) - p_{n-1}^2(t)} dt. \tag{1.15}$$

Then $\{d\nu_n\}$ satisfies the weak convergence in Theorem 1.6.

Using Corollary 1.3, we prove:

Theorem 1.9. *Let μ be an absolutely continuous positive measure supported on $[-1, 1]$ with μ' positive and continuous in $(-1, 1)$, and $\mu'(x)\sqrt{1 - x^2}$ bounded in $(-1, 1)$. Let $h \in L_1(\mathbb{R})$ with $\int_{-\infty}^{\infty} h \neq 0$, and let ν_n be the measure given by*

$$d\nu_n(t) = \left(\int_{-\infty}^{\infty} h \right)^{-1} \frac{1}{p_n(t)^2} h\left(\frac{p_n'(t)}{np_n(t)} \right) dt. \tag{1.16}$$

Then as $n \to \infty$, we have the weak convergence

$$d\nu_n(t) \to \frac{1}{2}\left(1 - t^2\right) \mu'(t)\, dt \tag{1.17}$$

in the same sense as in Theorem 1.6.

Thus, for example,

$$d\nu_n(t) = \frac{1}{\pi} \frac{1}{p_n^2(t) + \left(\frac{1}{n} p_n'(t)\right)^2} dt$$

satisfies (1.17). We prove the results in Section 2.

2. Proof of the results

We use Wendroff's Theorem [16] and Gauss type quadrature formulae as developed in [5]. It is possible to prove Theorem 1.1 without these, and without any reference to orthogonal polynomials, instead using Cauchy's integral theorem and Poisson integrals, as in [12]. It is also possible to use the theory of de Branges spaces. However, the proof given here is substantially shorter, even in the special case considered in [12].

Proof of Theorem 1.1. As mentioned previously, since the zeros of R and S interlace, Wendroff's theorem shows that there is a (nonunique) measure μ such that R and S are orthogonal polynomials of respective degrees $n - 1$ and n for μ [16], [1]. Let $p_j(x) = \gamma_j x^j + \cdots$ denote the jth orthonormal polynomial for μ, as in Section 1. We may multiply μ by a positive constant to ensure that $S = p_n$. Choose $\beta > 0$ so that $R = \beta p_{n-1}$. Now we use the theory of Gauss type quadratures as developed in Freud [5, pp. 19 ff.]. These amount to Gauss quadratures with one preassigned abscissa, and when that abscissa is an endpoint of an interval of orthogonality, give Radau type quadratures. Let $s \neq 0$. This corresponds to the ratio $s = p_{n-1}(\xi)/p_n(\xi)$ in [5, pp. 19 ff.]. There are n simple zeros of the polynomial $p_{n-1}(t) - sp_n(t)$, which we denote by $\{t_j(s)\}_{j=1}^n$. These interlace the zeros of p_n. Moreover, there is the Gauss type quadrature formula [5, p. 21]

$$\sum_{j=1}^{n} (\lambda_n P)(t_j(s)) = \int P d\mu, \tag{2.1}$$

for polynomials P of degree $\leq 2n-2$. Here $\lambda_n(t)$ is the nth Christoffel function for μ, given by

$$\lambda_n(t)^{-1} = \sum_{j=0}^{n-1} p_j^2(t) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n'(t)\, p_{n-1}(t) - p_{n-1}'(t)\, p_n(t)\right). \tag{2.2}$$

Next, order the zeros $\{x_j\}_{j=1}^n$ of $p_n = S$ as

$$-\infty = x_{n+1} < x_n < \cdots < x_1 < x_0 = \infty,$$

and for $0 \leq j \leq n$, let

$$I_j = (x_{j+1}, x_j), \quad 0 \leq j \leq n,$$

and

$$\phi_j(t) = \frac{p_{n-1}(t)}{p_n(t)}, \quad t \in I_j.$$

Now as the zeros of p_{n-1} and p_n interlace,

$$\frac{p_{n-1}(t)}{p_n(t)} = \sum_{j=1}^{n} \frac{c_j}{t - x_j}, \tag{2.3}$$

with all $c_j > 0$. (It follows from the Lagrange interpolation formula that $c_j = (p_{n-1}/p_n')(x_j)$, but we shall not need this.) Then for $1 \leq j \leq n-1$, ϕ_j is strictly decreasing in I_j, from ∞ to $-\infty$, so has an inverse $\phi_j^{[-1]} : (-\infty, \infty) \to I_j$. For $j = 0$, instead ϕ_j is strictly decreasing in I_0 from ∞ to 0, so $\phi_0^{[-1]} : (0, \infty) \to I_0$. For $j = n$, instead ϕ_j is strictly decreasing in I_n from 0 to $-\infty$, so $\phi_n^{[-1]} : (-\infty, 0) \to I_n$. Also, for $t \in I_j$,

$$\phi_j'(t) = \frac{\left(p_{n-1}' p_n - p_n' p_{n-1}\right)(t)}{p_n(t)^2} = -\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^{-1} \frac{1}{\lambda_n(t)\, p_n(t)^2}, \tag{2.4}$$

by (2.2). Next, if $s > 0$, we see that we can order the zeros $\{t_j(s)\}$ of $p_{n-1} - sp_n$ so that for $1 \leq j \leq n$,

$$t_j(s) = \phi_{j-1}^{[-1]}(s)$$

while for $s < 0$,

$$t_j(s) = \phi_j^{[-1]}(s).$$

Thus we can rewrite (2.1) as

$$\sum_{j=1}^{n} (\lambda_n P)\left(\phi_{j-\sigma}^{[-1]}(s)\right) = \int P d\mu,$$

where $\sigma = \sigma(s) = -1$ if $s > 0$, and $\sigma = 0$ if $s < 0$. We now multiply by $h(s)$ and integrate:

$$\sum_{j=1}^{n} \int_{-\infty}^{\infty} (\lambda_n P) \left(\phi_{j-\sigma}^{[-1]}(s) \right) h(s)\, ds = \left(\int_{-\infty}^{\infty} h(s)\, ds \right) \int P d\mu. \tag{2.5}$$

Note that $(\lambda_n P)(t) = P(t) / \sum_{j=0}^{n-1} p_j^2(t)$ is bounded, so the integral on the left is absolutely convergent. We can rewrite the left-hand side as

$$\begin{aligned}
&\sum_{j=1}^{n} \int_{-\infty}^{0} (\lambda_n P) \left(\phi_j^{[-1]}(s) \right) h(s)\, ds + \sum_{j=1}^{n} \int_{0}^{\infty} (\lambda_n P) \left(\phi_{j-1}^{[-1]}(s) \right) h(s)\, ds \\
&\quad = \sum_{j=1}^{n-1} \int_{-\infty}^{\infty} (\lambda_n P) \left(\phi_j^{[-1]}(s) \right) h(s)\, ds \\
&\qquad + \int_{0}^{\infty} (\lambda_n P) \left(\phi_0^{[-1]}(s) \right) h(s)\, ds + \int_{-\infty}^{0} (\lambda_n P) \left(\phi_n^{[-1]}(s) \right) h(s)\, ds.
\end{aligned}$$

We now make the substitution $s = \phi_j(t)$, with the relevant choice of j, and use (2.4) to continue this as

$$\begin{aligned}
&- \sum_{j=0}^{n} \int_{I_j} (\lambda_n P)(t)\, h(\phi_j(t))\, \phi_j'(t)\, dt \\
&= \left(\frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \sum_{j=0}^{n} \int_{I_j} \frac{P(t)}{p_n^2(t)} h\left(\frac{p_{n-1}(t)}{p_n(t)} \right) dt \\
&= \left(\frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\frac{R(t)}{\beta S(t)} \right) dt.
\end{aligned}$$

Using this and the usual Gauss quadrature for μ allows us to rewrite (2.5) as

$$\begin{aligned}
&\left(\frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\frac{R(t)}{\beta S(t)} \right) dt \\
&= \left(\int_{-\infty}^{\infty} h(s)\, ds \right) \sum_{j=1}^{n} (\lambda_n P)(x_j) \\
&= \left(\frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \left(\int_{-\infty}^{\infty} h(s)\, ds \right) \sum_{j=1}^{n} \frac{P(x_j)}{(p_n' p_{n-1})(x_j)},
\end{aligned}$$

by (2.2). Thus,

$$\int_{-\infty}^{\infty} \frac{P(t)}{S^2(t)} h\left(\frac{R(t)}{\beta S(t)} \right) dt = \left(\int_{-\infty}^{\infty} h(s)\, ds \right) \sum_{j=1}^{n} \frac{\beta P(x_j)}{(S'R)(x_j)}.$$

Finally, we replace $h(s)$ by $h(s\beta)$ and dilate the variable s in $\int_{-\infty}^{\infty} h(\beta s)\, ds$ to obtain the result. ■

Proof of Theorem 1.2. Since (1.4) is linear in the measure ν, and Theorem 1.1 deals with the absolutely continuous case, we may assume that ν is singular. Because of our assumption that this singular ν has compact support, the condition $\phi_j(t) \in$supp$[\nu]$ forces t to lie a positive distance from x_j or x_{j+1} (except for $x_0 = \infty$ or $x_n = -\infty$). Thus $d\nu(\phi_j(t))$ is a well defined measure on I_j. We can then follow the steps above: instead of integrating at (2.5) with respect to $h(s)\,ds$, we integrate with respect to $d\nu(s)$ and follow the same steps as above. ■

Proof of Corollary 1.3. We choose $R = \alpha S'$ so that $(S'R)(x_j) = \alpha S'(x_j)^2$. ■

Proof of Corollary 1.4. Theorem 1.1 and our definition of W give for P of degree $\leq 2n-2$,

$$\int_{-\infty}^{\infty} P(t)\,W(t)\,dt = \sum_{j=1}^{n} \frac{P(x_j)}{(RS')(x_j)}.$$

Then choosing $P = \frac{1}{\sqrt{\Delta}} RT$, where T has degree $\leq n-1$, we obtain

$$\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\Delta}} R(t)\right) T(t)\,W(t)\,dt = \frac{1}{\sqrt{\Delta}} \sum_{j=1}^{n} \frac{T(x_j)}{S'(x_j)}. \tag{2.6}$$

When $T = \frac{1}{\sqrt{\Delta}} R$, this last right-hand side becomes 1, recall (1.6). Now suppose that T has degree $\leq n-2$. Using the Lagrange interpolation formula at the zeros of S, gives for all z,

$$T(z) = \sum_{j=1}^{n} \frac{T(x_j)}{S'(x_j)} \frac{S(z)}{z - x_j},$$

so

$$\frac{zT(z)}{S(z)} = \sum_{j=1}^{n} \frac{T(x_j)}{S'(x_j)} \frac{z}{z - x_j}.$$

Since the left-hand side is $O\left(z^{-1}\right)$ at ∞, we can let $z \to \infty$ to obtain

$$0 = \sum_{j=1}^{n} \frac{T(x_j)}{S'(x_j)}.$$

Then for T of degree $\leq n-2$, (2.6) gives the desired orthogonality

$$\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{\Delta}} R(t)\right) T(t)\,W(t)\,dt = 0.$$

■

Proof of Corollary 1.5. Since τ is fixed, we shall abbreviate $\psi_n(t,\tau)$ as $\psi_n(t)$ in this proof. We choose $S = \beta\psi_n$ and $R = p_{n-1}$ in Theorem 1.1. By our hypothesis on β, both R and S have positive leading coefficients. Moreover, the zeros of R and S interlace. Indeed, let $\{x_{j,n-1}\}_{j=1}^{n-1}$ denote the zeros of p_{n-1}, where $-\infty = x_{n,n-1} < x_{n-1,n-1} < x_{n-2,n-1} < \cdots < x_{1,n-1} < x_{0,n-1} = \infty$. Then

$$\operatorname{sign}(\beta\psi_n(x_{j,n-1})) = \operatorname{sign}(p_n(x_{j,n-1})) = (-1)^j,\ 1 \leq j \leq n-1,$$

Moreover, $\beta\psi_n(x)$ is positive for large positive x, and has sign $(-1)^n$ for large negative x. It follows that ψ_n has a simple zero, which we denote by t_j, in $(x_{j,n-1}, x_{j-1,n-1})$, $1 \leq j \leq n$. Theorem 1.1 gives

$$\int_{-\infty}^{\infty} \frac{P(t)}{\beta^2\psi_n^2(t)} h\left(\frac{p_{n-1}(t)}{\beta\psi_n(t)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\,dt\right) \sum_{j=1}^{n} \frac{P(t_j)}{(\beta\psi_n' p_{n-1})(t_j)}. \tag{2.7}$$

Next, we show that this has the same abscissas as the Gauss type quadrature associated with the zeros of ψ_n. To this end, let

$$K_n(t,s) = \sum_{j=0}^{n-1} p_j(t)\, p_j(s) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(t)\, p_{n-1}(s) - p_{n-1}(t)\, p_n(s)}{t-s}$$

denote the nth reproducing kernel for μ, and let

$$L_n(t,s) = \frac{\gamma_{n-1}}{\gamma_n} \frac{\psi_n(t)\, p_{n-1}(s) - \psi_n(s)\, p_{n-1}(t)}{t-s}.$$

By substituting in the definition (1.8) of ψ_n, and cancelling terms, we see that

$$L_n(t,s) = K_n(t,s)\, p_{n-1}(\tau). \tag{2.8}$$

Moreover, l'Hospital shows that

$$L_n(t,t) = \frac{\gamma_{n-1}}{\gamma_n}\left(\psi_n'(t)\, p_{n-1}(t) - \psi_n(t)\, p_{n-1}'(t)\right).$$

In particular, combining this and (2.8), we obtain

$$L_n(t_j,t_j) = \frac{\gamma_{n-1}}{\gamma_n}\psi_n'(t_j)\, p_{n-1}(t_j) = K_n(t_j,t_j)\, p_{n-1}(\tau).$$

The last two parts of this identity and (2.7) yield

$$\int_{-\infty}^{\infty} \frac{P(t)}{\beta^2\psi_n^2(t)} h\left(\frac{p_{n-1}(t)}{\beta\psi_n(t)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \frac{\gamma_{n-1}}{\gamma_n} \frac{1}{\beta p_{n-1}(\tau)} \sum_{j=1}^{n} \frac{P(t_j)}{K_n(t_j,t_j)}.$$

The Gauss type quadrature for the abscissa $\{t_j\}$ [5, Thm. 3.2, p. 21] then gives

$$\int_{-\infty}^{\infty} \frac{P(t)}{\beta^2\psi_n^2(t)} h\left(\frac{p_{n-1}(t)}{\beta\psi_n(t)}\right) dt = \left(\int_{-\infty}^{\infty} h(t)\, dt\right) \frac{\gamma_{n-1}}{\gamma_n} \frac{1}{\beta p_{n-1}(\tau)} \int P(t)\, d\mu(t).$$

Then (1.9) follows. ■

Proof of Theorem 1.6. We first assume that h does not change sign in $\mathbb{R}$. In Corollary 1.5, we choose $\tau = \tau_n$ and $\beta = \beta_n$ such that

$$\frac{p_n(\tau_n)}{p_{n-1}(\tau_n)} = \frac{b_n}{a_n} \text{ and } \beta_n = \frac{a_n}{p_{n-1}(\tau_n)}.$$

This is possible even if $b_n = 0$, as p_n and p_{n-1} have no common zeros. Then we see that

$$\beta_n\psi_n(t,\tau_n) = a_n p_n(t) - b_n p_{n-1}(t) = q_n(t),$$

so

$$\frac{p_{n-1}(\tau_n)}{\beta_n\psi_n(t,\tau_n)^2} h\left(\frac{p_{n-1}(t)}{\beta_n\psi_n(t,\tau_n)}\right) = \frac{a_n}{q_n(t)^2} h\left(\frac{p_{n-1}(t)}{q_n(t)}\right).$$

Then with $d\nu_n$ defined by (1.11), we see from Corollary 1.5 that for polynomials P of degree $\leq 2n-2$,

$$\int_{-\infty}^{\infty} P(t)\, d\nu_n(t) = \int_{-\infty}^{\infty} P(t)\, d\mu(t).$$

We can now proceed as in [12]. Since h does not change sign, the measure ν_n is non-negative. Let f be Riemann-Stieltjes integrable with respect to μ and of polynomial growth at ∞, and let $\varepsilon > 0$. Since μ is determinate, there exist upper and lower polynomials P_u and P_ℓ such that

$$P_\ell \leq f \leq P_u \text{ in } (-\infty,\infty) \tag{2.9}$$

and

$$\int (P_u - P_\ell)\, d\mu < \varepsilon. \tag{2.10}$$

See, for example, [5, Theorem 3.3, p. 73]. Then for n so large that $2n-2$ exceeds the degree of P_u and P_ℓ, Corollary 1.5 gives

$$\int_{-\infty}^{\infty} f d\nu_n - \int f\ d\mu \le \int P_u d\nu_n - \int P_\ell d\mu = \int (P_u - P_\ell)\, d\mu < \varepsilon.$$

Similarly, for such n,

$$\int_{-\infty}^{\infty} f d\nu_n - \int f\ d\mu \ge \int P_\ell d\nu_n - \int P_u d\mu = \int (P_\ell - P_u)\, d\mu > -\varepsilon.$$

Thus for h of one sign, (1.12) holds. In the general case, we write $h = h_+ - h_-$ where h_+ and h_- are non-negative, and let $d\nu_n^+$ and $d\nu_n^-$ denote the corresponding measures. Our proof so far shows that as $n \to \infty$, we have weakly

$$\begin{aligned}\left(\int_{-\infty}^{\infty} h\right) d\nu_n &= \left(\int_{-\infty}^{\infty} h_+\right) d\nu_n^+ - \left(\int_{-\infty}^{\infty} h_-\right) d\nu_n^- \\ &\to \left(\int_{-\infty}^{\infty} h_+\right) d\mu - \left(\int_{-\infty}^{\infty} h_-\right) d\mu = \left(\int_{-\infty}^{\infty} h\right) d\mu.\end{aligned}$$

■

Proof of Example 1.7. We choose $h(t) = \left(1+t^2\right)^{-1-\rho}$. Then (cf. [6, p. 285, 3.194.3])

$$\int_{-\infty}^{\infty} h = \frac{\sqrt{\pi}\Gamma\left(\rho+\frac{1}{2}\right)}{\Gamma(\rho+1)}$$

and we can easily recast (1.11) as (1.14). ■

Proof of Example 1.8. We choose

$$h(x) = \frac{\log x^{-2}}{1-x^2}$$

which has $h \in L_1(\mathbb{R})$. Moreover, the fact that h is even and a substitution show that [6, p. 533, 4.231.13]

$$\int_{-\infty}^{\infty} h = 8\int_0^1 \frac{\log x^{-1}}{1-x^2} dx = \pi^2.$$

We can easily recast (1.11) as (1.15). ■

Proof of Theorem 1.9. Since $\mu'(x)\sqrt{1-x^2}$ is bounded, we have for $n \ge 1$ and $x \in [-1,1]$ [5, Lemma III.3.2, p. 103]

$$n\lambda_n(x) \le C. \tag{2.11}$$

Since μ' is continuous in $(-1,1)$, we have uniformly for x in compact subsets of $(-1,1)$, (cf. [13, pp. 104-5])

$$\lim_{n\to\infty} n\lambda_n(x) = \pi\sqrt{1-x^2}\mu'(x). \tag{2.12}$$

Moreover,

$$\lim_{n\to\infty} \frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2}. \tag{2.13}$$

As we vary n, we denote the zeros of p_n by $\{x_{jn}\}_{j=1}^{n}$. Then uniformly for x_{jn} in compact subsets of $(-1,1)$, we have from (2.2),

$$\begin{aligned} \frac{n}{p_n'(x_{jn})^2} &= \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 (n\lambda_n(x_{jn}))\,\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn}) \\ &= \frac{\pi}{4}\sqrt{1-x_{jn}^2}\mu'(x_{jn})\,\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn})\,(1+o(1)). \end{aligned} \tag{2.14}$$

Moreover, uniformly for $1 \le j \le n$,

$$\frac{n}{p_n'(x_{jn})^2} \le C\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn}). \tag{2.15}$$

Since for every bounded and Riemann integrable function g, [13, Theorem 3.2.3, page 17],

$$\lim_{n\to\infty}\sum_{j=1}^{n}\lambda_n(x_{jn})\,p_{n-1}^2(x_{jn})\,g(x_{jn}) = \frac{2}{\pi}\int_{-1}^{1}\sqrt{1-t^2}g(t)\,dt,$$

it follows easily from (2.14) and (2.15), that for every polynomial P,

$$\lim_{n\to\infty}\sum_{j=1}^{n}\frac{nP(x_{jn})}{p_n'(x_{jn})^2} = \frac{1}{2}\int_{-1}^{1}P(t)\left(1-t^2\right)\mu'(t)\,dt. \tag{2.16}$$

We now proceed as in the proof of Theorem 1.6. Let f be as there, let $\varepsilon > 0$, and as there, let P_ℓ and P_u satisfy (2.9) and (2.10). We may assume that $h \ge 0$ and also $\int_{-\infty}^{\infty} h = 1$. Then as $n \to \infty$,

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{f(t)}{p_n^2(t)}h\left(\frac{p_n'(t)}{np_n(t)}\right)dt - \frac{1}{2}\int_{-1}^{1}f(t)\left(1-t^2\right)\mu'(t)\,dt \\ &\le \int_{-\infty}^{\infty}\frac{P_u(t)}{p_n^2(t)}h\left(\frac{p_n'(t)}{np_n(t)}\right)dt - \frac{1}{2}\int_{-1}^{1}P_\ell(t)\left(1-t^2\right)\mu'(t)\,dt \\ &= \sum_{j=1}^{n}\frac{nP_u(x_{jn})}{p_n'(x_{jn})^2} - \sum_{j=1}^{n}\frac{nP_\ell(x_{jn})}{p_n'(x_{jn})^2} + o(1) \\ &= \frac{1}{2}\int_{-1}^{1}(P_u(t)-P_\ell(t))\left(1-t^2\right)\mu'(t)\,dt + o(1) \\ &\le \int (P_u - P_\ell)\,d\mu + o(1) < \varepsilon + o(1). \end{aligned}$$

The lower bound is similar. ■

References

[1] A. Beardon, K. Driver, K. Jordaan, *Zeros of polynomials embedded in an orthogonal sequence,* Numerical Algorithms, 57(2011), 399–403.

[2] A. Bultheel, P. Gonzalez-Vera, E. Hendriksen, O. Njastad, *Orthogonal Rational Functions,* Cambridge University Press, Cambridge, 1999.

[3] R. Carmona, *One-dimensional Schrödinger operators with random or deterministic potentials: New spectral types*, J. Funct. Anal. 51 (1983), 229–258.

[4] P. Dewilde and H. Dym, *Lossless inverse scattering, digital filters, and estimation theory*, IEEE Trans. Inf. Th., IT-30(1984), 644–662.

[5] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[6] I. S. Gradshteyn and I. M. Ryzhik, T*ables of Integrals, Series, and Products*, Academic Press, New York, 1980.

[7] P. Koosis, *Introduction to H_p Spaces*, Cambridge University Press, Cambridge, 2nd ed., 1998.

[8] D. Krutikov, C. Remling, *Schrödinger operators with sparse potentials: asymptotics of the Fourier transform of the spectral measure*, Comm. Math. Phys. 223 (2001), 509–532.

[9] D. S. Lubinsky, *Explicit Orthogonal Polynomials for Reciprocal Polynomial Weights on* $(-\infty, \infty)$, Proc. Amer. Math. Soc., 137(2009), 2317–2327.

[10] D. S. Lubinsky, *Applications of New Geronimus Type Identities for Real Orthogonal Polynomials*, Proc. Amer. Math. Soc., 138 (2010), 2125–2134.

[11] D. S. Lubinsky, *Old and New Geronimus Type Identities for Orthogonal Polynomials on the Real Line*, Jaen Journal of Approximation Theory, 2(2010), 289–301.

[12] D. S. Lubinsky, *New Integral Identities for Orthogonal Polynomials on the Real Line*, Proc. Amer. Math. Soc., 139(2011), 1743–1750.

[13] P. Nevai, *Orthogonal Polynomials*, Memoirs of the Amer. Math. Soc., no. 213, 18(1979).

[14] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.

[15] B. Simon, *Orthogonal Polynomials with Exponentially Decaying Recursion Coefficients*, (in) Probability and Mathematical Physics (eds. D. Dawson, V. Jaksic and B. Vainberg), CRM Proc. and Lecture Notes 42 (2007), 453–463.

[16] B. Wendroff, *On Orthogonal Polynomials*, Proc. Amer. Math. Soc., 12(1961), 554–555.

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