THE SIZE OF $(q;q)_n$ FOR q ON THE UNIT CIRCLE

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Abstract. There is increasing interest in $q-$series with $|q|=1$. In analysis of these, an important role is played by the behaviour as $n\to\infty$ of

$$(q;q)_n=(1-q)(1-q^2)...(1-q^n).$$

We show, for example, that for almost all q on the unit circle

$$\log|(q;q)_n|=O(\log n)^{1+\varepsilon}$$

iff $\varepsilon>0$. Moreover, if $q=\exp(2\pi i\tau)$ where the continued fraction of τ has bounded partial quotients, then the above relation is valid with $\varepsilon=0$. This provides an interesting contrast to the well known geometric growth as $n\to\infty$ of

$$\|\ (q;q)_n\ \|_{L_\infty(|q|=1)}\ .$$

1. Statement of Results

There are a growing number of applications of $q-$series with $|q|=1$ and $q\neq 1$ in number theory, Pade approximation, continued fractions, ... [3-7], [15-17], [19-20], [22-24]. In analysis of a continued fraction of Ramanujan [16], the author was confronted with the need to analyse the behaviour as $n\to\infty$ of

$$(q;q)_n:=(1-q)(1-q^2)...(1-q^n) \tag{1.1}$$

for q on the unit circle. Obviously, the size of $(q;q)_n$will play an important role in the development of q-series for $|q|=1$. To first order, the answer to this question is provided by an old identity:

$$\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}=\exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n(1-q^n)}\right). \tag{1.2}$$

Hardy and Littlewood showed [10] that this identity remains valid even for $|q|=1$, that is both power series above have the same radius of convergence. Thus

$$\liminf_{n\to\infty}|(q;q)_n|^{1/n}=\liminf_{n\to\infty}|1-q^n|^{1/n}\ .$$

It follows easily from the elementary theory of diophantine approximation, that if

$$q=\exp(2\pi i\tau),\tau\in[0,1) \tag{1.3}$$

then for almost all $\tau\in[0,1]$ (and in fact except for τ in a set of Hausdorff dimension 0 and logarithmic dimension 2 [14]),

$$\liminf_{n\to\infty}|(q;q)_n|^{1/n}=1.$$

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Using estimates for quadrature sums and results from the theory of uniform distribution, one can then show [17] that for almost all τ,

$$\lim_{n\to\infty} |(q;q)_n|^{1/n} = 1. \tag{1.4}$$

(See also [18]). It turns out that this is a rather crude estimate, and one can show much more: for $x \in \mathbb{R}$, we let $[x]$ denote the greatest integer $\leq x$ and let a.e. denote a.e. with respect to linear Lebesgue measure on $[0,1]$.

Theorem 1.1
Let $(c_m)_{m=1}^{\infty}$ be an increasing sequence of positive numbers such that for some $\beta > 0$,

$$\limsup_{j\to\infty} c_{[j^{1+\beta}]}/c_j < \infty. \tag{1.5}$$

Let $q = \exp(2\pi i\tau)$.
(I) *The following are equivalent:*
(a) For a.e. τ,

$$|\log|(q;q)_n|| = O\left((\log n)\left(c_{[\log n]}\right)\right); \tag{1.6}$$

(b) For a.e. τ, and some $A = A(\tau)$

$$\log \frac{1}{|(q;q)_n|} \leq A\left((\log n)\left(c_{[\log n]}\right)\right); \tag{1.7}$$

(c)

$$\sum_{n=1}^{\infty} \frac{1}{nc_n} < \infty. \tag{1.8}$$

(II) Moreover, if

$$\sum_{n=1}^{\infty} \frac{1}{nc_n} = \infty, \tag{1.9}$$

then for a.e. τ, we have

$$\limsup_{n\to\infty} \left(\log \frac{1}{|(q;q)_n|}\right) / \left((\log n)\left(c_{[\log n]}\right)\right) = \infty. \tag{1.10}$$

Thus for example, if $\varepsilon > 0$ and we choose $c_n := (\log(n+1))^{1+\varepsilon}$, then we have for a.e. τ,

$$|\log|(q;q)_n|| = O\left((\log n)(\log\log n)^{1+\varepsilon}\right).$$

but

$$\log \frac{1}{|(q;q)_n|} \neq O\left((\log n)(\log\log n)\right).$$

In particular, $|(q;q)_n|$ will decay to 0 for infinitely many n faster than any negative power of n. In the other direction, we can show that $(q;q)_n$ grows almost as fast as n for infinitely many n:

Theorem 1.2
For irrational τ and $q = \exp(2\pi i\tau)$,

$$\limsup_{n\to\infty} \frac{\log|(q;q)_n|}{\log n} \geq 1. \tag{1.11}$$

The set of measure 0 omitted by the first theorem includes all algebraic irrationals τ. We now attend to these. Any irrational $\tau \in (0,1)$ has the continued fraction expansion

$$\tau = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + ... \tag{1.12}$$

where the positive integers a_j are the *partial quotients* of τ. For algebraic irrationals τ the $\{a_j\}$ are periodic and in particular are bounded.

Theorem 1.3
Let $q = \exp(2\pi i\tau)$, where τ has continued fraction (1.12).
(I) If

$$\sup_{j\geq 1} a_j = \infty, \tag{1.13}$$

then

$$\liminf_{n\to\infty} \log|(q;q)_n| = -\infty. \tag{1.14}$$

(II) If

$$\sup_{j\geq 1} a_j < \infty, \tag{1.15}$$

then

$$|\log|(q;q)_n|| = O(\log n), n \to \infty; \tag{1.16}$$

Thus if the partial quotients $(a_j)_{j=1}^{\infty}$ are bounded, we have for some $C_1, C_2 > 0$,

$$n^{-C_2} \leq |(q;q)_n| \leq n^{C_1}. \tag{1.17}$$

This has the consequence that the associated $q-$exponential functions

$$\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} \text{ and } \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}z^n}{(q;q)_n}$$

grow no faster than $(1-|z|)^{-C_3}$ as $|z| \to 1-$, for some $C_3 > 0$. It is an interesting problem to determine the smallest C_2 in (1.17). The above result leaves open the question of whether (1.14) holds in the presence of bounded partial quotients; our proofs show that there exists K such that if infinitely many $a_j \geq K$, then this is the case, and we are certain that it is true in general.

The above theorems provide an interesting contrast to old results of Sadler and Wright which show that $\| (q;q)_n \|_{L_\infty(|q|=1)}$grows geometrically, that is,

$$\lim_{n\to\infty} \| (q;q)_n \|_{L_\infty(|q|=1)}^{1/n} = 1.217... > 1.$$

For recent developments around this, and its relation to a problem of Erdös-Szekeres, see [1], [2]. For a first order discussion of the possible behaviour of the more general $q-$Pochhammer symbol $(a;q)_n$ as $n \to \infty$, see [3-7], [20].

This paper is organised as follows: in Section 2, we discuss the Ostrowski representation and present some technical lemmas. Our basic estimate for $(q;q)_n$ is presented in Section 3. In Section 4, we estimate a certain trigonometric sum. In Section 5, we prove the theorems. The basic ideas are the Ostrowski representation of a positive integer [8], [12] and elementary theory of diophantine approximation. We note that the more obvious approach of estimating $\log|(q;q)_n|$, namely treating it as a quadrature sum, and applying (for example) Koksma-Hlawka's inequality and discrepancy estimates yield essentially weaker results. Likewise use of identities such as (1.2) yield much weaker results.

We shall derive estimates on $(q;q)_n$ that hold for all n, with explicit numerical constants. This involves more work, but we believe the explicit constants will be useful in analysing Ramanujan's continued fraction [16]. If we required estimates that hold only for large n, then the size of most of the constants could be reduced, and some proofs could be shortened.

2. The Ostrowski Representation and Technicalities

In this section, we present some background material and two technical lemmas. We begin by recalling some elementary properties of continued fractions, all of which can be found in Lang [13]. Throughout, let $\tau \in (0, \frac{1}{2})$ be irrational with c.f. (1.12) and let

$$\frac{\pi_n}{\chi_n} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + ... \frac{1|}{|a_n} \tag{2.1}$$

denote the nth convergent. (We do not use the more customary notation p_n/q_n to avoid confusion between q and q_n). The recurrence relations for π_n, χ_n are

$$\chi_n = a_n \chi_{n-1} + \chi_{n-2}; \pi_n = a_n \pi_{n-1} + \pi_{n-2}, n \geq 1, \tag{2.2}$$

where $\chi_{-1} = 0; \chi_0 = 1; \pi_{-1} = 1; \pi_0 = 0$. Successive convergents satisfy for $n \geq 0$,

$$\chi_n \pi_{n-1} - \chi_{n-1} \pi_n = (-1)^n . \tag{2.3}$$

For $x \in \mathbb{R}$, we let

$$\begin{aligned} &[x] := \text{ greatest integer } \leq x; \\ &\{x\} := x - [x] \in [0, 1); \\ &\| x \| := \min_{j \in \mathbf{Z}} |x - j| = \min\{\{x\}, 1 - \{x\}\} \in [0, \tfrac{1}{2}]. \end{aligned}$$

Then for $n \geq 0$,

$$\| \chi_n \tau \| = (-1)^n (\chi_n \tau - \pi_n) = \frac{\delta_n}{\chi_{n+1}}, \delta_n \in (\frac{1}{2}, 1). \tag{2.4}$$

(This is true even for $n = 0$ since we assumed $\tau < \frac{1}{2}$). There is the best approximation property, valid for $n \geq 0$,

$$\| \chi_n \tau \| \leq \| k\tau \| \leq |k\tau - j| , 0 < k < \chi_{n+1}, j \in \mathbb{Z}. \tag{2.5}$$

(We have strict inequality unless $k = \chi_n$). Moreover, for $n \geq 1$,

$$\| \chi_n \tau \| < \| \chi_{n-1} \tau \|; \tag{2.6}$$

$$\| \chi_{n-1} \tau \| = a_{n+1} \| \chi_n \tau \| + \| \chi_{n+1} \tau \|; \tag{2.7}$$

and

$$a_{n+1} = \left[\frac{\| \chi_{n-1}\tau \|}{\| \chi_n \tau \|} \right]. \tag{2.8}$$

We remark that in the 1966 edition of Lang's book [13,p.9], there is a misprint: a_{n+1} is replaced by a_n in (2.7) and (2.8). However it easily seen from the proofs - or just from (2.4) - that it should be a_{n+1}. It is also easily seen that (2.7-8) also hold for $n = 1$.

We shall use the *Ostrowski representation* of a positive integer n with respect to the basis provided by the c.f. of τ (see [8,p.48]). Assume that for some $m \geq 0$,

$$\chi_m \leq n < \chi_{m+1}. \tag{2.9}$$

(Note that since $\tau < \frac{1}{2}, \chi_0 < \chi_1$). Then n may be uniquely represented in the form

$$n = \sum_{j=0}^{m} b_j \chi_j \tag{2.10}$$

where

$$0 \leq b_j \leq a_{j+1} \text{ and } b_{j-1} = 0 \text{ if } b_j = a_{j+1}, 0 \leq j \leq m; b_m > 0; b_0 < a_1. \tag{2.11}$$

We shall use the convention that $b_{m+1} = 0$. The integers b_j may be determined by the following algorithm:

$$\begin{aligned}
&n_m := n = b_m \chi_m + n_{m-1}, 0 \leq n_{m-1} < \chi_m;\\
&n_{m-1} := b_{m-1}\chi_{m-1} + n_{m-2}, 0 \leq n_{m-2} < \chi_{m-1};\\
&\vdots\\
&n_0 := b_0 \chi_0; n_{-1} := 0.
\end{aligned}$$

For the given n, we set

$$m^{\#} := \#\{j : 0 \leq j \leq m \text{ and } b_j \neq 0\}. \tag{2.12}$$

Of course, $m^{\#} \geq 1$. Note that conversely, given any $\{b_j\}_{j=0}^{m}$ satisfying (2.11), n defined by (2.10) satisfies (2.9). We use the notation

$$\log^{+} x := \max\{0, \log x\}.$$

We need two lemmas, the first dealing with the size of fractional parts. To simplify notation in this and the next section, for a given n with representation (2.10), we fix j between 0 and m, and set

$$\ell := \chi_j; \ell' := \chi_{j+1}; b := b_j; J := n - n_j; s := \frac{\pi_j}{\chi_j}; r := e^{2\pi i s}. \tag{2.13}$$

Lemma 2.1
(a) Let $N \geq M \geq 0$ *and* $c_k \in \mathbb{Z}, k \geq 0$ *with*

$$0 \leq c_k \leq a_{k+1} \text{ and } c_{k-1} = 0 \text{ if } c_k = a_{k+1}, k \geq 0; c_0 < a_1. \tag{2.14}$$

(i) Then if $M > 0$ *and* $c_M \neq 0$,

$$0 < (-1)^M \sum_{k=M}^{N} c_k \left(\chi_k \tau - \pi_k\right) = \| \tau \sum_{k=M}^{N} c_k \chi_k \| < \| \chi_{M-1}\tau \| . \tag{2.15}$$

If $c_M = 0$, *the rightmost inequality persists.*
(ii)

$$\| \chi_M \tau \| (c_M - 1) < \| \tau \sum_{k=M}^{N} c_k \chi_k \| < \| \chi_M \tau \| (c_M + 1) . \tag{2.16}$$

(iii) If $c_M \neq 0$,
(2.17)

$$\| \tau \sum_{k=M}^{N} c_k \chi_k \| > (a_{M+2} - c_{M+1}) \| \chi_{M+1} \tau \| \geq \max\{ \| \chi_{M+1} \tau \|, \frac{1}{8 \max\{1, c_{M+1}\} \chi_{M+1}} \}.$$

(b) Let n *have the Ostrowski representation (2.9) and assume the above notation involving* J, L, ℓ, ℓ', b.
(i) We have for $0 < L \leq b$, *and* $j \geq 1$,

$$\| J\tau \| < \| \ell\tau \| ; \| (L\ell + J)\tau \| < \| \chi_{j-1}\tau \| . \tag{2.18}$$

The first inequality is still valid if $j = 0$.
(ii) Assume that $j < m$. *For* $0 \leq L \leq b$,

$$\| \ell\tau \| (L - 1) < \| (L\ell + J)\tau \| < \| \ell\tau \| (L + 1) . \tag{2.19}$$

Moreover, for $0 < L \leq b$ *and* $j < m$,

$$\| (L\ell + J)\tau \| > (a_{j+2} - b_{j+1}) \| \ell'\tau \| \geq \max\{ \| \ell'\tau \|, \frac{1}{8 \max\{1, b_{j+1}\} \ell'} \}. \tag{2.20}$$

(iii) If $1 \leq k < \ell$, *then for* $0 \leq L < b$,

$$\| (k + L\ell + J) \tau \| > \| \ell\tau \| . \tag{2.21}$$

Moreover, if $k \neq \chi_{j-1}$, *then*

$$\| (k + L\ell + J) \tau \| > \| \chi_{j-1}\tau \| \tag{2.22}$$

and if $k = \chi_{j-1}$, *then*

$$\| (k + L\ell + J) \tau \| > (a_{j+1} - L) \| \ell\tau \| . \tag{2.23}$$

Proof
(a) (i) This is contained in Proposition 1 in [21,p.248] and contained in the proof of Lemma 1.62 in [8,p.50] but we provide the details as some of those are needed below. Suppose that $M > 0$ and is even, the proof is similar when M is odd. We use the recurrence relation (2.2) and the fact that $\chi_k \tau - \pi_k$ has sign $(-1)^k$ as well as our restriction (2.14) on the c_k. We have

$$- (\chi_M \tau - \pi_M) = \sum_{i=1}^{\infty} \left(\left(\chi_{M+2i} - \chi_{M+2i-2} \right) \tau - \left(\pi_{M+2i} - \pi_{M+2i-2} \right) \right)$$

$$= \sum_{i=1}^{\infty} a_{M+2i} \left(\chi_{M+2i-1}\tau - \pi_{M+2i-1} \right) < \sum_{1 < i \leq (N-M+1)/2} c_{M+2i-1} \left(\chi_{M+2i-1}\tau - \pi_{M+2i-1} \right)$$

$$\leq \sum_{t=M}^{N} c_t \left(\chi_t \tau - \pi_t \right)$$

$$< \sum_{i=0}^{\infty} a_{M+2i+1}\left(\chi_{M+2i}\tau - \pi_{M+2i}\right)$$

(2.24)
$$= \sum_{i=0}^{\infty} \left(\left(\chi_{M+2i+1} - \chi_{M+2i-1}\right)\tau - \left(\pi_{M+2i+1} - \pi_{M+2i-1}\right)\right) = -\left(\chi_{M-1}\tau - \pi_{M-1}\right).$$

Since the first and last elements in this chain of inequalities have absolute value less than $\frac{1}{2}$ (recall (2.4)), we deduce that

$$\| \tau \sum_{k=M}^{N} c_k \chi_k \| = \left| \sum_{k=M}^{N} c_k \left(\chi_k \tau - \pi_k\right) \right| < \| \chi_{M-1}\tau \|.$$

Then by applying the last but one inequality with $M+1$,

$$\sum_{k=M}^{N} c_k\left(\chi_k\tau - \pi_k\right) \geq c_M\left(\chi_M\tau - \pi_M\right) - \left| \sum_{k=M+1}^{N} c_k\left(\chi_k\tau - \pi_k\right) \right|$$
$$> c_M \| \chi_M\tau \| - \| \chi_M\tau \| \geq 0,$$

if $c_M \neq 0$.

(ii) For the rightmost inequality, we note that

$$\| \tau \sum_{k=M}^{N} c_k\chi_k \| \leq c_M \| \tau\chi_M \| + \| \tau \sum_{k=M+1}^{N} c_k\chi_k \|$$

and then (i) gives the rightmost inequality in (2.16). The leftmost inequality is similar.

(iii) Let us call the quantity in the left-side of (2.17) Δ and assume that M is even. Then if $c_M \geq 2$, (2.16) gives

$$\Delta > \| \chi_M\tau \| \geq a_{M+2} \| \chi_{M+1}\tau \|,$$

by (2.8). This is stronger than the left inequality in (2.17). Now we suppose that $c_M = 1$ and prove the left inequality in (2.17). Note first that from (i), if $M \geq 1$,

$$(\chi_M + c_{M+1}\chi_{M+1})\tau - (\pi_M + c_{M+1}\pi_{M+1}) = \| \left(\chi_M + c_{M+1}\chi_{M+1}\right)\tau \|$$
$$= \| \chi_M\tau \| - c_{M+1} \| \chi_{M+1}\tau \|$$

(2.25)
$$= \left(a_{M+2} - c_{M+1}\right) \| \chi_{M+1}\tau \| + \| \chi_{M+2}\tau \|,$$

by (2.7). If $M = 0$, it is easy to check that this remains valid, recall that $\| \chi_0\tau \| = \tau < \frac{1}{2}$. We consider two subcases:

(I) $c_M = 1$ and $c_{M+2} \neq 0$

Then (i) shows that

$$\sum_{k=M+2}^{N} c_k\left(\chi_k\tau - \pi_k\right) > 0,$$

so

$$\Delta = \sum_{k=M}^{N} c_k(\chi_k\tau - \pi_k) > \sum_{k=M}^{M+1} c_k\left(\chi_k\tau - \pi_k\right)$$
$$> \left(a_{M+2} - c_{M+1}\right) \| \chi_{M+1}\tau \|,$$

by (2.25).
(II) $c_M = 1$ and $c_{M+2} = 0$
Then using (i),

$$\Delta \geq \| \tau \left(\chi_M + c_{M+1}\chi_{M+1}\right) \| - \| \tau \sum_{k=M+3}^{N} c_k \chi_k \|$$

$$> \| \tau \left(\chi_M + c_{M+1}\chi_{M+1}\right) \| - \| \chi_{M+2}\tau \|$$

$$= (a_{M+2} - c_{M+1}) \| \chi_{M+1}\tau \|,$$

by (2.25). So we have the left inequality in (2.17) in all cases.

We turn to the proof of the second inequality in (2.17). Now if $c_{M+1} \leq \frac{1}{2}a_{M+2}$, then

$$(a_{M+2} - c_{M+1}) \| \chi_{M+1}\tau \| \geq \frac{1}{2} a_{M+2} \| \chi_{M+1}\tau \|$$

$$\geq \frac{1}{4} \| \chi_M \tau \| \geq \frac{1}{8\chi_{M+1}} \geq \frac{1}{8\max\{1, c_{M+1}\}\chi_{M+1}}.$$

In the first inequality in the last line, we used a simple consequence of (2.8). Next, if $c_{M+1} > \frac{1}{2}a_{M+2}$, then as $c_{M+1} < a_{M+2}$ (since $c_M \neq 0$, see (2.14))

$$(a_{M+2} - c_{M+1}) \| \chi_{M+1}\tau \| \geq \| \chi_{M+1}\tau \|$$

$$\geq \frac{1}{2\chi_{M+2}} \geq \frac{1}{4a_{M+2}\chi_{M+1}} \geq \frac{1}{8c_{M+1}\chi_{M+1}}.$$

So in all cases, we have the second lower bound in (2.17).
(b) (i) Now

$$\ell = \chi_j \text{ and } J = n - n_j = \sum_{k=j+1}^{m} c_k \chi_k$$

so the upper bound follows from (2.15). The upper bound for $\| J\tau \|$ is valid even for $j = 0$.
(ii) The first inequality (2.19) follows from (2.16). The second inequality (2.20) follows from (2.17).
(iii) We can write

$$k + L\ell + J = \sum_{i=t}^{m} b_i^* \chi_i,$$

where $t \leq j - 1, b_t^* \neq 0$ and $b_j^* = L, b_i^* = b_i, i > j$. Now if $t \leq j - 2$, then (2.17) gives

$$\| (k + L\ell + J)\tau \| > \| \chi_{t+1}\tau \| \geq \| \chi_{j-1}\tau \| .$$

by (2.6). We then have both (2.21) and (2.22). If $t = j - 1$ then (2.21) follows similarly from (2.17). If $t = j - 1$ and $k \neq \chi_{j-1}$, then $b_{j-1}^* \geq 2$, so (2.16) gives

$$\| (k + L\ell + J)\tau \| > \left(b_{j-1}^* - 1\right) \| \chi_{j-1}\tau \| \geq \| \chi_{j-1}\tau \|$$

We have thus proved (2.22) in all the cases where $k \neq \chi_{j-1}$. Finally if $k = \chi_{j-1}$, then (2.23) follows from (2.17).□

Lemma 2.2

With the notation (2.13) involving s, ℓ, we have
(a)

$$\sum_{k=1}^{\ell-1} \frac{1}{\sin^2 k\pi s} = \sum_{k=1}^{\ell-1} \frac{1}{\sin^2 k\pi/\ell} \leq \frac{\ell^2}{3}; \tag{2.26}$$

and

$$\sum_{k=1}^{\ell-1} \frac{k^2}{\sin^2 k\pi s} \leq \frac{5\ell^4}{24}. \tag{2.27}$$

(b)

$$\sum_{k=1}^{\ell-1} \frac{1}{|\sin k\pi s|} = \sum_{k=1}^{\ell-1} \frac{1}{|\sin k\pi/\ell|} \leq \ell\left(1+\log \ell\right). \tag{2.28}$$

(c)

$$\sum_{k=1}^{\ell-1} \cot k\pi s = 0. \tag{2.29}$$

Proof
(a) We use the standard observation that since $s = \pi_j/\chi_j = \pi_j/\ell$ has coprime numerator and denominator,

$$\{ks(\operatorname{mod} 1) : 1 \leq k < \ell\} = \{k/\ell : 1 \leq k < \ell\}$$

and hence for any function f defined on the rationals,

$$\sum_{k=1}^{\ell-1} f(ks(\operatorname{mod} 1)) = \sum_{k=1}^{\ell-1} f(k/\ell). \tag{2.30}$$

Then the first equalities in (2.26) and (2.28) follow. Next we note the identity [2]

$$\sum_{k=1}^{\ell-1} \frac{1}{\sin^2 k\pi/\ell} = \frac{\ell^2-1}{3},$$

so we have (2.26). Next, since $|\sin(\ell-k)\pi s| = |\sin k\pi s|$,

$$\sum_{k=1}^{\ell-1} \frac{k^2}{\sin^2 k\pi s} \leq \left(\frac{\ell}{2}\right)^2 \sum_{k=1}^{[\ell/2]} \frac{1}{\sin^2 k\pi s} + \ell^2 \sum_{k=[\ell/2]+1}^{\ell-1} \frac{1}{\sin^2 k\pi s}$$
$$\leq \frac{5\ell^2}{8} \sum_{k=1}^{\ell-1} \frac{1}{\sin^2 k\pi s} \leq \frac{5\ell^4}{24}.$$

(b) We have

$$\sum_{k=1}^{\ell-1} \frac{1}{|\sin k\pi/\ell|} \leq 2\sum_{k=1}^{[\ell/2]} \frac{1}{|\sin k\pi/\ell|} \leq \sum_{k=1}^{[\ell/2]} \frac{1}{k/\ell} \leq \ell\left(1+\log \ell\right).$$

(c) This follows since $\cot\left(\ell-k\right)\pi s = -\cot k\pi s$.□

3. The Basic Estimate

The main result of this section is:

Theorem 3.1
For n with Ostrowski representation (2.10), (2.11), let
(3.1)

$$\Gamma := \log\left|(q;q)_n\right| - \left(\sum_{j=0}^{m} b_j \left(\log \frac{2\pi b_j \chi_j \parallel \chi_j \tau \parallel}{e}\right)\right) - \pi \sum_{j=0}^{m} b_j \left(\tau - \frac{\pi_j}{\chi_j}\right) \sum_{k=1}^{\chi_j - 1} k \cot k\pi\tau.$$

Then

$$-114 \sum_{j=0}^{m} b_j \leq \Gamma \leq 14 \sum_{j=0}^{m} b_j \left(\frac{\chi_j}{\chi_{j+1}}\right)^2 + \frac{3}{2} \sum_{j=0}^{m} \log^+ b_j + 3m^{\#}. \tag{3.2}$$

We now outline our steps towards the proof of (3.2). For an n given by (2.10), we write

$$\log\left|(q;q)_n\right| = \sum_{j=0}^{m} \sum_{k=n-n_j+1}^{n-n_{j-1}} \log\left|1-q^k\right| =: \sum_{j=0}^{m} S_j \tag{3.3}$$

where

$$S_j = \sum_{k=n-n_j+1}^{n-n_{j-1}} \log\left|1-q^k\right| = \sum_{k=1}^{b_j \chi_j} \log\left|1-q^{k+n-n_j}\right|. \tag{3.4}$$

Recall that to simplify notation, we fix j between 0 and m, and set

$$\ell := \chi_j; \ell' := \chi_{j+1}; b := b_j; J := n - n_j; s := \frac{\pi_j}{\chi_j}; r := e^{2\pi i s}. \tag{3.5}$$

We see that then

$$S_j = \sum_{L=0}^{b-1} \left[S_{j,L} + \log\left|1 - q^{(L+1)\ell + J}\right|\right] \tag{3.6}$$

where

$$S_{j,L} := \sum_{k=1}^{\ell-1} \log\left|1-q^{L\ell+J+k}\right|. \tag{3.7}$$

Note that as π_j and χ_j are coprime, r is a primitive ℓth$=\chi_j$th root of unity. Then

$$P(u) := \prod_{k=1}^{\ell-1}(u - r^k) = \frac{u^\ell - 1}{u-1}$$

so

$$\prod_{k=1}^{\ell-1}(1-r^k) = P(1) = \ell. \tag{3.8}$$

Then we deduce that

$$S_{j,L} - \log \ell = \sum_{k=1}^{\ell-1} \log \left| \frac{1 - q^{L\ell+J+k}}{1 - r^k} \right| . \tag{3.9}$$

We now prove:

Lemma 3.2

(3.10)

$$T - \left(\frac{\ell}{\ell'}\right)^2 \left(36 + 57\,(L+1)^2\right) \leq S_{j,L} - \log \ell - \pi\,(\tau - s) \sum_{k=1}^{\ell-1} k \cot k\pi\tau \leq 14 \left(\frac{\ell}{\ell'}\right)^2 ,$$

where for some $\sigma = \pm 1$,

(3.11)

$$T := \log \left| \frac{1 - q^{L\ell+J+\chi_{j-1}}}{1 - r^{\chi_{j-1}}} \right| - \left(\cot \pi\chi_{j-1} s\right) \pi \left(\chi_{j-1}\,(\tau - s) + \sigma \parallel (L\ell + J)\tau \parallel\right) .$$

Proof

If $j = 0$, then $\chi_j = 1$ and the sum $S_{j,L}$ is taken over an empty range, so we assume that $j \geq 1$. We use the Taylor series expansion

$$\log \sin v = \log \sin u + (\cot u)\,(v - u) - \frac{1}{2\sin^2 \xi}(v - u)^2 \tag{3.12}$$

for $u, v \in (0, \pi)$ and some ξ between u, v. We also use the facts that for $x, y \in \mathbb{R}$,

$$\begin{aligned} &|\sin \pi x| = \sin \pi\{x\} = \sin \pi \parallel x \parallel; \\ &\cot \pi x = \cot \pi\{x\}; \\ &|\sin \pi\,(x + y)| = |\sin \pi\,(\{x\} + \sigma \parallel y \parallel)| \end{aligned} \tag{3.13}$$

where $\sigma = 1$ if $\parallel y \parallel = \{y\}$ and $\sigma = -1$ if $\parallel y \parallel = 1 - \{y\}$. Then if $1 \leq k < \ell$, we note that from (2.18) and then the best approximation property (2.5),

$$\parallel (L\ell + J)\,\tau \parallel < \parallel \chi_{j-1}\tau \parallel \leq \parallel k\tau \parallel \leq \min\{\{k\tau\}, 1 - \{k\tau\}\}$$

so that

$$0 < \{k\tau\} \pm \parallel (L\ell + J)\,\tau \parallel < 1.$$

We choose $\sigma = \pm 1$ (independent of k) such that

$$|\sin\left(\pi\,(k + L\ell + J)\,\tau\right)| = \sin \pi\,(\{k\tau\} + \sigma \parallel (L\ell + J)\tau \parallel)$$

and set $u := \pi\,\{ks\}$ and $v := \pi\,(\{k\tau\} + \sigma \parallel (L\ell + J)\tau \parallel)$ above. Then for some ξ_k between u, v,

$$\log \left| \frac{1 - q^{L\ell+J+k}}{1 - r^k} \right| = \log \frac{\sin v}{\sin u}$$

$$= (\cot \pi\{ks\})\,\pi\,(\{k\tau\} - \{ks\} + \sigma \parallel (L\ell + J)\tau \parallel) - \frac{\pi^2}{2\sin^2 \xi_k}\,(\{k\tau\} - \{ks\} + \sigma \parallel (L\ell + J)\tau \parallel)^2 .$$

Now

$$\frac{1}{\ell} \leq \{ks\} \leq 1 - \frac{1}{\ell} \tag{3.14}$$

and by (2.4)

$$k\,|\tau - s| = \frac{k}{\ell} \parallel \ell\tau \parallel < \frac{k}{\ell\ell'} < \frac{1}{\ell} \tag{3.15}$$

so

$$\{k\tau\} = \{ks + k(\tau - s)\} = \{ks\} + k(\tau - s).$$

Then

$$\log\left|\frac{1-q^{L\ell+J+k}}{1-r^k}\right|$$

(3.16)

$$= (\cot \pi ks)\, \pi\, (k(\tau - s) + \sigma \parallel (L\ell + J)\tau \parallel) - \frac{\pi^2}{2\sin^2 \xi_k}\left(k(\tau - s) + \sigma \parallel (L\ell + J)\tau \parallel\right)^2.$$

Next, with u, v as above

$$\left|\frac{1}{\sin u} - \frac{1}{\sin v}\right| \le \left|\frac{u-v}{\sin u \sin v}\right| \le \frac{\pi\left(k\,|\tau - s| + \parallel (L\ell + J)\tau \parallel\right)}{(\sin u)\, |\sin \pi\, (\parallel (k + L\ell + J)\tau \parallel)|}. \tag{3.17}$$

In the last sin term, we used again the properties (3.13). Now if $k \neq \chi_{j-1}$, then by (2.22),

$$\sin \pi\, (\parallel (k + L\ell + J)\tau \parallel) \ge 2 \parallel (k + L\ell + J)\tau \parallel > 2 \parallel \chi_{j-1}\tau \parallel,$$

so (3.15) shows that

$$\left|\frac{1}{\sin u} - \frac{1}{\sin v}\right| \le \frac{\pi\left(\parallel \ell\tau \parallel + \parallel \chi_{j-1}\tau \parallel\right)}{2(\sin u) \parallel \chi_{j-1}\tau \parallel} \le \frac{\pi}{\sin u}.$$

Hence ξ_k between u and v satisfies

$$\frac{1}{\sin \xi_k} \le \frac{1+\pi}{|\sin u|} = \frac{1+\pi}{|\sin k\pi s|}. \tag{3.18}$$

Next, recall (2.29). Adding over k in (3.16) gives (recall σ is independent of k and recall that T is given by (3.11))

$$0 \ge \sum_{k=1}^{\ell-1} \log\left|\frac{1-q^{L\ell+J+k}}{1-r^k}\right| - \pi(\tau - s)\sum_{k=1}^{\ell-1} k \cot k\pi s$$

$$= T - \sum_{k=1, k\neq\chi_{j-1}}^{\ell-1} \frac{\pi^2}{2\sin^2 \xi_k}\left(k(\tau - s) + \sigma \parallel (L\ell + J)\tau \parallel\right)^2$$

$$\ge T - (\pi(1+\pi))^2\left(|\tau - s|^2 \sum_{k=1}^{\ell-1}\frac{k^2}{\sin^2 k\pi s} + \parallel (L\ell + J)\tau \parallel^2 \sum_{k=1}^{\ell-1}\frac{1}{\sin^2 k\pi s}\right)$$

$$\ge T - \left(\frac{\ell}{\ell'}\right)^2 (\pi(1+\pi))^2\left[\frac{5}{24} + \frac{(L+1)^2}{3}\right]. \tag{3.19}$$

Here we have used Lemma 2.2(a), (2.19) and (3.15). Next, for some ζ_k between $\pi\{ks\}$ and $\pi\{\kappa\tau\}$,

$$\cot \pi\{ks\} - \cot \pi\{k\tau\} = \frac{-k\pi(s-\tau)}{\sin^2 \zeta_k}.$$

Here

$$\left|\frac{1}{\sin \pi\{ks\}} - \frac{1}{\sin \pi\{k\tau\}}\right| \le \frac{\pi k\,|s - \tau|}{(\sin \pi\{ks\})(\sin \pi\{k\tau\})}$$

$$\le \frac{\pi \parallel \ell\tau \parallel}{(\sin \pi\{ks\})\, 2 \parallel k\tau \parallel} \le \frac{\pi \parallel \ell\tau \parallel}{(\sin \pi\{ks\})\, 2 \parallel \chi_{j-1}\tau \parallel} \le \frac{\pi}{2(\sin \pi\{ks\})}$$

by the best approximation property (2.5). Thus

$$\frac{1}{\sin \zeta_k} \leq \left(1+\frac{\pi}{2}\right) \frac{1}{|\sin k\pi s|}$$

and then

$$0 \leq (\tau - s)\,\pi \sum_{k=1}^{\ell-1} k\,(\cot k\pi s - \cot k\pi\tau) = \pi^2 (\tau - s)^2 \sum_{k=1}^{\ell-1} \frac{k^2}{\sin^2 \zeta_k}$$

$$\leq \left(\pi\left(1+\frac{\pi}{2}\right)\right)^2 \frac{5}{24}\left(\frac{\ell}{\ell'}\right)^2, \tag{3.20}$$

by (3.15) and (2.27). Combining this and (3.19) and estimating the constants gives (3.10).□

We now deduce

Lemma 3.3

$$\Delta := S_j - b\log \ell - b\pi\,(\tau - s) \sum_{k=1}^{\ell-1} k \cot k\pi\tau - \sum_{L=1}^{b} \log\left|1 - q^{L\ell+J}\right| \tag{3.21}$$

admits the estimate

$$T^* - \pi b - 93b^3 \left(\frac{\ell}{\ell'}\right)^2 \leq \Delta \leq 14b\left(\frac{\ell}{\ell'}\right)^2, \tag{3.22}$$

where $T^* = 0$ *if* $j = 0$ *and otherwise*

$$T^* := \sum_{L=0}^{b-1} \log\left|\frac{1 - q^{L\ell+J+\chi_{j-1}}}{1 - r^{\chi_{j-1}}}\right|. \tag{3.23}$$

Proof

For $j = 0$, the result holds trivially with $T^* = 0$ since then $\ell = \chi_0 = 1$ and so each $S_{j,L} = 0$. So assume that $j \geq 1$. The upper bound follows from (3.6) by adding over $L = 0, 1, ..., b-1$ in the previous lemma; the lower bound follows similarly, on noting that we can bound the following part of T in (3.11):

$$\left|\left(\cot \pi\chi_{j-1}s\right)\pi\left(\chi_{j-1}(\tau - s) + \sigma \parallel (L\ell + J)\tau \parallel\right)\right|$$

$$\leq \frac{\pi}{2 \parallel \chi_{j-1}s \parallel}\left(\frac{\chi_{j-1}}{\chi_j} \parallel \chi_j\tau \parallel + \parallel \chi_{j-1}\tau \parallel\right) \leq \frac{\pi \parallel \chi_{j-1}\tau \parallel}{\parallel \chi_{j-1}s \parallel}.$$

Here by (2.3),

$$\parallel \chi_{j-1}s \parallel = \parallel \chi_{j-1}\frac{\pi_j}{\chi_j} - \pi_{j-1} \parallel = \frac{1}{\chi_j} = \frac{1}{\ell}, \tag{3.24}$$

so

$$\sum_{L=0}^{b-1} \left|\left(\cot \pi\chi_{j-1}s\right)\pi\left(\chi_{j-1}(\tau - s) + \parallel (L\ell + J)\tau \parallel\right)\right| \leq \pi b\chi_j \parallel \chi_{j-1}\tau \parallel \leq \pi b. \tag{3.25}$$

□

We turn to estimation of the second sum in the right-hand side of (3.21) (recall we set $b_{m+1} = 0$):

Lemma 3.4

(3.26)
$$\sum_{L=1}^{b}\log\left|1-q^{L\ell+J}\right|-b\log\left(\frac{2\pi b\parallel\ell\tau\parallel}{e}\right)\left\{\begin{array}{l}\leq\frac{3}{2}\log b+3\\ \geq-\frac{1}{2}\log b-\log^{+}\left(4b_{j+1}\right)-\left(\pi/\ell'\right)^{2}b^{3}\end{array}\right. .$$

Proof
Let

$$g(u):=\left|\frac{\sin\pi u}{\pi u}\right|,u\in\mathbb{R}.$$

It is easily seen that

$$0\leq 1-g(u)\leq\frac{\pi^{2}}{6}u^{2},u\in\mathbb{R},$$

and hence that

$$0\geq\log g(u)\geq-\frac{3}{2}\left(1-g(u)\right)\geq-\frac{\pi^{2}}{4}u^{2},u\in[0,\frac{1}{2}]. \tag{3.27}$$

Then (recall that for all x, $\parallel x\parallel\in\left[0,\frac{1}{2}\right]$)

$$0\geq\sum_{L=1}^{b}\log\left|1-q^{L\ell+J}\right|-\sum_{L=1}^{b}\log\left(2\pi\parallel(L\ell+J)\tau\parallel\right)=\sum_{L=1}^{b}\log g\left(\parallel(L\ell+J)\tau\parallel\right)$$

$$\geq-\frac{\pi^{2}\parallel\ell\tau\parallel^{2}}{4}\sum_{L=1}^{b}\left(L+1\right)^{2}\geq-\left(\pi/\ell'\right)^{2}b^{3}, \tag{3.28}$$

recall (2.19). Next (2.19), (2.20) and Stirling's formula give if $j<m$,
(3.29)
$$\sum_{L=1}^{b}\log\left(2\pi\parallel(L\ell+J)\tau\parallel\right)\geq\sum_{L=2}^{b}\log\left(2\pi\left(L-1\right)\parallel\ell\tau\parallel\right)+\log\left(\frac{\pi}{4\max\left\{1,b_{j+1}\right\}\ell'}\right)$$

$$=b\log\left(2\pi\parallel\ell\tau\parallel\right)+\log\left((b-1)!\right)+\log\left(\frac{1}{8\max\left\{1,b_{j+1}\right\}\ell'\parallel\ell\tau\parallel}\right)$$

$$\geq b\log\left(\frac{2\pi b\parallel\ell\tau\parallel}{e}\right)+\log\left(\sqrt{2\pi b}\right)-\log b+\log\left(\frac{1}{8\max\left\{1,b_{j+1}\right\}\ell'\parallel\ell\tau\parallel}\right)$$

$$\geq b\log\left(\frac{2\pi b\parallel\ell\tau\parallel}{e}\right)-\frac{1}{2}\log b-\log^{+}\left(4b_{j+1}\right). \tag{3.30}$$

Next, by (2.19) and Stirling's formula,

$$\sum_{L=1}^{b}\log\left(2\pi\parallel(L\ell+J)\tau\parallel\right)\leq\sum_{L=1}^{b}\log\left(2\pi\left(L+1\right)\parallel\ell\tau\parallel\right)$$

$$\leq b\log\left(\frac{2\pi b\parallel\ell\tau\parallel}{e}\right)+\frac{3}{2}\log b+3.$$

Combining this with (3.28) and (3.29) gives the result if $j < m$. For $j = m$, we use $J = n - n_m = 0$ and hence $\| (L\ell + J)\tau \| = L \| \ell\tau \|$, so that

$$\sum_{L=1}^{b} \log\left(2\pi \| (L\ell + J)\tau \|\right) = b\log\left(2\pi \| \ell\tau \|\right) + \log(b!)$$

and this may be estimated as before, in a simpler fashion.□
We turn to the estimation of the term T^* :

Lemma 3.5
The term T^* defined by (3.23) satisfies

$$T^* \geq -3b. \tag{3.31}$$

Proof
Now by (3.24) and then (2.23),

$$\left| \frac{\sin \pi \| \left(\chi_{j-1} + L\ell + J\right)\tau \|}{\sin \pi \| \chi_{j-1} s \|} \right|$$

$$\geq \frac{2}{\pi}\ell \| \left(\chi_{j-1} + L\ell + J\right)\tau \| \geq \frac{2}{\pi}\left(a_{j+1} - L\right)\ell \| \ell\tau \| . \tag{3.32}$$

Then

$$\begin{aligned} T^* &= \sum_{L=0}^{b-1} \log \left| \frac{\sin \pi \| \left(\chi_{j-1} + L\ell + J\right)\tau \|}{\sin \pi \| \chi_{j-1} s \|} \right| \\ &\geq b\log\left(\frac{2}{\pi}\ell \| \ell\tau \|\right) + \log\frac{a_{j+1}!}{(a_{j+1} - b)!} \\ &\geq b\log\left(\frac{2}{\pi e}a_{j+1}\ell \| \ell\tau \|\right) - \frac{1}{12}, \end{aligned}$$

by Stirling's formula. Then the inequality $\chi_{j+1} \leq 2a_{j+1}\chi_j$, gives

$$T^* \geq -b\log\left(2\pi\exp\left(\frac{13}{12}\right)\right).$$

□

Proof of Theorem 3.1
Combining our estimates of the last three lemmas, we have for a fixed j, with $b = b_j > 0$,

$$S_j - b\log\left(\frac{2\pi b\ell \| \ell\tau \|}{e}\right) - b\pi(\tau - s)\sum_{k=1}^{\ell-1} k\cot k\pi\tau \begin{cases} \leq 14b\left(\frac{\ell}{\ell'}\right)^2 + \frac{3}{2}\log b + 3 \\ \geq -110b - \log^+(4b_{j+1}) \end{cases}.$$

(We have used $b^3/\ell'^2 \leq b; \log b \leq b$ and $b^3\left(\frac{\ell}{\ell'}\right)^2 \leq b$). Adding over j and using $\log^+ x \leq x$, as well as $b_{m+1} = 0$ gives the result.□

4. Estimate of a Certain Sum

Let

$$U_n := \sum_{k=1}^{n} \cot k\pi\tau. \tag{4.1}$$

$$V_n := \sum_{k=1}^{n} k \cot k\pi\tau. \tag{4.2}$$

The main result of this section is:

Theorem 4.1
Let $\chi_m \leq n < \chi_{m+1}$ and represent n in its Ostrowski representation (2.10). Then
(a)

$$|U_n| \leq \left(124 + 24\left(\max_{k\leq m} \log b_k\right)\right)\chi_{m+1}. \tag{4.3}$$

(b)

$$|V_n| \leq n\chi_{m+1}\left(248 + 48 \max_{k\leq m+1} \log a_k\right). \tag{4.4}$$

Corollary 4.2
If $n = \chi_m - 1$, then

$$|V_n| \leq \chi_m^2\left(248 + 48 \max_{k\leq m} \log a_k\right). \tag{4.5}$$

We remark that in contrast to (4.3), one expects,

$$\sum_{k=1}^{n} |\cot k\pi\tau| \geq C \sum_{k=1}^{n} \frac{1}{\| k\tau \|} \geq C\left(n\log n + \chi_{m+1}\left(1 + \log \frac{n}{\chi_m}\right)\right),$$

see [11, p.247, (77)]. The last right-hand side may grow essentially faster than that in (4.3) for infinitely many choices of n. We begin our proof with

The Deduction of Theorem 4.1(b) from Theorem 4.1(a)
A summation by parts shows that if we set $U_0 := 0$,

$$V_n = \sum_{k=1}^{n} k\left(U_k - U_{k-1}\right) = nU_n - \sum_{k=1}^{n-1} U_k.$$

Now apply (4.3) to deduce (4.4). Note that the Ostrowski representation of k with $k \leq n$ involves only $\chi_k, k \leq m$; recall too that $b_k \leq a_{k+1}$.□

The Proof of Corollary 4.2
Note that $n = \chi_m - 1$ satisfies $\chi_{m-1} \leq n < \chi_m$, so apply the theorem with m replaced by $m - 1$.□

We shall use the Ostrowski representation (2.10) and proceed similarly to the previous section. Thus we write (cf. (3.3), (3.4))

$$U_n = \sum_{j=0}^{m} S_j \tag{4.6}$$

where

$$S_j := \sum_{k=n-n_j+1}^{n-n_{j-1}} \cot k\pi\tau = \sum_{k=1}^{b_j\chi_j} \cot\left(k+n-n_j\right)\pi\tau. \tag{4.7}$$

As at (3.5), we fix j, and adopt the notation there. Then as at (3.6), we write

$$S_j = \sum_{L=0}^{b-1} \left(S_{j,L} + \cot\left(\left(\left(L+1\right)\ell + J\right)\pi\tau\right)\right) \tag{4.8}$$

with

$$S_{j,L} = \sum_{k=1}^{\ell-1} \cot\left(k + L\ell + J\right)\pi\tau. \tag{4.9}$$

We recall that

$$\frac{1}{2} < \ell' \parallel \ell\tau \parallel < 1.$$

Lemma 4.3

$$\left|S_{j,L}\right| \leq 5\frac{\left(L+2\right)\ell^2}{\ell'} + \frac{\pi\left(L+2\right)\ell}{2\left(a_{j+1}-L\right)}. \tag{4.10}$$

Proof
We use

$$\sum_{k=L\ell+1}^{(L+1)\ell-1} \cot k\pi s = \sum_{k=1}^{\ell-1} \cot k\pi s = 0. \tag{4.11}$$

Now for $1 \leq k < \ell$, we have

$$t_k := \left|\cot\left(k+L\ell+J\right)\pi\tau - \cot k\pi s\right| \leq \frac{k\pi\left|\tau - s\right| + \pi \parallel \left(L\ell+J\right)\tau \parallel}{\left|\sin\pi\left(\parallel \left(k+L\ell+J\right)\tau \parallel\right)\right|\left|\sin k\pi s\right|}.$$

Here if $k \neq \chi_{j-1}$, then as at (3.17-3.18) in the proof of Lemma 3.2, we obtain

$$\frac{1}{\left|\sin\pi\left(\parallel \left(k+L\ell+J\right)\tau \parallel\right)\right|} \leq \frac{1+\pi}{\left|\sin k\pi s\right|}$$

and hence from (2.19),

$$t_k \leq \frac{\left(1+\pi\right)\pi\left(L+2\right)}{\ell'\sin^2 k\pi s}.$$

If $k = \chi_{j-1}$, we use instead (2.23) and then (3.24) to deduce that

$$t_k \leq \frac{\pi\left(L+2\right)}{2\left(a_{j+1}-L\right)\ell' \parallel \ell\tau \parallel \left|\sin k\pi s\right|} \leq \frac{\pi\left(L+2\right)\ell}{2\left(a_{j+1}-L\right)}.$$

Adding over k and using (4.11) and (2.26) gives the result.□

With $S_{j,L}$ estimated, we now turn to the other term in (4.8):

Lemma 4.4
If $j < m$,

$$\left| \sum_{L=0}^{b-1} \cot \pi \left(\left((L+1)\, \ell + J \right) \tau \right) \right| \leq \ell' \left(\log b + 5 \max\{1, b_{j+1}\} \right). \tag{4.12}$$

Proof
We use (2.19) to deduce that for $L \geq 1$,

$$\left| \sin \pi \left((L+1)\, \ell + J \right) \tau \right| = \sin \pi \parallel \left((L+1)\, \ell + J \right) \tau \parallel \geq 2L \parallel \ell \tau \parallel$$

so that

$$\left| \sum_{L=1}^{b-1} \cot \pi \left(\left((L+1)\, \ell + J \right) \tau \right) \right| \leq \sum_{L=1}^{b-1} \frac{1}{2L \parallel \ell \tau \parallel} \leq \ell'(1 + \log b).$$

Next, for $L = 0$, we obtain from (2.20), if $j < m$,

$$\left| \cot \pi \left(\left((L+1)\, \ell + J \right) \tau \right) \right| \leq \frac{1}{2 \parallel (\ell + J)\, \tau \parallel} \leq 4 \max\{1, b_{j+1}\} \ell'.$$

If $j = m$, then $J = 0$, so the same estimate holds as we set $b_{m+1} = 0$. Combining the last two estimates gives the result.□

We summarize the results of the previous two lemmas and (4.8-9) in:

Lemma 4.5

$$|S_j| \leq \ell' \left(6 \log 2b + 20 \max\{1, b_{j+1}\} \right). \tag{4.13}$$

Proof
Adding over L the estimate in Lemma 4.3 to that in Lemma 4.4 gives

$$\begin{aligned} |S_j| &\leq 15 \frac{(\ell b)^2}{\ell'} + \frac{\pi \ell}{2} \sum_{L=0}^{b-1} \frac{L+2}{a_{j+1} - L} + \ell' \left(\log b + 5 \max\{1, b_{j+1}\} \right) \\ &\leq \ell' \left(\log b + 20 \max\{1, b_{j+1}\} \right) + \Sigma, \end{aligned}$$

where

$$\begin{aligned} \Sigma &:= \frac{\pi \ell}{2} \sum_{L=0}^{b-1} \frac{L+2}{a_{j+1} - L} \\ &= \frac{\pi \ell}{2} \left(-b + (a_{j+1} + 2) \sum_{k = a_{j+1} - b + 1}^{a_{j+1}} \frac{1}{k} \right) \\ &\leq \frac{\pi \ell}{2} (a_{j+1} + 2) \log \left(\frac{a_{j+1}}{a_{j+1} - b} \right) \leq 5 \ell' \log \left(\frac{a_{j+1}}{a_{j+1} - b} \right), \end{aligned}$$

at least if $b < a_{j+1}$. Here by considering the cases $b \leq a_{j+1}/2$ and $b > a_{j+1}/2$, we see that we can continue this as $\leq 5\ell' \log 2b$. The case $b = a_{j+1}$ is easier.□

We turn to

The Proof of Theorem 4.1(a)
Now if $j < m$, $b_{j+1}\chi_{j+1} \leq a_{j+2}\chi_{j+1} \leq \chi_{j+2}$, so from (4.6) and (4.13),

$$|U_n| \leq \sum_{j=0}^{m} |S_j| \leq 6\left(\max_{k\leq m} \log(2b_k)\right)\sum_{j=0}^{m}\chi_{j+1} + 20\sum_{j=0}^{m-1}\chi_{j+2} + 20\chi_{m+1}$$

$$\leq \chi_{m+1}\left(24\max_{k\leq m}\log(2b_k) + 80 + 20\right),$$

recall $\chi_{j+2} \geq 2\chi_j$.$\square$

5. Proof of the Theorems

We begin by combining the result of Theorem 3.1 and Theorem 4.1, with the notation as in Theorem 3.1. Recall that we assumed $q = \exp(2\pi i\tau)$ with $\tau \in \left(0, \frac{1}{2}\right)$. If $\tau \in \left(\frac{1}{2}, 1\right)$, we set $\tau' := 1 - \tau; q' := \exp(2\pi i\tau')$ and use $q' = \overline{q}$, so that

$$|(q;q)_n| = |(q';q')_n|$$

Thus in the sequel, we deal only with $\tau \in \left(0, \frac{1}{2}\right)$.

Proposition 5.1

$$\Gamma^* := \log|(q;q)_n| - \left(\sum_{j=0}^{m} b_j\left(\log\frac{2\pi b_j\chi_j \parallel \chi_j\tau \parallel}{e}\right)\right) \tag{5.1}$$

satisfies

$$\Gamma \leq m^{\#}\left(800 + 151\max_{0\leq j\leq m}\log a_j\right) + \frac{3}{2}\sum_{j=0}^{m}\log^{+} b_j \tag{5.2}$$

and

$$\Gamma \geq -900\sum_{j=0}^{m} b_j - 151m^{\#}\max_{k\leq m}\log a_k. \tag{5.3}$$

Moreover, we may replace the terms involving $\max_{k\leq m}\log a_k$ *in the last two right-hand sides by*

$$151\sum_{j=0}^{m}\frac{b_j}{a_{j+1}}\max_{k\leq j}\log a_k. \tag{5.4}$$

Proof
We have from Corollary 4.2 that

$$\left|\pi\sum_{j=0}^{m} b_j\left(\tau - \frac{\pi_j}{\chi_j}\right)\sum_{k=1}^{\chi_j - 1} k\cot k\pi\tau\right| \leq \pi\sum_{j=0}^{m}\frac{b_j\chi_j}{\chi_{j+1}}\left(248 + 48\max_{k\leq j}\log a_k\right) \tag{5.5}$$

$$\leq \left(248\pi + 48\pi\max_{k\leq m}\log a_k\right)m^{\#}.$$

(Recall that $m^{\#}$ is the number of $b_j, j \leq m$ with $b_j \neq 0$). An alternative upper bound is

$$248\pi m^{\#} + 48\pi \sum_{j=0}^{m} \frac{b_j}{a_{j+1}} \max_{k \leq j} \log a_k$$

since $\chi_{j+1} \geq a_{j+1}\chi_j$. If we add this to the upper estimate in (3.2) in Theorem 3.1, and use $b_j \leq a_{j+1}$, we obtain

$$\Gamma \leq m^{\#} \left(248\pi + 17 + 48\pi \max_{k \leq m} \log a_k \right) + \frac{3}{2} \sum_{j=0}^{m} \log^{+} b_j$$

and hence (5.2). The lower bound follows similarly from (5.5) and (3.2).□

We need a simple lemma:

Lemma 5.2
Let $(c_k)_{k=1}^{\infty}$ be an increasing sequence of positive numbers satisfying (1.5) and (1.8). Then

$$\lim_{k \to \infty} \frac{c_k}{\log k} = \infty. \tag{5.6}$$

Proof
Let $\varepsilon > 0$. For large j, the convergence (1.8) gives

$$\varepsilon \geq \sum_{k=2^j}^{[2^{j(1+\beta)}]} \frac{1}{kc_k}$$

$$\geq \frac{1}{\max\{\frac{c_k}{\log k} : 2^j \leq k \leq [2^{j(1+\beta)}]\}} \sum_{k=2^j}^{[2^{j(1+\beta)}]} \frac{1}{k \log k} \geq \frac{C}{c_{2^j} / \log 2^j}$$

with C independent of j, ε by (1.5) and some simple estimation. Then (5.6) follows.□

We turn to the

Proof of Theorem 1.1 (I)
Clearly (a) ⇒ (b). We shall show that (b) ⇒ (c) and (c) ⇒ (a). We first recall from [11,pp.234-5] some well known results in the theory of diophantine approximation: let $(c_j)_{j=1}^{\infty}$ be an increasing sequence of positive numbers. The following are equivalent:
(i) (1.8) holds;
(ii) For a.e. τ,

$$a_j = O(jc_j), j \to \infty; \tag{5.7}$$

(iii) For a.e. τ,

$$\sum_{k=0}^{j} a_k = O(jc_j), j \to \infty. \tag{5.8}$$

Moreover, a theorem of Khintchine-Levy asserts that for a.e. τ,

$$\lim_{k\to\infty}\frac{\log\chi_k}{k}=\frac{\pi^2}{12\ln 2}.$$

It follows that if for a given n, we determine $m=m(n)$ by (2.9), then

$$\lim_{n\to\infty}\frac{\log n}{\log\chi_m}=1;\ \lim_{n\to\infty}\frac{\log n}{m}=\frac{\pi^2}{12\ln 2}. \tag{5.9}$$

We turn to

(b) ⇒ (c)

Fix $0<\alpha<1$ and for $m\geq 1$ define $n=n(m)$ by

$$n:=b_m\chi_m \text{ where } b_m:=[\alpha a_{m+1}].$$

Of course it is possible that $b_m=0=n$. Proposition 5.1 gives (since then $b_j=0, j<m$ for the given n and so $m^{\#}=1$),

$$\log|(q;q)_n|\leq b_m\log\frac{2\pi b_m\chi_m\parallel\chi_m\tau\parallel}{e}+O\left(\max_{0\leq j\leq m+1}\log a_j\right)+O(1).$$

Here for a.e. τ, we have

$$\max_{0\leq j\leq m+1}\log a_j=O(\log m)$$

and

$$b_m\chi_m\parallel\chi_m\tau\parallel\leq b_m\chi_m/\chi_{m+1}\leq[\alpha a_{m+1}]/a_{m+1}\leq\alpha,$$

so if α is small enough (independently of m,n or even τ), we obtain

$$\log|(q;q)_n|\leq\frac{1}{2}b_m\log\alpha+O(\log m).$$

Our choice of b_m and hypothesis then show that for a.e. τ, as $m\to\infty$, and for some $C_1,C_2>0$ independent of m,

$$(\log n)\left(c_{[\log n]}\right)\geq C_1\log\frac{1}{|(q;q)_n|}\geq C_2a_{m+1}+O(\log m).$$

Then (1.5) and (5.9) show that

$$a_m=O(mc_m).$$

Since this is true for a.e. τ, the quoted results from [11] show that (1.8) must hold.

(c) ⇒ (a)

We see from Proposition 5.1 and then (5.7), (5.8) that for a.e. τ,

$$\left|\log|(q;q)_n|-\sum_{j=0}^{m}b_j\left(\log\frac{2\pi b_j\chi_j\parallel\chi_j\tau\parallel}{e}\right)\right|=O\left(\sum_{j=0}^{m+1}a_j\right)+O(m\max_{k\leq m+1}\log a_k)$$

$$=O(mc_m)+O(m\log m)=O(mc_m), \tag{5.10}$$

by Lemma 5.2. Next, letting

$$t_j:=b_j\chi_j/\chi_{j+1}\in(0,1]$$

we have

$$b_j\left(\log\frac{2\pi b_j\chi_j\parallel\chi_j\tau\parallel}{e}\right)=\frac{\chi_{j+1}}{\chi_j}t_j\left(\log t_j+O(1)\right)$$

by (2.4), so as $t\log t$ is bounded for $t\in[0,1]$, we obtain from (5.10)

$$|\log|(q;q)_n|| = O(mc_m) + O\left(\sum_{j=0}^{m}\frac{\chi_{j+1}}{\chi_j}\right) = O(mc_m) + O\left(\sum_{j=0}^{m} a_{j+1}\right) = O(mc_m)$$

by (5.8). Finally (1.5) and (5.9) give (1.6).□

Proof of Theorem 1.1 (II)
Suppose that for τ in a set of positive measure, we have

$$\limsup_{n\to\infty}\left(\log\frac{1}{|(q;q)_n|}\right)/\left((\log n)\left(c_{[\log n]}\right)\right) < \infty.$$

Then the proof of (b) ⇒ (c) above shows that for τ in a set of positive measure,

$$a_m = O(mc_m).$$

Now under the hypothesis (1.9), this can be true only for τ in a set of measure 0, [11,p.234] and so we have a contradiction.□

We turn to the

Proof of Theorem 1.2
Let us set $n := \chi_m$. Then in the Ostrowski representation of n, we have $b_m = 1$ and $b_j = 0, j < m$. Moreover, (5.3) and (5.4) of Proposition 5.1 give

$$\log|(q;q)_n| \geq \log\frac{2\pi\chi_m \parallel \chi_m\tau \parallel}{e} - 151\frac{\max_{k\leq m}\log a_k}{a_{m+1}} - O(1).$$

If the partial quotients $(a_j)_{j=1}^{\infty}$ are bounded, then the middle term in the last right-hand side is bounded. If they are unbounded, we restrict ourselves to those m for which

$$a_{m+1} \geq a_k, k < m. \tag{5.11}$$

In either case, we obtain infinitely many integers $n = \chi_m$ for which

$$\log|(q;q)_n| \geq \log\left(\chi_m \parallel \chi_m\tau \parallel\right) - O(1).$$

Now

$$\log|1-q^n| = \log|2\sin\pi \parallel \chi_m\tau \parallel| = \log \parallel \chi_m\tau \parallel + O(1)$$

so we deduce that for infinitely many n,

$$\log|(q;q)_{n-1}| \geq \log\chi_m + O(1) \geq \log(n-1) + O(1),$$

so we have (1.11) in a stronger form.□

Proof of Theorem 1.3(a)
As in the previous proof, we choose $n = \chi_m$, and since the partial quotients are unbounded, we may restrict ourselves to those n for which (5.11) holds for $m = m(n)$. Of course then the corresponding $a_{m+1}\to\infty$. From Theorem 3.1,

$$\log|(q;q)_n| - \left(\log\frac{2\pi\chi_m \parallel \chi_m\tau \parallel}{e}\right) - \pi\left(\tau - \frac{\pi_m}{\chi_m}\right)\sum_{k=1}^{\chi_m-1} k\cot k\pi\tau \leq 17. \tag{5.12}$$

Moreover, from (2.4) and Corollary 4.2, for some C independent of m,

$$\left|\pi\left(\tau - \frac{\pi_m}{\chi_m}\right)\sum_{k=1}^{\chi_m - 1} k\cot k\pi\tau\right| \leq \frac{C\chi_m}{\chi_{m+1}}\max_{k\leq m}\log a_k + O(1) = O(1),$$

in view of (5.11) and $\chi_{m+1} \geq a_{m+1}\chi_m$. Then (5.12) gives

$$\log|(q;q)_n| \leq \log\frac{\chi_m}{\chi_{m+1}} + O(1) \leq -\log a_{m+1} + O(1)$$

and so we have (1.14) in a sharper form.□

Proof of Theorem 1.3(b)
The fact that all a_j and hence all b_j are bounded and Proposition 5.1 give

$$\left|\log|(q;q)_n| - \left(\sum_{j=0}^{m} b_j\left(\log\frac{2\pi b_j\chi_j \parallel \chi_j\tau \parallel}{e}\right)\right)\right| = O(m).$$

Here also $\chi_j \parallel \chi_j\tau \parallel$ is bounded above and below by positive constants independent of j, so we obtain

$$|\log|(q;q)_n|| = O(m).$$

Finally, as χ_m grows geometrically, we obtain $m = O(\log\chi_{m-1}) = O(\log n)$.□

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