ED SAFF AT THREE SCORE AND TEN

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Ed Saff was born in 1944 in Brooklyn, New York, to Irving and Rose Saff. They gave him and his two brothers Harvey and Donald a warm Jewish upbringing, and a solid work ethic - all three have successful careers. Ed's primary school was PS164 and his middle school was Parson's Junior High School, both in Queens. High school was split between Forest Hills High in Queens and South Broward High School in Hollywood, Florida.

Loretta Saff records that "Ed started college at the Georgia Institute of Technology at the tender age of 16. His parents accompanied him to the train station to say goodbye but did not personally deliver him to the campus. He faced the very long train ride from Hollywood, Florida to Atlanta, Georgia - by himself. All was going smoothly when just over the Georgia line Ed felt the train come to a halt. After a few minutes an announcement was made: 'Ladies and Gentlemen, due to yesterday's hurricane, there is debris on the tracks and we will be delayed several hours. As soon as the track is cleared, we will continue to Atlanta.' Ed had no choice – he just had to sit and wait. The wait included spending the night on the train.

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FIGURE 1. Ed as a cowboy

By the time he arrived at Georgia Tech and located his dormitory, someone else had been assigned to his room and had taken his bed. In addition, Ed missed out on taking the required exams given during the first day of orientation. These frustrations were offset, however, by Ed's introduction to campus fraternity life (aka parties with the opposite gender and many samples of the local beverages). Fortunately for Ed, several fraternities were on academic probation and in need of a grade-point-average boost. Hence Ed was sought after and even offered 'a grant' to join one. After a rough start, Ed's college life was a joy both intellectually and socially." And most importantly, while at Tech, he met his future wife, Loretta, then a high school student.

Ed clearly shone at Georgia Tech, so much so, that he was asked as a senior to teach freshmen: "When I was 19 years old, a college senior at Georgia Tech, the Chairman of the Mathematics Department asked me to teach a course for freshmen, a responsibility usually reserved for graduate students or post-docs. Naturally I was flattered, but I was also a bit insecure about being the instructor of students around my own age—how would I get them to respect me?

I told my father about my trepidations and he responded with a very special gift, my grandfather's pocket watch. It was an unpretentious but clearly antique timepiece that came with an ornate chain that I could attach to my belt loop. It became my official time keeper for the course, helping to offset my youth and inexperience with an aura of ancestral authority whenever I glanced at it during class. Recently I returned to Georgia Tech to give a colloquium lecture, in a room close to where I first taught, and I brought that watch with me, not only to measure how many minutes I was exceeding the allotted time, but to reminisce and express my appreciation for the exciting opportunity I had had to teach my first mathematics course there."

Ed recalls how Joseph Walsh became his doctoral adviser: "When I first arrived as a graduate student at the University of Maryland in 1964, I met W.E. (Brit) Kirwan who was just starting his career as an Assistant Professor in the Mathematics Department. His outgoing personality and my fascination with complex variables, a subject to which Kirwan contributed significantly in his early years, was the basis for a close friendship, one that continues to this very day. I took several courses from Brit and approached him about becoming my Ph.D. adviser. He told me, somewhat to my disappointment, that he felt himself a bit too inexperienced to take on such a responsibility and that my career would likely get a bigger boost from a true mathematical star and new addition to the Math Department – Joseph L. Walsh who had just arrived from Harvard.

I quickly scanned the offerings for the next semester and found Prof. Walsh listed as instructor for a course in approximation theory. Brit recommended that I talk with him about the content and background needed for that course, so I shyly knocked on his door to ask, among other naive things, whether there would be a textbook we would be following. "Sure" he said as he handed me a substantial blue monograph that I opened cautiously. Of course that book was his famous "Approximation and Interpolation in the Complex Domain", an American Mathematical Society Colloquium publication. As I tried to conceal my embarrassment at seeing his name as the author, he reassured me that the course would contain a strong dose of complex variables and that my background in that area was more than sufficient for taking his class.

FIGURE 2. Ed and Loretta with the Walshes

It took only a few weeks of Walsh's course to convince me that approximation was the subject I wanted to be the focal point of my mathematical career. And Walsh was indeed an inspiring and attentive mentor—always approachable, always willing to listen to whatever I had discovered and even appear to be truly interested. I owe a great deal to both Brit Kirwan and Joseph Walsh. They instilled in me a true appreciation for the aesthetic qualities of mathematics–the unrivaled beauty of a dynamic subject. Kirwan went on to become President of the University of Maryland, College Park, then President of Ohio State University, and then Chancellor of the entire Maryland University System. Recently I was honored to participate in a special one-day workshop in his honor at the University of Maryland and had an opportunity to publicly thank him for his impact on my career.

For Joseph Walsh, a veteran of two world wars, a member of the US National Academy of Sciences, a former President of the American Mathematical Society, and whose name is often co-joined with Fourier regarding orthogonal series, how truly fortunate I was to be his student. It is but a small repayment that I co-edited (with Ted Rivlin) a volume of Walsh's Selected Works. And, at the invitation of the National Academy of Sciences, I will be writing in the next few years, a biography of Walsh for their archives."

After postdoctoral positions at Maryland and Imperial College, Ed joined the University of South Florida in 1969. By then he was already a seasoned and tough instructor, able to handle tricky situations: "One afternoon Ed walked into a mass lecture auditorium to teach his calculus class. The room had huge double-paneled blackboards that slid up to reveal the next available writing space.

Ed was deeply into the material for the day when he pushed up the first board to gain access to what was underneath. What was underneath was a large pin-up of a Playboy centerfold. Never one to miss a beat, Ed smiled at the hysterical students and said, 'Well, unfortunately, these are not the types of figures we are studying today.' Then he discretely removed the photo, folded it, and put it in his attaché for future reference."

Over the space of 32 years, from 1969 to 2001, Ed raised the profile and quality of mathematics at the University of South Florida. He founded centers and institutes, was heavily involved in outreach to primary and high schools, and cofounded two research journals: *Constructive Approximation* and *Computational Methods* *and Function Theory.* Also during that period he coauthored three research monographs, coedited numerous conference proceedings, and coauthored popular textbooks on a variety of topics, including differential equations and complex analysis. Ed supervised doctoral students from several countries, who have gone onto successful careers in both academe and industry. His energy, drive, and success earned the respect - and sometimes envy - of his colleagues.

A hallmark of his entire professional career, has been his ability to engage people from all over the world, to get them to work together, and to succeed. I recall reading that in the area of nuclear physics, Enrico Fermi and Niels Bohr played a crucial role in the years between the two world wars, by hosting visiting scientists, and creating an atmosphere in which ideas could be discussed and advanced. Ed has consistently done that in approximation theory and its ramifications, especially encouraging young talent. He has the ability to gather a group of people and get them to work together. A semester or two working with Ed Saff is a formative experience in a research career - it certainly was in mine. Ed has also been prepared to venture to all corners of the world to start new and often long term collaborations, such as current projects with Laurent Baratchart's group at INRIA in France, and Ian Sloan's group at the University of New South Wales in Australia. Throughout, Loretta played a wonderfully supportive role, constantly entertaining visitors, making them feel at home, warmly putting up with all their foibles and little difficulties.

Ed organized many groundbreaking conferences - one was that in Tampa in 1976, at which G.G. Lorentz delivered his seminal address on incomplete polynomials (more on that below!). Another was the first US-USSR conference in Tampa in 1990, which brought together the great Russian school of approximators led by A.A. Gonchar, with many from the US, Europe and beyond. In organizing this and other events, Ed had to surmount political barriers, but he has always felt that mathematics transcends politics. It is also a principle of the CMFT conferences, which he coorganizes with Stephan Ruscheweyh, held whenever possible in developing countries. After moving to Vanderbilt in 2001, Ed has continued all these activities, in addition to serving a term as Executive Dean of the College of Arts and Sciences.

And now to research - where does one start in surveying a research career that has covered such a broad swathe of topics? It would take a collected works, with a lengthy introduction, to properly cover Ed's work. In this short and hastily prepared review, we necessarily select a few of the main topics, and focus on Ed's papers. We apologize for not doing justice to the contributions of Ed's many students and collaborators, let alone assessing the work of competing researchers. We do not present the most general forms of results, nor the state of the art, and in some cases, do not provide a complete formulation.

As noted above, Ed's doctoral adviser was Joseph Walsh, the premier American approximator of his time, most of whose research focused on interpolation and approximation in the complex plane. This naturally influenced Ed's first paper [30], published in 1968, which characterized a class of functions analytic in the unit ball, that satisfy some smoothness conditions on the boundary, in terms of sequences of interpolating rational functions of type $(n-1, n)$ (that is with respective numerator and denominator degrees at most $n-1$ and n).

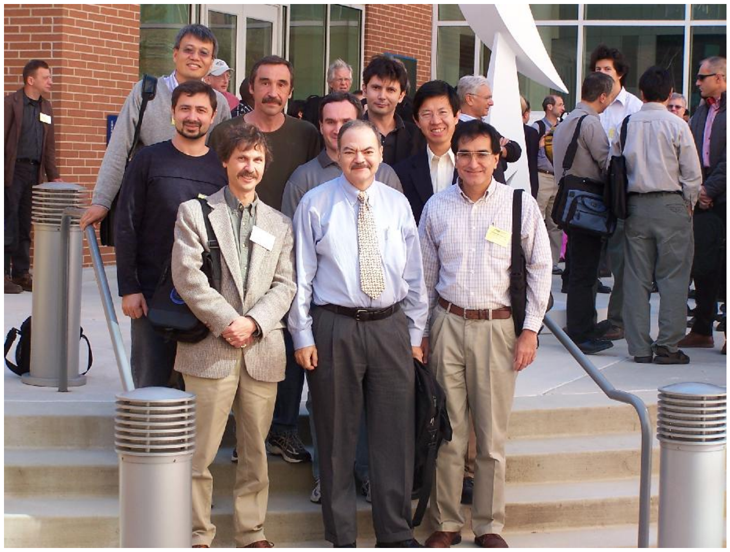

FIGURE 3. Ed and his PhD students, 60th birthday conference, 2004

However, it was sequences of rational functions with *bounded* denominator degree that mark his early fundamental contributions. His second paper [31] gave sufficient conditions for a continuous function defined on a Jordan curve to be the restriction of a function meromorphic inside Γ with a bounded number of poles. His fifth paper [31] resolved difficult issues about *intermediate rows*.

Recall the setting: if f is a continuous complex valued function on a compact set $E \subset \mathbb{C}$, and $m, n \geq 0$, then the best uniform approximant r_{nm}^* is a rational function of type (n, m) satisfying

$$\|f - r_{nm}^*\|_{L_\infty(E)} = \inf \left\{ \|f - r\|_{L_\infty(E)} : r \text{ of type } (n, m) \right\}.$$

These are arranged into an infinite table, called the *Walsh array*, in which the rows are indexed by m, the upper bound on the denominator degree, while the columns are indexed by n, the upper bound on the numerator degree.

The Walsh array is a cousin of the *Padé table*. Given a function f with a Maclaurin series (or even just a formal power series) at 0, its n, m Padé approximant $[n/m] = p/q$ is a rational function of type (n, m) with q not the zero polynomial, satisfying

$$(fq - p)(z) = O\left(z^{n+m+1}\right).$$

The Padé table is the array of Padé approximants, again with m as the row index and n as the column index. Of course, some take the transpose, and one can debate which arrangement is more natural.

de Montessus de Ballore's classic 1902 theorem gives conditions for convergence of the $(m+1)$st row of the Padé table. If f is meromorphic in the ball $B_R = \{z : |z| < R\}$, with poles of total multiplicity m, none at 0, then the sequence of Padé approximants with denominator degree $\leq m$, namely $\{[n/m]\}_{n=1}^{\infty}$, converges uniformly to f in compact subsets of B_R that exclude the poles of f. This theorem, in which it is essential that the number of poles of the approximants matches that of f, initiated a field that is still being explored today.

In the late 1920's, R. Wilson explored the topic of intermediate rows of the Padé table, not covered by de Montessus' theorem. Suppose, for example, that $0 < R < S$, that f is meromorphic in $|z| < S$, and has poles of total multiplicity 4 in $|z| < R$, and poles of total multiplicity 7 in $|z| < S$, with the 3 new poles all on the boundary circle $|z| = R$. de Montessus asserts that $\{[n/4]\}_{n=1}^{\infty}$ converges in $|z| < R$ away from the poles, and $\{[n/7]\}_{n=1}^{\infty}$ converges in $|z| < S$ away from the poles. What about the intermediate rows, namely, $\{[n/5]\}_{n=1}^{\infty}$ and $\{[n/6]\}_{n=1}^{\infty}$? They may converge or diverge, depending on a variety of factors.

Ed's important 5th paper dealt with analogues of Wilsons' theorem for best rational approximants, resolving the situation when all the new boundary poles coincide. His supervisor, Joseph Walsh had earlier proved an analogue of de Montessus' theorem for the rows of the Walsh array.

Theorem [32] *Let E be a closed set whose boundary is a set of finitely many C^2 mutually exterior Jordan curves. Let G be the Green's function for $\mathbb{C}\backslash E$, with pole at ∞, and for $\rho > 1$, let $\Gamma_\rho = \{z : G(z) = \log \rho\}$, and E_ρ denote its interior. Let f be analytic on E, and meromorphic in E_ρ with poles of total multiplicity m in E_ρ, and analytic on Γ_ρ, except for a pole of order ℓ at $a \in \Gamma_\rho$, which is not a critical point of the Green's function G. Then for $0 \le k < \ell$, we have*

$$\inf_{r \text{ of type } (n,m+k)} \|f - r\|_{L_\infty(E)} \le A n^{\ell-2m-1}/\rho^n.$$

Moreover, if for $n \ge 1$, r_n is a rational function of type $(n, m+k)$ satisfying the weaker requirement

$$\|f - r_n\|_{L_\infty(E)} = o\left(n^{\ell-2m+1}/\rho^n\right),$$

then for large enough n, r_n has precisely $m + k$ finite poles, m of which approach the poles of f in E_ρ, and k approach a. In addition $\{r_n\}$ converges uniformly to f on compact subsets of E_ρ containing no poles of f. In particular, this is the case for the $(m + k + 1)$st row of the Walsh array.

Ed's next paper [33] marked a transition to a new topic, namely approximation of the exponential function e^z, the beginning of a long love affair! For $\rho > 0, m, n \ge 0$, let

$$E_{nm}(e^z; \rho) = \inf \left\{ \|e^z - r(z)\|_{L_\infty(|z| \le \rho)} : r \text{ of type } (n, m) \right\}.$$

For fixed m, Ed proves that there exist positive constants $A_1, A_2 > 0$ such that for $n \ge 0$,

$$A_1 \le (n + 2m + 1)! \rho^{-n} E_{nm}(e^z; \rho) \le A_2.$$

Moreover, given a sequence of rational functions $\{r_n\}$ of type (n, m) satisfying the weaker approximation rate

$$\|e^z - r(z)\|_{L_\infty(|z| \le \rho)} = o\left([(n + 2m - 2)! \rho^n]^{-1} \right),$$

for large enough n, r_n has at least $m - 1$ finite poles that approach ∞ as $n \to \infty$, and $\{r_n\}$ converges uniformly in compact subsets of the plane to e^z. In particular this is the case for the $(m + 1)$st row of the Walsh array. Similar results hold for columns of the Walsh array, and also when one considers best L_p, rather than best L_∞, approximants. In a follow up paper, Ed showed [35] that *any* sequence $\{r^*_{nm}\}$ of best uniform rational approximants to e^z on the unit disk, with $n + m \to \infty$, converges uniformly in compact subsets of the plane to e^z.

FIGURE 4. Ed and Gonchar's school, 2005

Another 1972 paper of Ed [36] provided an important breakthrough in two directions: a new elementary (contour integral) way to prove de Montessus theorems, and the first de Montessus theorem for multipoint Padé approximants. It started a whole new chapter in multipoint Padé approximation, and in particular, was noticed by Gonchar's Russian school of Padé enthusiasts. In a related vein, Ed's 1971 paper [34] on determining regions of meromorphy from the rate of best rational approximation, inspired a paper of Gonchar [9]. Back to that de Montessus paper:

Theorem *Let E be a compact set whose complement is connected and possesses a Green's function G with pole at ∞. Let $cap(E)$ denote its capacity, and let $\rho > 1$, and Γ_ρ and E_ρ be as above. Let there be given a triangular array of points $\left\{\beta_j^{(n)}\right\}_{n\geq 0, 1\leq j\leq n+1}$ with no limit point outside E. Assume that*

$$\lim_{n\to\infty}\left|\prod_{j=1}^{n+1}\left(z-\beta_j^{(n)}\right)\right|^{1/n} = cap\,(E)\, e^{G(z)},$$

uniformly in compact subsets of $\mathbb{C}\backslash E$. Suppose that f is analytic in E and meromorphic in E_ρ with poles of total multiplicity ν. Then for large enough n, there exists a unique rational function r_n of type (n,ν) that interpolates to f in the points $\left\{\beta_j^{(n+\nu)}\right\}_{1\leq j\leq n+\nu+1}$. Moreover, r_n has ν finite poles that approach those of f in E_ρ, and $\{r_n\}$ converges uniformly to f on compact subsets of E_ρ omitting poles of f.

Subsequent years saw a steady stream of papers making fundamental contributions both to approximation of e^z and the general theory of rational approximation. In a 1973 paper, Ed [37] improved his earlier estimates for best rational approximation of e^z into asymptotics: let $E_{nm}\,(e^z;\rho)$ be as above, let r_{nm}^* denote a best rational function of type (m,n) attaining the inf, and

$$\varepsilon_{nm} = \frac{m!n!}{(m+n)!\,(m+n+1)!}.$$

Ed proved that for each fixed m, we have as $n \to \infty$

$$E_{nm}\left(e^{z};\rho\right) = \varepsilon_{nm}\rho^{n+m+1}\left(1+o\left(1\right)\right),$$

possibly the first such precise asymptotics in best approximation of any special function by rational functions. Moreover,

$$\lim_{m\to\infty}\frac{e^{z}-r_{mn}^{*}\left(z\right)}{\left(-1\right)^{m}\varepsilon_{mn}z^{m+n+1}} = 1$$

uniformly in compact subsets of $|z| > \rho$. That same year saw Ed's first paper on the distribution of zeros of best polynomial approximants and error functions, for a class of entire functions.

One aside was a short 1974 joint paper with Sheil-Small, that solved a 30+ year old problem of Paul Erdős as well as partially resolving a conjecture of Walter Hayman. They proved the truth of Erdős' 1940 conjecture that if T_n is a trigonometric polynomial of degree n with all its $2n$ zeros in $[0, 2\pi)$, then

$$\int_0^{2\pi}\left|T_n\left(\theta\right)\right| d\theta \leq 4\left\|T_n\right\|_{L_\infty[0,2\pi]}.$$

The famous Saff and Varga collaboration began in 1975, giving rise to some 35 papers in about a decade. What with Ed's interest in rational approximation to e^z, and Richard Varga's interest in numerical analysis, it was natural that they should work on Padé approximants to e^z - they play an important role in analyzing stability of numerical solutions of certain types of ordinary differential equations. In their first paper [44], they proved that $\{[n-1/n]\}$ and $\{[n-2/n]\}$ converge uniformly to e^{-z} on the *unbounded* sector $\left\{z = re^{i\theta} : |\theta| \leq \frac{\pi}{2} - \delta\right\}$, for any $0 < \delta < \frac{\pi}{2}$, the first result of its type. They also established similar results for other sequences over smaller unbounded regions. In that same year, extending work of Ehle and Van Rossum, they showed in [45] that for $n \geq 2, \nu \geq 0$, $[n/\nu]$ for e^z has no zeros in the sector $\left\{z : |\arg\left(z\right)| \leq \cos^{-1}\left(\frac{n-\nu-2}{n+\nu}\right)\right\}$. This was the first of three famous papers on the zeros and poles of Padé approximants to e^z.

The following year, [46] they studied zeros of sequences of polynomials satisfying a certain recurrence relation, showing that there are no zeros in a parabolic region. This can be applied to hypergeometric function, Bessel polynomials, and Padé approximants. For example, they deduced that the Padé numerator $P_{n,m}$ for e^z has no zeros in

$$\left\{z = x+iy : y^2 \leq 4\left(m+1\right)\left(x+m+1\right), x > -m-1\right\};$$

the Padé denominator $Q_{n,m}$ has no zeros in

$$\left\{z = x+iy : y^2 \leq 4\left(n+1\right)\left(n+1-x\right), x < n+1\right\}.$$

Subsequent papers explored the sharpness of these regions, and also convergence

In their second paper on zeros and poles of Pade approximants to e^z, Saff and Varga proved [47] that for $n \geq 1, \nu \geq 0$, all the zeros of $[n/\nu]$ lie in the half plane $\operatorname{Re} z < n - \nu$, and also in the annulus

$$\left(n+\nu\right)\lambda < |z| < n+\nu+4/3$$

where $\lambda = 0.278465...$ is the unique positive root of $\lambda e^{1+\lambda} = 1$. They prove that this choice of λ is best possible.

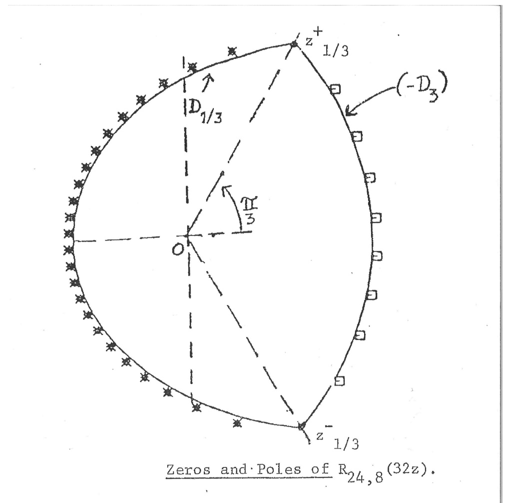

FIGURE 5. Saff Varga Szegő curve

However, it is in their third paper on zeros and poles [48] that they made their deepest contribution, with a wonderful extension of *Szegő's curve.* Recall that if s_n is the nth partial sum of the Maclaurin series of e^z, Szegő proved that the set of limit points of zeros of the scaled sequence $\{s_n(nz)\}$ is the Szegő curve, namely

$$\left\{z : |z| \leq 1 \text{ and } \left|ze^{1-z}\right| = 1\right\}.$$

Let $0 < \sigma < \infty$, let

$$z_\sigma^\pm = \exp\left(\pm i \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right)$$

and define the vertical rays that start from $z_\sigma^\pm$,

$$\mathcal{R}_\sigma = \left\{z_\sigma^+ + is, s \geq 0\right\} \cup \left\{z_\sigma^- - is, s \geq 0\right\}.$$

Let

$$g_\sigma(z) = \sqrt{1 + z^2 - 2z\frac{1-\sigma}{1+\sigma}},$$

a function that is analytic in single valued in $\mathbb{C}\backslash\mathcal{R}_\sigma$ with branchpoints at $z_\sigma^\pm$. Define in $\mathbb{C}\backslash\mathcal{R}_\sigma$,

$$w_\sigma(z) = \frac{4\sigma^{\frac{\sigma}{1+\sigma}} ze^{g_\sigma(z)}}{(1+\sigma)(1+z+g_\sigma(z))^{\frac{2}{1+\sigma}}(1-z+g_\sigma(z))^{\frac{2\sigma}{1+\sigma}}}.$$

Note that as $\sigma \to 0+$, this converges to ze^{1-z}, for $\operatorname{Re} z < 1$, the function defining the Szegő curve. Let

$$S_\sigma = \left\{z : |\arg(z)| > \cos^{-1}\left(\frac{1-\sigma}{1+\sigma}\right)\right\}.$$

Using steepest descent, they proved:

Theorem *Let* $\lim_{j\to\infty} n_j = \infty$ *and* $\lim_{j\to\infty}(m_j/n_j) = \sigma$, *for some* $0 < \sigma \leq \infty$.
(I) $\hat{z}$ *is a limit point of zeros of* $\{[n_j/m_j]((n_j+m_j)z)\}_{j=1}^{\infty}$ *iff*

$$\hat{z} \in \overline{S_\sigma} \cap \{z : |z| \leq 1 \text{ and } |w_\sigma(z)| = 1\}.$$

(II) If $\sigma > 0$, $\hat{z}$ *is a limit point of poles of* $\{[n_j/m_j]((n_j+m_j)z)\}_{j=1}^{\infty}$ *iff*

$$\hat{z} \in \overline{\mathbb{C}\backslash S_\sigma} \cap \{z : |z| \leq 1 \text{ and } |w_\sigma(z)| = 1\}.$$

(III) $\hat{z}$ *is a limit point of nontrivial zeros of the remainders* $\left\{e^{(n_j+m_j)z} - [n_j/m_j]((n_j+m_j)z)\right\}_{j=1}^{\infty}$ *iff*

$$\hat{z} \in \{z : |z| \geq 1 \text{ and } |w_\sigma(z)| = 1\}.$$

In addition to this remarkable result, they established asymptotic densities of the proportion of zeros or poles on each subarc of the curves above. In many ways, this last paper marked the high point of analysis of zeros and poles of Padé approximants to e^z, and is still state of the art.

The mid 1970's marked another transition, to a series of results that are probably the most influential in Ed's research career. In 1976, the doyenne of approximators, G.G. Lorentz gave an invited address at a conference organized by Ed in Tampa. He presented results and conjectures about *incomplete polynomials*,

$$p_n(x) = \sum_{\theta n \leq k \leq n} c_{nk} x^k \tag{1}$$

where $\theta \in (0,1)$. The work of Ed, his collaborators, and others on this topic gave rise to a deep theory of potentials with external fields, that has applications in asymptotics of orthogonal polynomials, weighted approximation, random matrices, and many other topics.

The main theorem announced by G.G. Lorentz in 1976 on incomplete polynomials stated that if for $n \geq 1$, p_n is a polynomial of the form (1), with

$$|p_n(x)| \leq 1, \ x \in [0,1],$$

then

$$\lim_{n\to\infty} p_n(x) = 0,$$

uniformly in compact subsets of $\left(0, \theta^2\right)$. Lorentz then raised the question of whether the θ^2 is the largest number with this property. Saff and Varga not only established this fact, but went much further by determining which functions can be uniformly approximated on $[0,1]$ by such incomplete polynomials. Indeed, they proved in [49] the following:

Theorem *Let* $\theta \in (0,1)$. *Let* $f : [0,1] \to \mathbb{R}$ *be continuous, and not a polynomial of the form (1). Then a necessary and sufficient condition that* f *be the uniform limit as* $n \to \infty$ *of a sequence of incomplete polynomials of the form (1), is that* $f = 0$ *in* $\left[0, \theta^2\right]$.

This breakthrough was followed by a series of papers in which they study weighted Chebyshev polynomials that attain the infimum in

$$\inf_{p \text{ monic of degree } m} \|x^s p(x)\|_{L_\infty[0,1]},$$

and asymptotics of zeros of Jacobi polynomials $\left\{P_n^{(\alpha_n,\beta_n)}(x)\right\}$ under appropriate asymptotic conditions on $\{\alpha_n\}, \{\beta_n\}$, incomplete polynomials that vanish at both endpoints. There were additional collaborators on these topics, including Michael Lachance, Ed's first doctoral student, and Joe Ullman. The techniques became successively more sophisticated with subharmonic function theory and maximum principles playing a key role.

Here is a sample from [17]: let $\pi(s_1, s_2, m)$ denote the set of all polynomials of the form $(x-1)^{s_1}(x+1)^{s_2}q(x)$, where q is of degree $\leq m$. Let $-1 < a < b < 1$, and

$$\phi(z) = \frac{\sqrt{z-a} + \sqrt{z-b}}{\sqrt{z-a} - \sqrt{z-b}}, z \in \mathbb{C} \setminus [a, b]$$

the conformal map of the exterior of $[a, b]$ onto the exterior of the unit ball. Define for given θ_1, θ_2

$$G(z; \theta_1, \theta_2) = |\phi(z)| \left| \frac{\phi(z) - \phi(1)}{\phi(1)\phi(z) - 1} \right|^{\theta_1} \left| \frac{\phi(z) - \phi(-1)}{\phi(-1)\phi(z) - 1} \right|^{\theta_2},$$

and $G = 1$ on $[a, b]$. Note that G is harmonic in $\mathbb{C} \setminus [-1, 1]$, and has zeros of multiplicity θ_1 and θ_2 at 1 and -1 respectively. A fundamental bound is

Theorem *Let $p \in \pi(s_1, s_2, m)$ and set $n = s_1 + s_2 + m$. Then for $z \in \mathbb{C}$,*

$$|p(z)| \leq \|p\|_{L_\infty[-1,1]} G(z; s_1/n, s_2/n)^n.$$

Lachance, Saff and Varga studied the set $\Lambda(\theta_1, \theta_2)$ in the complex plane, where $|G| < 1$, where sequences of incomplete polynomials decay exponentially. It looks like two tennis rackets stuck together at the handles. Tennis racket shaped regions would become a recurring subtheme in Ed's work on incomplete polynomials. The threesome also studied constrained Chebyshev polynomials, their zeros, and asymptotics.

1980 saw another groundbreaking paper, with Joe Ullman and Richard Varga [43], where they obtained an explicit expression for the equilibrium density associated with incomplete polynomials, by first solving a discrete energy problem, showing that the solution involves Jacobi polynomials, and then taking limits. They also applied these to obtain the asymptotic zero distribution of weighted Chebyshev polynomials.

There are many high points in Ed's research career, but his 1984 paper with Hrushikesh Mhaskar [23] must rank at the very top of the high points. This seminal and celebrated paper and its sequels, along with papers of EA Rakhmanov, laid the groundwork for a comprehensive analysis of orthogonal and extremal polynomials for exponential weights. It is in this paper that they introduced the Mhaskar-Rakhmanov-Saff number, established nth root asymptotics of orthogonal polynomials, and employed potential theory with external fields. This paper is definitely worth discussing in some detail.

Let $\alpha > 0$, $W_\alpha(x) = e^{-|x|^\alpha}$, let $\phi(z) = z + \sqrt{z^2 - 1}$, and define the *Ullman* (or *Nevai-Ullman*) distribution

$$v_\alpha(t) = \frac{\alpha}{\pi} \int_{|t|}^{1} \frac{s^{\alpha-1}}{\sqrt{s^2 - t^2}} ds, t \in [-1, 1].$$

Let

$$\lambda_\alpha = \frac{\Gamma(\alpha)}{2^{\alpha-2}\Gamma(\alpha/2)^2}$$

and define the weighted Green's function

$$G_\alpha(z) = \exp\left(\lambda_\alpha\left[\int_{-1}^{1}\log|z-t|\,v_\alpha(t)\,dt + \log 2 + \frac{1}{\alpha} - \log|\phi(z)| - \frac{|z|^\alpha}{\lambda_\alpha}\right]\right).$$

Using the maximum principle for subharmonic functions and other tools, they proved a basic majorization theorem:

Theorem *For any polynomial P of degree $\leq n$, for $a>0$, and all $z\in\mathbb{C}$,*

$$W_\alpha(|z|)\,|P_n(z)| \leq G_\alpha\left(\frac{z}{a}\right)^{a^\alpha}\left|\phi\left(\frac{z}{a}\right)\right|^n \|P_n W_\alpha\|_{L_\infty[-a,a]}.$$

By analyzing G_α, they obtain the famous Mhaskar-Saff identity: define the nth *Mhaskar-Rakhmanov-Saff number*

$$a_n(\alpha) = (n/\lambda_\alpha)^{1/\alpha}.$$

Theorem *For polynomials P_n of degree $\leq n$,*

$$\|P_n W_\alpha\|_{L_\infty(\mathbb{R})} = \|P_n W_\alpha\|_{L_\infty[-a_n(\alpha),a_n(\alpha)]}$$

and

$$|P_n W_\alpha|(x) < \|P_n W_\alpha\|_{L_\infty(\mathbb{R})}, |x| > a_n(\alpha).$$

Such a result is known as an infinite-finite range inequality. Geza Freud and others had established such results, but the novelty here is that $a_n(\alpha)$ is sharp. They also obtained asymptotics of extremal errors and polynomials, and their zero distribution: for $n\geq 1$, define

$$E_n(\alpha) = \inf\left\{\|W_\alpha(x)(x^n - p(x))\|_{L_\infty(\mathbb{R})} : \deg(p) < n\right\},$$

and let $T_{n,\infty}$ denote a monic polynomial of degree n attaining the infimum, so that it is a weighted Chebyshev polynomial:

Theorem *(I)*

$$\lim_{n\to\infty} n^{-1/\alpha} E_n(\alpha)^{1/n} = \frac{1}{2}\left(\frac{1}{e\lambda_\alpha}\right)^{1/\alpha}.$$

(II) Uniformly for z in compact subsets of $\mathbb{C}\backslash[-1,1]$,

$$\lim_{n\to\infty} a_n(\alpha)^{-1}\left|T_{n,\infty}(a_n(\alpha)z)\right|^{1/n} = \exp\left(\int_{-1}^{1}\log|z-t|\,v_\alpha(t)\,dt\right).$$

(III) Let $N_n([c,d])$ denote the total number of zeros of $T_{n,\infty}(a_n(\alpha)z)$ in an interval $[c,d]\subset[-1,1]$. Then

$$\lim_{n\to\infty}\frac{1}{n}N_n([c,d]) = \int_c^d v_\alpha(t)\,dt.$$

It is difficult to top the 1984 paper, but their 1985 paper "Where Does the Sup Norm of a Weighted Polynomial Live" [24] does that, greatly extending the theory, and laying the groundwork for a potential theory with external fields, replete with a Frostman type theorem. It deals with weighted polynomials of the form $w^n P_n$. This is a natural extension of the 1984 paper, since

$$W_\alpha(x)^n = W_\alpha\left(n^{1/\alpha}x\right).$$

It is also here that the notion of an *admissible weight* was first defined, though it was later generalized.

Definition *Let* $w:\mathbb{R}\to[0,\infty)$. *We say that* w *is an* admissible weight function *if all of the following hold:*
(i) $\Sigma =$*supp*$[w]$ *has positive capacity*
(ii) $w_{|\Sigma}$ *is continuous on* Σ
(iii) $Z=\{x\in\Sigma : w(x)=0\}$ *has zero capacity*
(iv) If Σ *is unbounded, then* $|x|\,w(x)\to 0$ *as* $x\to\infty, x\in\Sigma$.

Their fundamental extension of Otto Frostman's classical theorem of potential theory involves an energy integral

$$I_w(\mu)=\int\int\log\frac{1}{|x-t|}d\mu(x)\,d\mu(t)+2\int Q(t)\,d\mu(t)\,, \tag{2}$$

where μ is a probability measure on Σ. We let $\mathcal{M}(\Sigma)$ denote the set of all such probability measures.

Theorem *Let* $w=e^{-Q}$ *be an admissible weight with support* Σ*, and let*

$$V_w=\inf\{I_w(\mu):\ \mu\in M(\Sigma)\}\,.$$

(a) V_w *is finite.*
(b) There exists a unique measure $\mu_w\in\mathcal{M}(\Sigma)$*, called the equilibrium measure for* w*, such that* $I_w(\mu_w)=V_w$.
(c) $\mathcal{S}_w=$supp$[\mu_w]$ *is compact and* $\mathcal{S}_w\subset\Sigma\backslash Z$ *has positive capacity.*
(d) Let

$$U^{\mu_w}(x)=\int\log\frac{1}{|x-t|}d\mu_w(t)$$

and

$$F_w=V_w-\int Qd\mu_w.$$

Then

$$\begin{aligned} U^{\mu_w}(x)+Q(x) &\geq F_w,\text{ q.e. on }\Sigma;\\ U^{\mu_w}(x)+Q(x) &\leq F_w,\text{ on }\mathcal{S}_w. \end{aligned}$$

(e) For any polynomial P_n *of degree* $\leq n$ *and all* $z\in\mathbb{C}$,

$$|P_n(z)|\leq\|w^nP_n\|_{L_\infty(\mathcal{S}_w)}\exp\left(n\left[-U^{\mu_w}(z)+F_w\right]\right).$$

As a consequence, one can majorize weighted polynomials:

Theorem *Let* $w=e^{-Q}$ *be an admissible weight function with support* Σ.
(I) For $n\geq 1$ *and polynomials* P *of degree* $\leq n$*, we have for q.e.* $x\in\Sigma$,

$$w(x)^n\,|P(x)|\leq\|w^nP\|_{L_\infty(\mathcal{S}_w)}.$$

If also Σ *is a regular set,*

$$\|w^nP\|_{L_\infty(\Sigma)}=\|w^nP\|_{L_\infty(\mathcal{S}_w)}.$$

(II) $\mathcal{S}_w$ *maximizes the* F *functional*

$$F(K)=\log\operatorname{cap}(K)-\int_K Qd\nu_K,$$

where the sup is taken over all compact subsets K of $\Sigma\backslash Z$ with equilibrium density ν_K.
(III)

$$E_n(w) = \inf\left\{\|w(x)^n[x^n - p(x)]\|_{L_\infty(\Sigma)} : \deg(P) \le n-1\right\} \tag{3}$$

satisfies

$$E_n(w)^{1/n} \ge \exp(F(\mathcal{S}_w)).$$

Under additional conditions, they could say something about the structure of $\mathcal{S}_w$:

Theorem *Assume that $\Sigma\backslash Z$ is the finite union of nondegenerate intervals, and Q is convex in each of the components of $\Sigma\backslash Z$.*
(I) $\mathcal{S}_w$ is the finite union of nondegenerate disjoint closed intervals.
(II)

$$\lim_{n\to\infty} E_n(w)^{1/n} = \exp(F(\mathcal{S}_w)).$$

(III) If $T_{n,\infty}$ is the monic polynomial of degree n attaining the infimum in (3), then

$$\lim_{n\to\infty} |T_{n,\infty}(z)|^{1/n} = \exp(-U^{\mu_w}(z)),$$

and the zero counting measure of $T_{n,\infty}$ converges weakly to μ_w as $n \to \infty$.

We have already seen above that Saff and Varga studied approximation of continuous functions by incomplete polynomials. In a 1983 survey paper [38], Ed formulated this problem for weighted polynomials involving W_α:

Conjecture *Let $\alpha > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous with $f(x) = 0$ for $|x| \ge \lambda_\alpha^{-1/\alpha}$. Do there exist for $n \ge 1$, polynomials P_n of degree $\le n$ such that*

$$\lim_{n\to\infty} \|P_n W_\alpha^n - f\|_{L_\infty(\mathbb{R})} = 0?$$

This was subsequently generalized and since there does not seem to be a single formulation, we pose it in the following way:

Saff's Weighted Approximation Problem *Let w be an admissible weight. Find a "smallest" closed set $S \subset \Sigma$ with the following property: for every function $f : \Sigma \to \mathbb{R}$ that is continuous and is 0 in $\Sigma\backslash S$, there exist for $n \ge 1$, polynomials P_n of degree $\le n$ such that*

$$\lim_{n\to\infty} \|P_n w^n - f\|_{L_\infty(\Sigma)} = 0.$$

Mhaskar and Saff were the first to prove Saff's Conjecture for Laguerre weights [25], and as a consequence, for the Hermite weight $W_2(x) = \exp(-x^2)$. It was a great privilege of the author's to work with Ed on this problem - we positively resolved it for the weights $W_\alpha, \alpha > 1$, with S taken as a suitably scaled Mhaskar-Saff interval. For $\alpha = 1$, Totik and the author showed that the same S works, while for $\alpha < 1$, one needs the approximated function f to vanish at 0. All this was part of a long series of papers, with important contributions by many authors, notably Kuijlaars, and especially, Totik. The state of the art appears in [50] and [51].

One important consequence of the work on Saff's approximation problem, was the resolution of Geza Freud's 1976 Conjecture, on the recurrence coefficients of orthogonal polynomials for the weight W_α. Let $p_n(W_\alpha, x) = \gamma_n(W_\alpha) x^n + ...$ denote the nth orthonormal polynomial for the weight W_α, so that for $m, n \geq 0$,

$$\int_{-\infty}^{\infty} p_n(W_\alpha, x) p_m(W_\alpha, x) W_\alpha^2(x) dx = \delta_{mn}.$$

Inasmuch as W_α is an even weight, the recurrence relation for $\{p_n(W_\alpha, \cdot)\}$ takes the form

$$xp_n(W_\alpha, x) = \frac{\gamma_n(W_\alpha)}{\gamma_{n+1}(W_\alpha)} p_{n+1}(W_\alpha, x) + \frac{\gamma_{n-1}(W_\alpha)}{\gamma_n(W_\alpha)} p_{n-1}(W_\alpha, x), n \geq 1.$$

Freud conjectured that

$$\lim_{n\to\infty} \frac{\gamma_{n-1}(W_\alpha)}{\gamma_n(W_\alpha)} / a_n(a) = \frac{1}{2}.$$

The author, Mhaskar, and Saff showed [19] that the conjecture is true for W_α for all $\alpha > 0$, and in fact admits a generalization to a larger class of weights. Later applications of Saff's weighted polynomial approximation problem included Szegő type asymptotics for orthogonal and extremal polynomials [20].

The third of the four classic papers of Mhaskar and Saff posed and answered the question "Where Does the L_p-Norm of a Weighted Polynomial Live?" [26]. They proved L_p analogues of the L_∞ results in their 1984 paper. We shall not formulate them precisely: suppose that $0 < p < \infty$, that w is "strongly admissible", and satisfies a few other conditions, that are certainly satisfied for $W_\alpha, \alpha > 0$. A first result is a restricted range inequality, that we can chuck away a tail in the L_p norm: let $\eta > 0$. Then there exist $c_1, c_2 > 0$ and a compact set Δ with measure $< \eta$, such that for $n \geq 1$ and $\deg(P) \leq n$,

$$\|w^n P\|_{L_p(\Sigma)} \leq \left(1 + c_1 e^{-c_2 n}\right) \|w^n P\|_{L_p(\mathcal{S}_w \cup \Delta)}.$$

Let

$$E_{n,p}(w) = \inf\left\{\|(x^n - P(x)) w(x)\|_{L_p(\Sigma)} : \deg(P) < n\right\}$$

and let $T_{n,p}(x) = x^n + ...$ be a monic polynomial of degree n attaining the inf. Then

$$\lim_{n\to\infty} E_{n,p}(w)^{1/n} = \exp(F(\mathcal{S}_w)).$$

Moreover, if I is an interval containing Σ, then uniformly in compact subsets of $\mathbb{C} \backslash I$

$$\lim_{n\to\infty} |T_{n,p}(z)|^{1/n} = \exp[-U^{\mu_w}(z)],$$

and the zero counting measures of $T_{n,p}$ converge weakly to μ_w as $n \to \infty$.

The fourth and final classic paper of Mhaskar and Saff, [27] deals with weighted analogues of capacity and transfinite diameter. For an admissible weight function w defined on a closed subset E of $\mathbb{C}$, the authors define a weighted capacity $\text{cap}(w, E)$, a weighted nth diameter $\delta_n(w, E)$, and a weighted Chebyshev constant $\text{cheb}(w, E)$, as follows. Let $I_w(\sigma)$ denote the energy integral associated with a probability measure σ supported on E, as at (2). The $w-$modified capacity of E is

$$\text{cap}(w, E) = \exp\left(-\inf_{\sigma \in \mathcal{M}(E)} I_w(\sigma)\right).$$

The nth diameter is

$$\delta_n(w,E)=\sup_{z_1..z_n\in E}\left\{\prod_{1\le i<j\le n}|z_j-z_i|\,w(z_j)\,w(z_i)\right\}^{\frac{2}{n(n-1)}}$$

and the $w-$modified transfinite diameter is

$$\tau(w,E)=\lim_{n\to\infty}\delta_n(w,E).$$

The $w-$modified Chebyshev constant of E is

$$\operatorname{cheb}(w,E)=\lim_{n\to\infty}\left(\inf_{P\text{ monic of deg }n}\|w^nP\|_{L_\infty(E)}\right)^{1/n}.$$

Under mild conditions on w, they prove that

$$\tau(w,E)=\operatorname{cap}(w,E)$$

while

$$\operatorname{cheb}(w,E)=\exp\left(\int Qd\mu_w\right)\operatorname{cap}(w,E),$$

where $Q=\log\frac{1}{w}$ and μ_w is the equilibrium measure for the external field Q. These identities are the weighted analogues of the classical Fekete-Szegő Theorem for the unweighted case. They also prove a host of properties for their weighted capacity and provide many examples.

The fundamental works above deserved a thorough exposition. It came in 1997, in the form of Saff and Totik's celebrated monograph "Logarithmic Potentials with External Fields" [42]. It provides a masterly and complete treatment of this circle of ideas that blossomed in the 1980's, and even its treatment of classical unweighted potential theory, especially balayage, is the best in the literature. It also contains a host of new examples and results, providing the definitive and sharpest form of almost every theorem. One measure of its impact is its over 560 citations according to MathSciNet.

Ed's devotion to weighted approximation did not deter him from working on another favorite topic: zeros of best approximating polynomials and rational functions, and their alternation or extreme points. This has historical roots in the theorems of Jentsch and Szegő for the partial sums $\{s_n\}$ of the Maclaurin series of a function f analytic in the unit ball, but with some singularity on the unit circle. Jentsch showed that every point on the unit circle is a limit point of zeros of $\{s_n\}$, while Szegő showed that for an infinite subsequence of integers, the zero counting measures of s_n converge weakly as $n\to\infty$ to normalized Lebesgue measure on the unit circle.

It was in 1988, that Ed formulated [39] his principle of contamination, although he and his collaborators had earlier, for example in [1], proved results illustrating special cases:

Principle of Contamination *Let E be a compact set with connected and regular complement. Let f be continuous on E and analytic in E^o. If f has one or more singularities on the boundary of E, then these adversely affect the behavior over the whole boundary of E, at least for a subsequence of the best approximants p_n^* to f on E.*

FIGURE 6. Ed, Vili Totik, and Herbert Stahl at Oberwolfach in 2011

A 1986 Jentzsch type result with Hans-Peter Blatt [2] illustrates the principle:

Theorem *Let E be compact with connected and regular complement. Suppose f is continuous in E, analytic in E^o, but not on E. Assume, moreover, that f does not vanish identically on any component of E^o. Let p_n^* be the polynomial of best uniform approximation to f on E. Then each point in ∂E is a limit point of zeros of $\{p_n^*\}$. Moreover, if we write*

$$p_n^*(z) = a_n^* z^n + ..., \, n \geq 1, \tag{4}$$

then the non-analyticity of f on E is equivalent to

$$\limsup_{n\to\infty} |a_n^*|^{1/n} = \frac{1}{\operatorname{cap}(E)}.$$

Blatt and Saff also established related results for near-best approximants, and weak convergence of the counting measures of a subsequence of $\{p_n^*\}$, although there were more powerful results in a subsequent paper of the two authors and Simkani [3]:

Theorem *Let E be a compact set in the plane with positive capacity, and equilibrium measure μ_E. Let f be continuous on E and analytic in E^o, but not on E. With the notation (4), assume that $\mathcal{N}$ is an infinite sequence of integers such that*

$$\lim_{n\to\infty, n\in\mathcal{N}} |a_n^*|^{1/n} = \frac{1}{\operatorname{cap}(E)}.$$

Then as $n \to \infty, n \in \mathcal{N}$, the zero counting measures of p_n^* converge weakly to μ_E.

In the case when f is analytic on E, but not entire, the authors show that a subsequence of the zero counting measures converges weakly to the equilibrium distribution of the largest E_ρ inside which f is analytic. The authors also prove analogues for best rational approximants with a bounded number of poles, and for

FIGURE 7. Ed's 60th Birthday Conference in 2004

best polynomial approximations in L_p norm. In later work with Rene Grothmann, [10], Ed showed that weak convergence does not hold for the full sequence of zero counting measures. Together with Andras Kroo [14], Ed showed a Jentzsch type theorem for extreme points:

Theorem *Let $E \subset \mathbb{C}$ be a compact set, with connected and regular complement. If f is continuous on E and analytic in its interior, then the set of extreme points of the best polynomial approximants, namely*

$$A_n = \left\{ z \in E : \left| f - p_n^* \right| (z) = \| f - p_n^* \|_{L_\infty(E)} \right\}$$

becomes dense in ∂E for a subsequence:

$$\liminf_{n \to \infty} \left(\sup_{\zeta \in \partial E} \inf_{z \in A_n} |\zeta - z| \right) = 0.$$

They also give an example of an entire function for which the lim sup is positive.

In a 1989 paper with Blatt and Totik, [4], Ed established a Szegő type result: there is a subsequence of the best polynomial approximants to a function f on a compact set E such that the zero counting measures of a Fekete set formed from their extreme points, converges weakly to the equilibrium distribution μ_E of E, as $n \to \infty$. Remarkably, similar results hold for polynomials that minimize a weighted norm under very mild conditions on the weight w.

Ed kept up his interest in zeros of orthogonal polynomials through the 1990's. A particularly nontrivial setting is that of *Bergman polynomials.* These are polynomials orthogonal with respect to Lebesgue measure over some region G in the plane. Of course, Lebesgue measure is often replaced by something more general. In an elegant 1990 survey [40], Ed begins by proving that when the convex hull of the support of the underlying measure is not a straight line segment, all the zeros lie in the interior of the convex hull, not on the boundary. Of course how they distribute themselves inside that interior is intriguing.

In a 2003 paper, Ed and Victor Maymeskul [22] analyzed the zeros of $\{Q_n\}_{n=0}^{\infty}$ when G is a regular $N-$gon G_N. They proved some conjectures of Eiermann and Stahl: for $N = 3, 4$, all the zeros are located exactly on the diagonals of G_N. They also show that for fixed j, all real zeros of $\{Q_{N\ell+j}\}_{\ell=1}^{\infty}$ interlace on $(0, 1)$. For $N \geq 5$, results of Levin, Saff, Stylianopoulos, [18] Andrievskii, and Blatt, show that the zeros are not all on the diagonals, and moreover, a subsequence of the zero counting measures of $\{Q_n\}$ converges weakly to the equilibrium measure for ∂G_N. In a 2009 paper [11], Ed and Bjorn Gustafsson, Mihai Putinar, and Nikos Stylianopoulos investigated Bergman polynomials on an "archipelago", that is a fine union of disjoint Jordan domains. They obtained bounds and asymptotics for leading coefficients, distribution of zeros of the orthogonal polynomials, and investigated how to reconstruct the shape of the archipelago.

Ed also contributed to the theory of orthogonal polynomials associated with purely discrete measures. In a 1997 paper [7], Peter Dragnev and Ed advanced potential theory in a constrained setting. In particular, they showed how to convert constrained energy problems into unconstrained ones with an external field, and deduced asymptotics for Krawtchouk polynomials.

1994 marks another transition in Ed's research career, the publication of his first paper on distribution of points on a sphere, joint with E.A. Rakhmanov and Y.M. Zhou [28]. This basic question has connections to the over 100 year old Thomson's problem on arrangement of elections in an atom; to Nobel prize winning work on stable carbon-60 molecules; to studies of Smale and Shub on computational complexity; and to problems of spherical designs and numerical integration.

Let $d \geq 2, N \geq 2$ and $\omega_N = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N\}$ be a set of N points on the unit sphere S^d in $\mathbb{R}^{d+1}$. The $s-$energy associated with ω_N is defined by

$$E_d(s, \omega_N) = \sum_{1 \leq i < j \leq N} \|\mathbf{x}_i - \mathbf{x}_j\|^{-s} \text{ if } s \neq 0,$$

while

$$E_d(0, \omega_N) = \sum_{1 \leq i < j \leq N} \log \frac{1}{\|\mathbf{x}_i - \mathbf{x}_j\|}.$$

The extremal energy for a given N and s

$$\begin{aligned} \mathcal{E}_d(s, N) &= \inf_{\omega_N \subset S^d} E_d(s, \omega_N) \text{ if } s \geq 0; \\ \mathcal{E}_d(s, N) &= \sup_{\omega_N \subset S^d} E_d(s, \omega_N) \text{ if } s < 0. \end{aligned}$$

Understandably, the sphere S^2 in three dimensions, has received the most attention. The determination of $\mathcal{E}_2(1, N)$ is called J.J. Thomson's Problem. It is known that $\mathcal{E}_2(-2, N) = N^2$ while as $s \to \infty$, the problem becomes the Best-Packing Problem on the sphere, also called Tammes' problem. This asks for the spherical radius of N identical spherical caps that can be packed onto the surface of the unit sphere.

Saff, Rakhmanov, and Zhou give improved estimates for $\mathcal{E}_2(0, N)$ as well asymptotics for $E_2(s, \omega_N)$ for special point sets. They present conjectures for the asymptotics of $\mathcal{E}_2(s, N)$ as $N \to \infty$, based on numerical experiments. For example, they conjecture that for some constants B_s, C_s, as $N \to \infty$

$$\mathcal{E}_2(0, N) = -\frac{N^2}{4} \log\left(\frac{4}{e}\right) - \frac{1}{4} N \log N + B_0 N + C_0 \log N + o(1)$$

while for $s \in (-2,2) \setminus \{0\}$,

$$\mathcal{E}_2(s,N) = \frac{2^{-s}}{2-s}N^2 + B_s N^{1+s/2} + C_s N^{s/2} + O\left(N^{-1+s/2}\right).$$

They prove theoretically that

$$\mathcal{E}_2(0,N) = -\frac{N^2}{4}\log\left(\frac{4}{e}\right) - \frac{1}{4}N\log N + B_{0,N}N$$

where $\{B_{0,N}\}$ is bounded above and below by certain negative constants. In a subsequent paper [29], they prove that for $s=0$, the equilibrium points $\{\mathbf{x}_i\}$ for a given N, are well-separated in the sense that

$$\delta_n = \min_{i\neq j}\|\mathbf{x}_i - \mathbf{x}_j\| \geq \frac{3}{5}\frac{1}{\sqrt{N}}, N \geq 2.$$

In the late 1990's, Ed began a collaboration on this topic with Arno Kuijlaars. Define the energy integral

$$I_{d,s}(\mu) = \int\int \frac{1}{\|x-y\|^s}d\mu(x)\,d\mu(y),$$

for probability measures supported on S^d. The measure that minimizes this for $0<s<d$ is normalized Lebesgue measure σ on S^d, and the minimum is

$$V_d(s) = I_{d,s}(\sigma) = \frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma(d-s)}{\Gamma\left(\frac{d-s+1}{2}\right)\Gamma\left(d-\frac{s}{2}\right)}.$$

Building on earlier results, Ed and Arno proved [16]:

Theorem *Let $d \geq 2$.*
(I) Let $0<s<d$. There exists $C>0$ such that

$$\mathcal{E}_d(s,N) \leq \frac{1}{2}V_{d,s}N^2 - CN^{1+s/d}.$$

(II) Let $s>d$. Then

$$C_1 N^{1+s/d} \leq \mathcal{E}_d(s,N) \leq C_2 N^{1+s/d}.$$

(III)

$$\lim_{N\to\infty}\left(N^2\log N\right)^{-1}\mathcal{E}_d(d,N) = \frac{1}{2d}\frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma\left(\frac{d}{2}\right)\sqrt{\pi}}.$$

They deduce separation estimates:

Corollary *Let $s \geq d \geq 2$ and $\omega_N = \{\mathbf{x}_i\}_1^N$ minimize $E_d(s,N)$. Then*

$$\min_{i\neq j}\|\mathbf{x}_i - \mathbf{x}_j\| \geq CN^{-1/d} \text{ if } s>d$$

and

$$\min_{i\neq j}\|\mathbf{x}_i - \mathbf{x}_j\| \geq CN^{-1/d}(\log N)^{-1./d} \text{ if } s=d.$$

In a paper published in Mathematical Intelligencer in 1997 [15], Ed and Arno provided a lively introduction to the topic. In addition to asymptotics and estimates, they discuss the geometry of extremal set of points, using the notion of

Dirichlet (or Voronoi) cells: let ω_N be a set of N points on the sphere. The associated Dirichlet cells $D_j, 1 \leq j \leq N$, are defined by

$$D_j = \left\{ \mathbf{x} \in S^2 : |\mathbf{x} - \mathbf{x}_j| = \min_{1 \leq k \leq N} |\mathbf{x} - \mathbf{x}_k| \right\}.$$

The Dirichlet cells are closed subsets of S^2 whose union is the sphere, and with $D_j \cap D_i$ having empty interior if $i \neq j$.

For large numbers N of points, numerical observations indicate that at least 12 of the Dirichlet cells are pentagons, while the vast majority are hexagons. For example, the classical soccer ball has 20 hexagonal faces, and 12 pentagonal faces. Remarkably, the number 12 follows from the Euler characteristic formula $F - E + V = 2$. The authors go onto review asymptotics, spacing, best packing, spherical designs, and spiral points. In a 2004 joint paper [21] with Andre Martinez-Finkelshtein, Viktor Maymeskul, and Evgenii Rakhmanov**,** Ed investigated distribution of points on curves in $\mathbb{R}^d$, establishing asymptotics for extremal energies and analyzing distribution of the extremal points. In a 2007 paper, Peter Dragnev and Ed [8] proved that extremal points are well separated:

Theorem *Let S^d denote the unit sphere in $\mathbb{R}^{d+1}$, let $d-2 < s < d$, $N \geq 2$, and ω_N be points on the sphere minimizing the Riesz $s-$energy. There exist explicit constants (described in terms of beta functions) $A_{d,s} > 0$ such that for all N,*

$$\min_{i \neq j} |\mathbf{x}_i - \mathbf{x}_j| \geq A_{d,s} N^{-1/d}.$$

It was natural that one should progress from distributing points on the sphere to distributing points on manifolds and even rectifiable sets, including a weight in the energy. Motivations include computer aided geometric design, finite element tesselations, and statistical sampling. One of Ed's main collaborators on this topic has been Doug Hardin. They presented a beautiful survey on this in a 2004 article [12]. The theory was first developed by Ed and Doug in [13] and further developed with Sergiy Borodachov in [5]. To state these results we introduce the following notation.

Let A be a compact set in $\mathbb{R}^{d'}$ whose d-dimensional Hausdorff measure $\mathcal{H}_d(A)$ is finite. Let $w : A \times A \to [0, \infty)$ and $s > 0$. The weighted Riesz $s-$energy of a point set $\omega_N \subset A$ is

$$E_s^w(\omega_N) = \sum_{1 \leq i < j \leq N} \frac{w(x_i, x_j)}{|x_i - x_j|^s}.$$

Assume that w is symmetric, $w(y, x) = w(x, y)$. We set

$$\mathcal{E}_s^w(A, N) = \inf \{ E_s^w(\omega_N) : \omega_N \subset A, |\omega_N| = N \}.$$

When $w = 1$, we drop the superscript.

Theorem *Let A be a compact set in $\mathbb{R}^{d'}$ which is d-rectifiable, that is, is the image of a bounded set under a Lipschitz mapping. Then for $s > d$,*

$$\lim_{N \to \infty} \frac{\mathcal{E}_s(A, N)}{N^{1+s/d}} = \frac{C_{s,d}}{\mathcal{H}_d(A)^{s/d}}.$$

Here $C_{s,d}$ is a positive constant independent of A.

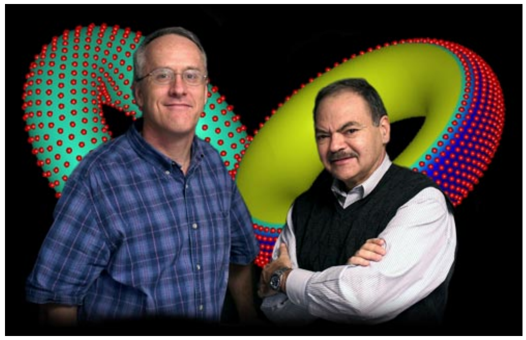

FIGURE 8. Ed and Doug Hardin with Tori

The constant $C_{s,d}$ is known to be equal to $2\zeta(s)$ for $d=1$, but for $d>1$ its value remains a fascinating open problem, especially for the special dimensions $d=2,8,$ and 24, where it is conjectured to be given by the Epstein zeta function for the equi-triangular, E_8, and the Leech lattices, respectively.

For appropriate weights, define the weighted Hausdorff measure of Borel sets $B\subset A$ by

$$\mathcal{H}_d^{s,w}(B)=\int_B w(x,x)^{-d/s}\,d\mathcal{H}_d(x)$$

and its normalized form

$$h_d^{s,w}(B)=\mathcal{H}_d^{s,w}(B)/\mathcal{H}_d^{s,w}(A).$$

Theorem *Let A be a compact set in $\mathbb{R}^{d'}$ which is d-rectifiable. Suppose $s>d$ and that w is a "CPD weight". Then*

$$\lim_{N\to\infty}\frac{\mathcal{E}_s^w(A,N)}{N^{1+s/d}}=\frac{C_{s,d}}{\mathcal{H}_d^{s,w}(A)^{s/d}}.$$

The counting measures of extremal sets converge weakly to $h_d^{s,w}$.

A particularly interesting case is that of the torus, with the weight $w=1$, which Doug and Ed brought to life in the following way: how do you evenly distribute poppy seeds on a bagel? This formulation led to Ed being interviewed by National Public Radio, and to articles in popular scientific magazines such as Science & Vie. They showed that when s is small the equilibrium points act as if they are responding to a long-range force, distributing themselves on the outer ring of the torus. When s is large, they act as if they are subject to a short-range force, eventually distributing themselves all over the torus. Ed, Doug and Johann Brauchart explored distribution of points on more general sets of revolution in [6].

Distribution of points on spheres and manifolds remains a major focus of Ed's research. It will be the subject of a forthcoming monograph by Borodachov, Hardin and Saff - one that should do for this area what Saff-Totik did for potential theory. We look forward to many more great papers and monographs in coming years!

There once was a guy named Ed,

About whom an awful lot can be said

He began as a Yellow Jacket
With the aim of making a packet
But off to Maryland he went
And by Joseph Walsh was he bent

Walsh made him a rational fellow
so de Montessus should have made him mellow
But Varga's exponential obsession
For a while was his virtual profession

Still, somehow he felt incomplete
Could GG Lorentz have him beat?
His potential was not fulfilled
Until Mhaskar and he double billed

The Mhaskar-Rakhmanov-Saff number
caught everyone else aslumber
It solved so many problems extremal
That even Freud was left adrenal

Before Ed could totally weary
Of the power of potential theory
He created a principle of contamination
Which, upon close examination,
Gave a comprehensive solution
to zero asymptotic distribution

And then Ed came around to the ball
To see how the poppyseeds fall
His work on that foreign sport soccer
Made some think he's gone off his rocker

But you see there is a common theme
That all his research does redeem
It is a great unifying synergy
Of polynomials, rationals, and potential energy

So as Ed reaches three score and ten
We know he's the wisest of men
With books, papers, students, and more,
An all rounded person at core.

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