Sobolev orthogonal polynomials on the unit ball via outward normal derivatives

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Abstract

We analyse a family of mutually orthogonal polynomials on the unit ball with respect to an inner product which involves the outward normal derivatives on the sphere. Using their representation in terms of spherical harmonics, algebraic and analytic properties will be deduced. First, we deduce explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. Then explicit representations for the norms and the kernels will be obtained. Finally, the asymptotic behaviour of the corresponding Christoffel functions is studied.

Key words: Sobolev orthogonal polynomials, unit ball, normal derivative
2010 MSC: 33C50, 42C10

1. Introduction

The term *Sobolev orthogonal polynomials* usually refers to a family of polynomials which are orthogonal with respect to an inner product which simultaneously involves functions and their derivatives. In the one variable case this kind of orthogonality has been studied during the last 25 years, and it constitutes the main subject of a vast literature (see [**?**] and the references therein).

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Sobolev orthogonal polynomials in several variables have a considerably shorter history. There are very few references on the subject and most of them deal with Sobolev orthogonality on the unit ball $\mathbb{B}^d$ of $\mathbb{R}^d$. Usually, the inner product considered is some modification of the classical inner product on the ball

$$\langle f, g\rangle_\mu = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx,$$

where $W_\mu(x) = (1-\|x\|^2)^\mu$ on $\mathbb{B}^d$, $\mu > -1$, and ω_μ is a normalizing constant such that $\langle 1, 1\rangle_\mu = 1$.

One of the first works on this subject was a paper by Y. Xu [**?**], where the inner product

$$\langle f, g\rangle_I = \frac{\lambda}{\sigma_d}\int_{\mathbb{B}^d} \nabla f(x)\cdot\nabla g(x)dx + \frac{1}{\sigma_d}\int_{\mathbb{S}^{d-1}} f(\xi)g(\xi)d\sigma(\xi), \quad \lambda > 0,$$

was considered. Here, $d\sigma$ denotes the surface measure on the sphere $\mathbb{S}^{d-1}$ and σ_d denotes the surface area. In the same article, the author studied another inner product where the second term on the right hand side was replaced by $f(0)g(0)$. In both cases, the central symmetry of the inner products plays an essential role and using spherical polar coordinates a mutually orthogonal polynomial basis is constructed. The polynomials in this basis are expressed in terms of Jacobi polynomials and spherical harmonics mimicking the standard construction of the classical ball polynomials.

In the present paper, we study orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_\mu^S = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)dx + \frac{\lambda}{\sigma_d}\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial \mathbf{n}}(\xi)\frac{\partial g}{\partial \mathbf{n}}(\xi)d\sigma(\xi),$$

where $\lambda > 0$ and $\frac{\partial}{\partial \mathbf{n}}$ stands for the outward normal derivative operator.

Using again spherical polar coordinates, we shall construct a sequence of mutually orthogonal polynomials with respect to $\langle \cdot, \cdot\rangle_\mu^S$, which depends on a family of Sobolev orthogonal polynomials of one variable. The latter are usually called a *non–diagonal Jacobi Sobolev–type* family of orthogonal polynomials and can be expressed in terms of Jacobi polynomials (see [**?**]).

Standard techniques provide us explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. The explicit representations for the norms and the kernels will be obtained.

A very interesting problem in the theory of multivariate orthogonal polynomials is that of finding asymptotic estimates for the Christoffel functions, because these estimates are related to the convergence of the Fourier series. Asymptotics for Christoffel functions associated to the classical orthogonal polynomials on the ball were obtained by Y. Xu in 1996 (see [**?**]). Recently, more general results on the asymptotic behaviour of the Christoffel functions were established by Kroó and Lubinsky [**?** **?**]. Those results include estimates in a quite general case where the orthogonality measure satisfies some regularity conditions.

Since our orthogonal polynomials do not fit into the above mentioned case, the asymptotic of the Christoffel functions deserves special attention. Not surprisingly, our results show that in any compact subset of the interior of the unit ball Christoffel functions in the Sobolev case behave exactly as in the classical case, see Theorem 4. On the sphere the situation is quite different and we can perceive the influence of the outward normal derivatives in the inner product, see Theorem 3.

The paper is organized as follows. In the next section, we state the background materials on orthogonal polynomials on the unit ball and spherical harmonics that we will need later. In Section 3, using spherical polar coordinates we construct explicitly a sequence of mutually orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_\mu^S$. Those polynomials are given in terms of spherical harmonics and a family of univariate Sobolev orthogonal polynomials in the radial part, their properties are studied in Section 4. In Section 5, we deduce explicit connection formulas relating classical multivariate ball polynomials and our family of Sobolev orthogonal polynomials. Moreover, an explicit representation for the kernels is obtained. The asymptotic behaviour of the corresponding Christoffel functions is studied in Section 6. And finally, in Section 7, we consider the special case $d = 2$.

2. Preliminaries

In this section we describe background materials on orthogonal polynomials and spherical harmonics. The first subsection is devoted to recall some properties on the Jacobi polynomials that we shall need later. Second subsection recalls the basic results on spherical harmonics and classical orthogonal polynomials on the unit ball.

2.1. Classical Jacobi polynomials

First, we collect some properties of classical Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$. All of them are well known and can be found in [**?** , Chapt. 22] and [**?**]. For $\alpha, \beta > -1$, these polynomials are orthogonal with respect to the Jacobi inner product

$$(f, g)_{[\alpha,\beta]} = \int_{-1}^{1} f(t)\, g(t)\, w_{\alpha,\beta}(t) dt,$$

where the weight function is defined as

$$w_{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta}, \qquad -1 < t < 1.$$

Jacobi polynomials are normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{1}$$

The squares of the L^2 norms are expressed as

$$h_n^{(\alpha,\beta)} = \left(P_n^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}\right)_{[\alpha,\beta]} = \frac{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\,\Gamma(n+\alpha+\beta+1)}. \tag{2}$$

The polynomial $P_n^{(\alpha,\beta)}(t)$ is of degree n and its leading coefficient $k_n^{(\alpha,\beta)}$ is given by

$$k_n^{(\alpha,\beta)} = \frac{1}{2^n}\binom{2n+\alpha+\beta}{n}. \tag{3}$$

The derivative of a Jacobi polynomial is again a Jacobi polynomial,

$$\frac{d}{dt} P_n^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(t). \tag{4}$$

The following relation between different families of the Jacobi polynomials also hold:

$$P_n^{(\alpha,\beta)}(t) = \frac{n+\alpha+\beta+1}{2\,n+\alpha+\beta+1} P_n^{(\alpha+1,\beta)}(t) - \frac{n+\beta}{2\,n+\alpha+\beta+1} P_{n-1}^{(\alpha+1,\beta)}(t). \qquad (5)$$

As usual we will denote by $p_n^{(\alpha,\beta)}(t)$ the orthonormal Jacobi polynomial of degree n. Moreover, using (1), (2) and (4), we get

$$p_n^{(\alpha,\beta)}(1) = \left(\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}} \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+\beta+1)}{\Gamma(n+1)\,\Gamma(n+\beta+1)} \right)^{1/2} \frac{1}{\Gamma(\alpha+1)}, \qquad (6)$$

$$p_n^{(\alpha,\beta)\prime}(1) = \left(\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+3}} \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+\beta+1)\,n}{\Gamma(n)\,\Gamma(n+\beta+1)} \right)^{1/2} \times \frac{n+\alpha+\beta+1}{\Gamma(\alpha+2)}. \qquad (7)$$

In addition to the Jacobi polynomials we will use the corresponding kernel polynomials defined as

$$K_n(t,u;\alpha,\beta) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(t)\,P_k^{(\alpha,\beta)}(u)}{h_k^{(\alpha,\beta)}}, \qquad (8)$$

which are symmetric functions. When it is clear from the context, we will omit the parameters α and β in the notation. We also denote the partial derivatives

$$K_n^{(0,1)}(t,u) = \frac{\partial}{\partial u} K_n(t,u), \qquad K_n^{(1,1)}(t,u) = \frac{\partial^2}{\partial t\,\partial u} K_n(t,u).$$

It is well known (see [**?** , p. 71]) that

$$K_n(t,1) = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(t). \qquad (9)$$

On the other hand, taking derivatives in the Christoffel–Darboux formula for the kernels in [**?** , (4.5.2) p. 71], and expressing the derivative of the kernel in terms of the Jacobi polynomials of parameters $(\alpha+2,\beta)$, it can be shown that

$$K_n^{(0,1)}(t,1) = 2^{-\alpha-\beta-2} \frac{\Gamma(n+\alpha+\beta+3)}{\Gamma(\alpha+2)\Gamma(n+\beta+1)} \times \left(\frac{n(n+\alpha+\beta+1)}{2n+\alpha+\beta+2} P_n^{(\alpha+2,\beta)}(t) - \frac{(n+1)(n+\beta)}{2n+\alpha+\beta+2} P_{n-1}^{(\alpha+2,\beta)}(t) \right). \qquad (10)$$

In this way, we can compute the values of the kernels at the point $(1,1)$.

Lemma 1. *For $n \geqslant 0$, we get*

$$
\begin{aligned}
K_n(1,1) &= \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\,\Gamma(\alpha+2)}, \\
K_n^{(0,1)}(1,1) &= \frac{2^{-\alpha-\beta-2}}{\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+\beta+3)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n)\,\Gamma(\alpha+3)}, \\
K_n^{(1,1)}(1,1) &= \frac{2^{-\alpha-\beta-3}}{\Gamma(\alpha+2)} \frac{\Gamma(n+\alpha+\beta+3)}{\Gamma(n+\beta+1)} \frac{\Gamma(n+\alpha+2)}{\Gamma(n)\,\Gamma(\alpha+4)} \\
&\quad \times ((\alpha+2)n(n+\alpha+\beta+2)+\beta).
\end{aligned}
$$

2.2. Orthogonal polynomials on the unit ball and spherical harmonics

For a multi–index $\kappa = (\kappa_1, \ldots, \kappa_d) \in \mathbb{N}_0^d$, and $x = (x_1, \ldots, x_d)$, a monomial in the variables $x_1, \ldots, x_d$ is a product

$$x^\kappa = x_1^{\kappa_1} \ldots x_d^{\kappa_d}.$$

The number $|\kappa| = \kappa_1 + \ldots + \kappa_d$ is called the total degree of x^κ. A polynomial P in d variables is a finite linear combination of monomials.

Let Π^d denote the space of polynomials in d real variables. For a given non negative integer n, let Π_n^d denote the linear space of polynomials in several variables of total degree at most n, and let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n. It is well known that

$$\dim \Pi_n^d = \binom{n+d}{n} \quad \text{and} \quad \dim \mathcal{P}_n^d = \binom{n+d-1}{n} = r_n^d.$$

For $x, y \in \mathbb{R}^d$, we use the standard notation of $\|x\|$ for the Euclidean norm of x, and $\langle x, y\rangle$ for the Euclidean product of x and y. The unit ball and the unit sphere in $\mathbb{R}^d$ are denoted, respectively, by

$$\mathbb{B}^d = \{x \in \mathbb{R}^d : \|x\| \leqslant 1\} \qquad \text{and} \qquad \mathbb{S}^{d-1} = \{\xi \in \mathbb{R}^d : \|\xi\| = 1\}.$$

For $\mu \in \mathbb{R}$, let W_μ be the weight function defined by

$$W_\mu(x) = (1 - \|x\|^2)^\mu, \qquad \|x\| < 1.$$

The function W_μ is integrable on the unit ball if $\mu > -1$, for which we denote the normalization constant of W_μ by

$$\omega_\mu = \int_{\mathbb{B}^d} W_\mu(x)\, dx = \frac{\pi^{d/2}\Gamma(\mu+1)}{\Gamma(\mu+d/2+1)}. \tag{11}$$

The weight W_μ is a radial and centrally symmetric function, that is, $W_\mu(-x) = W_\mu(x)$, for all $x \in \mathbb{B}^d$.

Let us consider the classical inner product on the unit ball

$$\langle f, g\rangle_\mu = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)\, g(x)\, W_\mu(x)\, dx,$$

which is normalized so that $\langle 1, 1\rangle_\mu = 1$.

A polynomial $P \in \Pi_n^d$ is called orthogonal with respect to W_μ on the ball if $\langle P, Q\rangle_\mu = 0$ for all $Q \in \Pi_{n-1}^d$. Let $\mathcal{V}_n^d(W_\mu)$ denote the linear space of orthogonal polynomials of total degree n with respect to W_μ. Then $\dim \mathcal{V}_n^d(W_\mu) = r_n^d$.

For $n \geqslant 0$, let $\{P_\nu^n(x) : |\nu| = n\}$ denote a basis of $\mathcal{V}_n^d(W_\mu)$. Notice that every element of $\mathcal{V}_n^d(W_\mu)$ is orthogonal to polynomials of lower degree. If the elements of the basis are also orthogonal to each other, that is, $\langle P_\nu^n, P_\eta^n\rangle_\mu = 0$ whenever $\nu \neq \eta$, we call the basis mutually orthogonal. If, in addition, $\langle P_\nu^n, P_\nu^n\rangle_\mu = 1$, we call the basis orthonormal.

Since the weight function $W_\mu(x)$ is centrally symmetric, then an orthogonal polynomial on the ball of degree n is a sum of monomials of even degree if n is even, and sum of monomials of odd degree if n is odd ([**?** , p. 78]).

Harmonic polynomials of degree n in d–variables are polynomials in $\mathcal{P}_n^d$ that satisfy the Laplace equation $\Delta Y = 0$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}$$

is the usual Laplace operator.

If $Y(x)$ is a harmonic polynomial of degree n, by Euler's equation for homogeneous polynomials, we deduce

$$\langle x, \nabla\rangle Y(x) = \sum_{i=1}^{d} x_i \frac{\partial}{\partial x_i} Y(x) = nY(x). \tag{12}$$

Let $\mathcal{H}_n^d$ denotes the space of harmonic polynomials of degree n. It is well known that

$$a_n^d = \dim \mathcal{H}_n^d = \binom{n+d-1}{n} - \binom{n+d-3}{n}.$$

Spherical harmonics are the restriction of harmonic polynomials to the unit sphere. If $Y \in \mathcal{H}_n^d$, then in spherical–polar coordinates $x = r\xi$, $r = \|x\| \geqslant 0$, and $\xi \in \mathbb{S}^{d-1}$, we get $Y(x) = r^n Y(\xi)$, so that Y is uniquely determined by its restriction to the sphere. We shall also use $\mathcal{H}_n^d$ to denote the space of spherical harmonics of degree n.

Let $d\sigma$ denote the surface measure on $\mathbb{S}^{d-1}$ and let σ_d denote the surface area,

$$\sigma_d = \int_{\mathbb{S}^{d-1}} d\sigma = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}. \tag{13}$$

Using Green's formula on the sphere it is easy to see that spherical harmonics of different degrees are orthogonal with respect to the inner product

$$\langle f, g\rangle_{\mathbb{S}^{d-1}} = \frac{1}{\sigma_d}\int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\sigma(\xi).$$

In spherical–polar coordinates a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$ can be given in terms of the Jacobi polynomials and spherical harmonics (see for example, [**?**]).

Lemma 2. *For $n \in \mathbb{N}_0$ and $0 \leqslant j \leqslant n/2$, let $\{Y_\nu^{n-2j}(x) : 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ denote an orthonormal basis for $\mathcal{H}_{n-2j}^d$. Define*

$$P_{j,\nu}^n(x;\mu) = P_j^{(\mu,\beta_j^n)}(2\,\|x\|^2 - 1)\, Y_\nu^{n-2j}(x), \tag{14}$$

where $\beta_j^n = n - 2j + \frac{d-2}{2}$.

Then the set $\{P_{j,\nu}^n(x;\mu) : 0 \leqslant j \leqslant n/2, 1 \leqslant \nu \leqslant a_{n-2j}^d\}$ is a mutually orthogonal basis of $\mathcal{V}_n^d(W_\mu)$.

More precisely,

$$\langle P_{j,\nu}^n, P_{k,\eta}^m\rangle_\mu = H_{j,n}^\mu \delta_{n,m}\, \delta_{j,k}\, \delta_{\nu,\eta},$$

where $H_{j,n}^\mu = \langle P_{j,\nu}^n, P_{j,\nu}^n\rangle_\mu$ is given by

$$H_{j,n}^\mu = \frac{1}{2^{\mu+\beta_j^n+2}}\,\frac{\sigma_d}{\omega_\mu}\, h_j^{(\mu,\beta_j^n)}.$$

3. A Sobolev inner product on the ball

Let us define the Sobolev inner product

$$\langle f, g\rangle_\mu^S = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} f(x)g(x)W_\mu(x)\,dx + \frac{\lambda}{\sigma_d}\int_{\mathbb{S}^{d-1}} \frac{\partial f}{\partial \mathbf{n}}(\xi)\frac{\partial g}{\partial \mathbf{n}}(\xi)d\sigma(\xi), \tag{15}$$

where $W_\mu(x) = (1-\|x\|^2)^\mu$, $\mu > -1$, is the classical weight function on the ball, ω_μ and σ_d are given by (11) and (13), respectively, and $\frac{\partial}{\partial \mathbf{n}}$ stands for the outward normal derivative operator, which on the sphere $\mathbb{S}^{d-1}$ is given by

$$\frac{\partial f}{\partial \mathbf{n}} = \sum_{i=1}^{d} x_i \frac{\partial f}{\partial x_i}.$$

We observe that the above inner product is centrally symmetric, in the sense that $\langle x^\kappa, x^\tau\rangle_\mu^S = 0$ whenever $|\kappa| + |\tau|$ odd. This implies that an orthogonal polynomial of degree n is a sum of monomials of even degree if n is even, and a sum of monomials of odd degree if n is odd.

In next theorem we will construct a mutually orthogonal basis relative to the previous Sobolev inner product, which will be given explicitly in terms of spherical harmonics and a family of Sobolev orthogonal polynomials in one variable.

Theorem 1. *Let $\{q_j^{(\alpha,\beta;M)}(t)\}_{j\geqslant 0}$ denote the univariate Sobolev orthogonal polynomials orthogonal with respect to the Sobolev inner product*

$$(f,g)^S_{[\alpha,\beta;M]} = \int_{-1}^{1} f(t)g(t)(1-t)^\alpha(1+t)^\beta\,dt + \mathbf{f}(1)M\mathbf{g}(1)^t, \tag{16}$$

where $\mathbf{f}(1) = (f(1), f'(1))$ and M is a 2×2 symmetric positive semidefinite matrix. Let $\{Y_\nu^{n-2j}(x) \,:\, 1 \leqslant \nu \leqslant a^d_{n-2j}\}$ be an orthonormal basis of the spherical harmonics $\mathcal{H}^d_{n-2j}$. Let us define the polynomials in d variables

$$Q^n_{j,\nu}(x) = q_j^{(\mu,\beta_j^n;M_{n-2j})}(2\|x\|^2-1)Y_\nu^{n-2j}(x), \tag{17}$$

with $\beta_j^n = n-2j+\delta$, $\delta = \frac{d-2}{2}$,

$$A_0 = \lambda\, 2^{\delta+\mu+2}\frac{\omega_\mu}{\sigma_d}, \tag{18}$$

and

$$M_{n-2j} = 2^{n-2j}\, A_0 \begin{bmatrix} (n-2j)^2 & 4(n-2j) \\ 4(n-2j) & 16 \end{bmatrix}. \tag{19}$$

Then, for $n \geqslant 0$*, the set* $\{Q^n_{j,\nu}(x) \,:\, 0 \leqslant j \leqslant n/2,\ 1 \leqslant \nu \leqslant a^d_{n-2j}\}$ *is a mutually orthogonal basis of* $\mathcal{V}^d_n(W_\mu, S)$*, the linear space of polynomials of degree* n *which are orthogonal with respect to the Sobolev inner product (15).*

Moreover,

$$\tilde{H}^\mu_{j,\nu} = \langle Q^n_{j,\nu}, Q^n_{j,\nu}\rangle^S_\mu = \frac{\lambda}{2^{n-2j}\, A_0}\, \tilde{h}_j^{(\mu,\beta^n_j;M_{n-2j})}, \tag{20}$$

where $\tilde{h}_j^{(\mu,\beta;M)} = (q_j^{(\mu,\beta;M)},\, q_j^{(\mu,\beta;M)})^S_{[\mu,\beta;M]}$.

Proof. In order to check the orthogonality, we need to compute the product

$$\begin{aligned} \langle Q^n_{j,\nu}, Q^m_{k,\eta}\rangle^S_\mu &= \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} Q^n_{j,\nu}(x) Q^m_{k,\eta}(x) W_\mu(x)\, dx \\ &\quad + \frac{\lambda}{\sigma_d}\int_{\mathbb{S}^{d-1}} \frac{\partial Q^n_{j,\nu}}{\partial \mathbf{n}}(\xi)\frac{\partial Q^m_{k,\eta}}{\partial \mathbf{n}}(\xi)\, d\sigma(\xi). \end{aligned} \tag{21}$$

Let us start with the computation of the first integral.

$$I_1 = \frac{1}{\omega_\mu}\int_{\mathbb{B}^d} Q^n_{j,\nu}(x) Q^m_{k,\eta}(x) W_\mu(x)\, dx.$$

Using spherical–polar coordinates and the orthogonality of the spherical harmonics we obtain

$$\begin{aligned} I_1 &= \frac{\sigma_d}{\omega_\mu}\int_0^1 q_j^{(\mu,\beta^n_j)}(2r^2-1) q_k^{(\mu,\beta^m_k)}(2r^2-1)(1-r^2)^\mu\, r^{n-2j+m-2k+d-1}\, dr \\ &\quad \times \delta_{n-2j,m-2k}\delta_{\nu\eta} \\ &= \frac{\sigma_d}{\omega_\mu}\int_0^1 q_j^{(\mu,\beta^n_j)}(2r^2-1) q_k^{(\mu,\beta^n_j)}(2r^2-1)(1-r^2)^\mu\, r^{2(n-2j)+d-1}\, dr \\ &\quad \times \delta_{n-2j,m-2k}\delta_{\nu\eta}, \end{aligned}$$

where $q_j^{(\mu,\beta^n_j)} \equiv q_j^{(\mu,\beta^n_j;M_{n-2j})}$ and $q_k^{(\mu,\beta^m_k)} \equiv q_k^{(\mu,\beta^m_k;M_{m-2k})}$.

Finally, the change of variables $t = 2r^2 - 1$ moves the integral to the interval $[-1, 1]$,

$$\begin{aligned} I_1 &= \frac{1}{2^{\beta^n_j+\mu+2}}\frac{\sigma_d}{\omega_\mu}\int_{-1}^1 q_j^{(\mu,\beta^n_j)}(t) q_k^{(\mu,\beta^n_j)}(t)(1-t)^\mu(1+t)^{\beta^n_j}\, dt \\ &\quad \times \delta_{n-2j,m-2k}\delta_{\nu\eta}. \end{aligned} \tag{22}$$

Let us now compute the second integral in (21),

$$I_2 = \frac{\lambda}{\sigma_d} \int_{\mathbb{S}^{d-1}} \frac{\partial Q^n_{j,\nu}}{\partial \mathbf{n}}(\xi) \frac{\partial Q^m_{k,\eta}}{\partial \mathbf{n}}(\xi)\, d\sigma(\xi).$$

In order to easily get that integral, we need some previous results.

Computing the normal derivatives

$$\frac{\partial}{\partial \mathbf{n}} \Big(q_j^{(\mu,\beta;M)}(2\|\xi\|^2 - 1) \Big) = 4\|\xi\|^2 (q_j^{(\mu,\beta;M)})'(2\|\xi\|^2 - 1),$$

and using Euler's formula (12), we deduce

$$\begin{aligned} &\frac{\partial}{\partial \mathbf{n}} \Big(q_j^{(\mu,\beta;M)}(2\|\xi\|^2 - 1)\, Y_\nu^{n-2j}(\xi) \Big) \\ &= \Big(4\|\xi\|^2 (q_j^{(\mu,\beta;M)})'(2\|\xi\|^2 - 1) + (n-2j) q_j^{(\mu,\beta;M)}(2\|\xi\|^2 - 1) \Big)\, Y_\nu^{n-2j}(\xi). \end{aligned}$$

Thus, the second integral splits into four terms,

$$\begin{aligned} I_2 &= \frac{\lambda}{\sigma_d} \Big(16\, q_j'(1)\, q_k'(1) + 4\,(n-2j)\, q_j'(1)\, q_k(1) + 4\,(n-2j)\, q_j(1)\, q_k'(1) \\ &\quad + (n-2j)^2\, q_j(1)\, q_k(1) \Big) \int_{\mathbb{S}^{d-1}} Y_\nu^{n-2j}(\xi)\, Y_\eta^{n-2k}(\xi)\, d\sigma(\xi) \\ &= \lambda \Big(16\, (q_j'(1)\, q_k'(1) + 4\,(n-2j)\, q_j'(1)\, q_k(1) + 4\,(n-2j)\, q_j(1)\, q_k'(1) \\ &\quad + (n-2j)^2\, q_j(1)\, q_k(1) \Big)\, \delta_{n-2j,m-2k}\, \delta_{\nu,\eta}, \end{aligned}$$

where we have omitted the superscript in $q_j^{(\mu,\beta_j^n;M_{n-2j})}$ for brevity.

Finally, this can be written in matrix form as follows

$$I_2 = \lambda \mathbf{q}_j^{(\mu,\beta_j^n;M_{n-2j})}(1)\, \tilde{M}_{n-2j}\, \mathbf{q}_k^{(\mu,\beta_j^n;M_{n-2j})}(1)^t \tag{23}$$

where $\mathbf{q}_j^{(\mu,\beta_j^n;M_{n-2j})}(1) = \Big(q_j^{(\mu,\beta_j^n;M_{n-2j})}(1), (q_j^{(\mu,\beta_j^n;M_{n-2j})})'(1) \Big)$ and

$$\tilde{M}_{n-2j} = \begin{bmatrix} (n-2j)^2 & 4(n-2j) \\ 4(n-2j) & 16 \end{bmatrix}.$$

Observe that $M_{n-2j} = 2^{n-2j}\, A_0\, \tilde{M}_{n-2j}$.

To end the proof, we just have to take together (22) and (23) to get the value of (21) in terms of the Sobolev inner product (16) as

$$\begin{aligned} \langle Q^n_{j,\nu}, Q^m_{k,\eta} \rangle^S_\mu &= \frac{\lambda}{2^{n-2j}\, A_0} \left(q_j^{(\mu,\beta_j^n;M_{n-2j})}, q_k^{(\mu,\beta_j^n;M_{n-2j})} \right)^S_{[\mu,\beta_j^n;M_{n-2j}]} \\ &\quad \times \delta_{n-2j,m-2k}\, \delta_{\nu,\eta}. \end{aligned}$$

Then the result follows from the orthogonality of the univariate Sobolev orthogonal polynomials.

■

4. The univariate non–diagonal Sobolev inner product

In this section we will explore some properties of the univariate Sobolev orthogonal polynomials involved in (17).

Let $(\cdot,\cdot)^S_{[\alpha,\beta;M]}$ be the non–diagonal Sobolev inner product defined in (16) by

$$(f,g)^S_{[\alpha,\beta;M]} = \int_{-1}^{1} f(t)\, g(t)(1-t)^\alpha (1+t)^\beta dt + \mathbf{f}(1) M \mathbf{g}(1)^t,$$

where M is a positive semidefinite matrix and $\mathbf{f}(1) = (f(1), f'(1))$.

Let $\{q_n^{(\alpha,\beta;M)}(t)\}_{n\geqslant 0}$ be the orthogonal polynomials with respect to this inner product, normalized with leading coefficient $k_n^{(\alpha,\beta)}$ given in (3). Some properties for the monic orthogonal polynomials with respect to this inner product can be found in [**?** **?**].

In what follows, when not confusing, we will simplify the notations $q_n^{(\alpha,\beta;M)} \equiv q_n^{(\alpha,\beta)} \equiv q_n$, and $(f,g)^S_{[\alpha,\beta;M]} \equiv (f,g)^S$.

These univariate Sobolev orthogonal polynomials can be expressed in terms of the classical Jacobi polynomials as follows.

Lemma 3. *For $\alpha, \beta > -1$, it holds*

$$q_j^{(\alpha,\beta)}(t) = b_{j,j}^{(\alpha,\beta)}\, P_j^{(\alpha+2,\beta)}(t) + b_{j,j-1}^{(\alpha,\beta)}\, P_{j-1}^{(\alpha+2,\beta)}(t) + b_{j,j-2}^{(\alpha,\beta)}\, P_{j-2}^{(\alpha+2,\beta)}(t), \qquad (24)$$

where

$$\begin{aligned} b_{j,j}^{(\alpha,\beta)} &= \frac{(j+\alpha+\beta+2)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+2)(2j+\alpha+\beta+1)}, \\ b_{j,j-1}^{(\alpha,\beta)} &= \frac{(j+\alpha+\beta+1)}{2j+\alpha+\beta}\left(-\frac{2(j+\beta)}{2j+\alpha+\beta+2} - c_{j-1}^{j}\right), \\ b_{j,j-2}^{(\alpha,\beta)} &= \frac{j+\beta-1}{2j+\alpha+\beta}\left(\frac{j+\beta}{2j+\alpha+\beta+1} + c_{j-2}^{j}\right), \end{aligned}$$

with

$$
\begin{aligned}
c_{j-1}^{j} &= 2^{-\alpha-\beta-2}\frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(j+\beta)}\mathbf{P}_j(1)\,\Lambda_{j-1}\begin{bmatrix} 2 \\ (j-1)(j+\alpha+\beta) \end{bmatrix}, \\
c_{j-2}^{j} &= 2^{-\alpha-\beta-2}\frac{\Gamma(j+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(j+\beta)}\mathbf{P}_j(1)\,\Lambda_{j-1}\begin{bmatrix} 2 \\ j(j+\alpha+\beta+1) \end{bmatrix},
\end{aligned}
$$

and

$$
\mathbf{P}_j(1) \equiv \mathbf{P}_j^{(\alpha,\beta)}(1) = \left(P_j^{(\alpha,\beta)}(1), (P_j^{(\alpha,\beta)})'(1)\right),
$$

$$
\Lambda_{j-1} \equiv \Lambda_{j-1}^{(\alpha,\beta;M)} = (I + M\mathcal{K}_{j-1})^{-1}\,M, \tag{25}
$$

$$
\mathcal{K}_{j-1} \equiv \mathcal{K}_{j-1}^{(\alpha,\beta)} = \begin{bmatrix} K_{j-1}(1,1) & K_{j-1}^{(1,0)}(1,1) \\ K_{j-1}^{(0,1)}(1,1) & K_{j-1}^{(1,1)}(1,1) \end{bmatrix}. \tag{26}
$$

Proof. If we expand $q_j^{(\alpha,\beta)}$ in terms of Jacobi polynomials,

$$
q_j^{(\alpha,\beta)}(t) = \sum_{i=0}^{j} b_{j,i}^{(\alpha,\beta)} P_i^{(\alpha+2,\beta)}(t),
$$

and using standard techniques

$$
b_{j,i}^{(\alpha,\beta)} = \frac{(q_j^{(\alpha,\beta)}(t), P_i^{(\alpha+2,\beta)}(t))_{[\alpha+2,\beta]}}{h_i^{(\alpha+2,\beta)}} = \frac{(q_j^{(\alpha,\beta)}(t), P_i^{(\alpha,\beta)}(t)\,(1-t)^2)^S}{h_i^{(\alpha+2,\beta)}}.
$$

Then, for $i < n-2$, $b_{j,i}^{(\alpha,\beta)} = 0$, and so relation (24) holds. Moreover, the coefficient $b_{j,j}^{(\alpha,\beta)}$ can be determined using the leading coefficients of the polynomials $q_j^{(\alpha,\beta)}(t)$ and $P_j^{(\alpha+2,\beta)}(t)$ both given by (3)

$$
b_{j,j}^{(\alpha,\beta)} = \frac{k_j^{(\alpha,\beta)}}{k_j^{(\alpha+2,\beta)}} = \frac{(j+\alpha+\beta+2)(j+\alpha+\beta+1)}{(2j+\alpha+\beta+2)(2j+\alpha+\beta+1)}.
$$

We determine the other two coefficients using Proposition 2 in [**?**],

$$
q_j^{(\alpha,\beta)}(t) = P_j^{(\alpha,\beta)}(t) - \mathbf{P}_j^{(\alpha,\beta)}(1)\,\Lambda_{j-1}\mathbf{K}_{j-1}^{\alpha,\beta}(t,1), \tag{27}
$$

where

$$
\mathbf{K}_{j-1}^{\alpha,\beta}(t,1) = \begin{bmatrix} K_{j-1}(t,1;\alpha,\beta) \\ K_{j-1}^{(0,1)}(t,1;\alpha,\beta) \end{bmatrix}. \tag{28}
$$

If we apply equation (5) twice we obtain

$$\begin{aligned} P_j^{(\alpha,\beta)}(t) = & \frac{(j+\alpha+\beta+1)(j+\alpha+\beta+2)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta+2)} P_j^{(\alpha+2,\beta)}(t) \\ & - \frac{2(j+\alpha+\beta+1)(j+\beta)}{(2j+\alpha+\beta+2)(2j+\alpha+\beta)} P_{j-1}^{(\alpha+2,\beta)}(t) \\ & + \frac{(j+\beta)(j+\beta-1)}{(2j+\alpha+\beta+1)(2j+\alpha+\beta)} P_{j-2}^{(\alpha+2,\beta)}(t). \end{aligned}$$

Substituting in (27), and using (9) and (10), the result follows.

■

For $n \geqslant 0$, we denote by

$$\tilde{K}_n(t,u) = \sum_{j=0}^{n} \frac{q_j^{(\alpha,\beta)}(t)\, q_j^{(\alpha,\beta)}(u)}{\tilde{h}_j^{(\alpha,\beta)}}$$

the reproducing kernels associated with the polynomials $q_j^{(\alpha,\beta)}(t)$.

We need to establish a relationship between these kernels and the kernels of Jacobi polynomials defined in Section 2.1. To this end we need the following lemmas.

Lemma 4. *The matrix $\Lambda_j = (I + M\mathcal{K}_j)^{-1} M$ is symmetric. Moreover,*

$$\Lambda_{j-1}\mathbf{P}_j(1)^t h_j^{-1}\mathbf{P}_j(1)\Lambda_j = \Lambda_{j-1} - \Lambda_j,$$

where $h_j = h_j^{(\alpha,\beta)}$ is given in (2).

Proof. Using Sherman–Morrison–Woodbury identity (see [**?**]), we get

$$(I + M\mathcal{K}_j)^{-1} = I - M(\mathcal{K}_j^{-1} + M)^{-1},$$

thus

$$(I + M\mathcal{K}_j)^{-1}M = M - M(\mathcal{K}_j^{-1} + M)^{-1}M,$$

and the symmetry of Λ_j follows from the symmetry of M and $\mathcal{K}_j^{-1}$. On the other hand,

$$
\begin{aligned}
&\Lambda_{j-1}\mathbf{P}_j(1)^t h_j^{-1}\mathbf{P}_j(1)\Lambda_j = \\
&\quad = \left(I + M\mathcal{K}_{j-1}\right)^{-1}\left(M\mathcal{K}_j - M\mathcal{K}_{j-1}\right)\left(I + M\mathcal{K}_j\right)^{-1} M \\
&\quad = \left(I + M\mathcal{K}_{j-1}\right)^{-1}\left(\left(I + M\,\mathcal{K}_j\right) - \left(I + M\mathcal{K}_{j-1}\right)\right)\left(I + M\mathcal{K}_j\right)^{-1} M \\
&\quad = \left(I + M\mathcal{K}_{j-1}\right)^{-1} M - \left(I + M\mathcal{K}_j\right)^{-1} M.
\end{aligned}
$$

■

Lemma 5. *For $j \geqslant 1$,*

$$(\tilde{h}_j^{(\alpha,\beta)})^{-1} = (h_j^{(\alpha,\beta)})^{-1} - (h_j^{(\alpha,\beta)})^{-2}\,\mathbf{P}_j^{(\alpha,\beta)}(1)\Lambda_j\mathbf{P}_j^{(\alpha,\beta)}(1)^t.$$

Proof. First, we get the relation between the norms, using (27)

$$\tilde{h}_j^{(\alpha,\beta)} = (q_j, q_j)^S = (q_j, p_j)^S = h_j^{(\alpha,\beta)} + \mathbf{q}_j^{(\alpha,\beta)}(1)M\mathbf{P}_j^{(\alpha,\beta)}(1)^t.$$

Taking into account that $\mathbf{q}_j^{(\alpha,\beta)}(1) = \mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\mathcal{K}_{j-1}\right)^{-1}$, we get

$$\tilde{h}_j^{(\alpha,\beta)} = h_j^{(\alpha,\beta)} + \mathbf{P}_j^{(\alpha,\beta)}(1)\,\Lambda_{j-1}\,\mathbf{P}_j^{(\alpha,\beta)}(1)^t.$$

On the other hand, from

$$M\mathbf{P}_j^{(\alpha,\beta)}(1)^t h_j^{-2}\mathbf{P}_j^{(\alpha,\beta)}(1) = h_j^{-1}M(\mathcal{K}_j - \mathcal{K}_{j-1}) = h_j^{-1}[(I+M\mathcal{K}_j) - (I+M\mathcal{K}_{j-1})],$$

we get

$$
\begin{aligned}
&\mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\,\mathcal{K}_{j-1}\right)^{-1} M\,\mathbf{P}_j^{(\alpha,\beta)}(1)^t\,h_j^{-2}\,\mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\mathcal{K}_j\right)^{-1} M\mathbf{P}_j^{(\alpha,\beta)}(1)^t \\
&\quad = h_j^{-1}\,\mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\mathcal{K}_{j-1}\right)^{-1}\left(\left(I + M\,\mathcal{K}_j\right) - \left(I + M\,\mathcal{K}_{j-1}\right)\right) \\
&\qquad \times \left(I + M\,\mathcal{K}_j\right)^{-1} M\,\mathbf{P}_j^{(\alpha,\beta)}(1)^t \\
&\quad = h_j^{-1}\,\mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\,\mathcal{K}_{j-1}\right)^{-1} M\,\mathbf{P}_j^{(\alpha,\beta)}(1)^t \\
&\qquad - h_j^{-1}\,\mathbf{P}_j^{(\alpha,\beta)}(1)\left(I + M\,\mathcal{K}_j\right)^{-1} M\,\mathbf{P}_j^{(\alpha,\beta)}(1)^t.
\end{aligned}
$$

Then, it is easy to show that

$$\tilde{h}_j^{(\alpha,\beta)} \left(h_j^{-1} - (h_j^{-1})^2 \, \mathbf{P}_j^{(\alpha,\beta)}(1) \, (I + M \, \mathcal{K}_j)^{-1} \; M \, \mathbf{P}_j^{(\alpha,\beta)}(1)^t \right) = 1,$$

and the result follows.

■

Now we are ready to derive an explicit formula for the univariate kernels.

Proposition 1. *For $j \geqslant 0$, we get*

$$\begin{aligned} q_j^{(\alpha,\beta)}(t) \, q_j^{(\alpha,\beta)}(u) \, (\tilde{h}_j^{(\alpha,\beta)})^{-1} &= P_j^{(\alpha,\beta)}(t) \, P_j^{(\alpha,\beta)}(u) \, (h_j^{(\alpha,\beta)})^{-1} \\ &- \quad \mathbf{K}_j(t,1)^t \, \Lambda_j \, \mathbf{K}_j(u,1) + \mathbf{K}_{j-1}(t,1)^t \, \Lambda_{j-1} \, \mathbf{K}_{j-1}(u,1). \end{aligned} \tag{29}$$

As a consequence, for $n \geqslant 0$,

$$\tilde{K}_n(t,u) = K_n(t,u) - \mathbf{K}_n(t,1)^t \, \Lambda_n \, \mathbf{K}_n(u,1). \tag{30}$$

Proof. Using (27) and Lemma 5, we get

$$\begin{aligned} q_j(t) \, q_j(u) \, \tilde{h}_j^{-1} &= (P_j(t) - \mathbf{P}_j(1) \, \Lambda_{j-1} \, \mathbf{K}_{j-1}(t,1)) \\ & \quad \times (P_j(u) - \mathbf{P}_j(1) \, \Lambda_{j-1} \, \mathbf{K}_{j-1}(u,1)) \\ & \quad \times (h_j^{-1} - (h_j^{-1})^2 \, \mathbf{P}_j(1) \, \Lambda_j \, \mathbf{P}_j(1)^t). \end{aligned}$$

Taking into account Lemma 4 and $\mathbf{P}_j(1) h_j^{-1} P_j(u) = \mathbf{K}_j(u,1)^t - \mathbf{K}_{j-1}(u,1)^t$, we obtain

$$\begin{aligned} q_j(t) \, q_j(u) \, \tilde{h}_j^{-1} &= P_j(t) \, P_j(u) \, h_j^{-1} - (\mathbf{K}_j(t,1)^t - \mathbf{K}_{j-1}(t,1)^t) \Lambda_{j-1} \mathbf{K}_{j-1}(u,1) \\ &- (\mathbf{K}_j(t,1)^t - \mathbf{K}_{j-1}(t,1)^t) \Lambda_j (\mathbf{K}_j(u,1) - \mathbf{K}_{j-1}(u,1)) \\ &+ (\mathbf{K}_j(t,1)^t - \mathbf{K}_{j-1}(t,1)^t)(\Lambda_{j-1} - \Lambda_j) \mathbf{K}_{j-1}(u,1) \\ &- (\mathbf{K}_j(u,1)^t - \mathbf{K}_{j-1}(u,1)^t) \Lambda_{j-1} \mathbf{K}_{j-1}(t,1) \\ &+ \mathbf{K}_{j-1}(t,1)^t \Lambda_{j-1} \mathbf{P}_j(1)^t \mathbf{P}_j(1) \Lambda_j \mathbf{K}_j(u,1). \end{aligned}$$

Therefore, we get (29) and a telescopic sum gives (30).

■

5. Multivariate Sobolev orthogonal polynomials

In this section, we will express multivariate Sobolev orthogonal polynomials in terms of classical ball polynomials. To this end, using the following lemmas we will simplify the matrix $\Lambda_m^{(\mu,k+\delta;M_k)} = (I + M_k\mathcal{K}_m)^{-1}M_k$ defined in (25), where M_k was introduced in (19), and $\mathcal{K}_m = \mathcal{K}_m^{(\mu,k+\delta)}$ was given in (26).

Lemma 6. *Let M and $\mathcal{K}$ be 2×2 matrices with* $\det(M) = 0$*. Then*

$$(I + M\mathcal{K})^{-1} M = \frac{1}{\Delta}M,$$

where

$$\Delta = 1 + \operatorname{trace}(M\mathcal{K})$$

is assumed non–zero.

PROOF. This is a straightforward calculation. ■

Lemma 7. *Let $k \geqslant 0$, then*

$$\Lambda_m^{(\mu,k+\delta;M_k)} = \frac{1}{\Delta_{k,m}}M_k,$$

where

$$\Delta_{k,m} = 1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 8kK_m^{(1,0)}(1,1) + 16K_m^{(1,1)}(1,1) \right\}. \tag{31}$$

PROOF. We see that $\det(M_k) = 0$, and

$$\begin{aligned} &1 + \operatorname{trace}(M_k\mathcal{K}_m) \\ &= 1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 4kK_m^{(1,0)}(1,1) + 4kK_m^{(0,1)}(1,1) + 16K_m^{(1,1)}(1,1) \right\}. \end{aligned}$$

Then Lemma 6 gives the result.

■

If we replace $t = 2\|x\|^2 - 1$ in equation (24), multiply the result times $Y_\nu^{n-2j}(x)$, and use (14) and (17), we can express Sobolev orthogonal polynomials in terms of ball polynomials. This representation is given in next theorem.

Theorem 2. *Let $n \in \mathbb{N}_0$, $0 \leqslant j \leqslant n/2$, and $1 \leqslant \nu \leqslant a_{n-2j}^d$. Then,*

$$Q_{j,\nu}^n(x) = b_{j,j}^{(\mu,\beta_j^n)}\, P_{j,\nu}^n(x;\mu+2) + b_{j,j-1}^{(\mu,\beta_j^n)}\, P_{j-1,\nu}^{n-2}(x;\mu+2) + b_{j,j-2}^{(\mu,\beta_j^n)}\, P_{j-2,\nu}^{n-4}(x;\mu+2),$$

where $P_{j,\nu}^n(x;\mu+2)$ are the polynomials in the ball orthogonal with respect to $W_{\mu+2}(x)$, and

$$\begin{aligned}
b_{j,j}^{(\mu,\beta_j^n)} &= \frac{(n-j+\mu+\delta+2)(n-j+\mu+\delta+1)}{(n+\mu+\delta+2)(n+\mu+\delta+1)},\\
b_{j,j-1}^{(\mu,\beta_j^n)} &= \frac{n-j+\mu+\delta+1}{n+\mu+\delta}\Big(-\frac{2(n-j+\delta)}{n+\mu+\delta+2} - d_j^n\, a_{j-1}^{n-2}\Big),\\
b_{j,j-2}^{(\mu,\beta_j^n)} &= \frac{n-j+\delta-1}{n+\mu+\delta}\Big(\frac{n-j+\delta}{n+\mu+\delta+1} + d_j^n\, a_j^n\Big),
\end{aligned}$$

with

$$\begin{aligned}
d_j^n &= \frac{A_0\Gamma(n-j+\mu+\delta+1)\,\Gamma(j+\mu+1)}{2^{\mu+\delta+2}\Gamma(\mu+1)\Gamma(\mu+2)\Gamma(n-j+\delta)\,j!}\\
&\quad \times ((\mu+1)(n-2j) + 2j(n-j+\mu+\delta+1)),\\
a_j^n &= 2\,((n-2j) + 2j(n-j+\mu+\delta+1)).
\end{aligned}$$

Let us define the kernels of the ball orthogonal polynomials and the kernels of the Sobolev orthogonal polynomials in the usual way,

$$\mathbb{L}_n(x,y) = \sum_{m=0}^{n}\sum_{j=0}^{[m/2]}\sum_{\nu=1}^{a_{m-2j}^d} P_{j,\nu}^m(x)\, P_{j,\nu}^m(y)\,(H_{j,\nu}^m)^{-1}, \tag{32}$$

$$\tilde{\mathbb{L}}_n(x,y) = \sum_{m=0}^{n}\sum_{j=0}^{[m/2]}\sum_{\nu=1}^{a_{m-2j}^d} Q_{j,\nu}^m(x)\, Q_{j,\nu}^m(y)\,(\tilde{H}_{j,\nu}^m)^{-1}. \tag{33}$$

Then, we can establish a relation between these kernels by means of the kernels of univariate Jacobi polynomials. From now on, let C_k^δ denote the usual ultraspherical polynomial ([**?** , (4.7.1) in p. 80]).

Proposition 2. *For $n \geqslant 0$ and $d \geqslant 3$, we get*

$$\begin{aligned}
\tilde{\mathbb{L}}_n(x,y) &= \mathbb{L}_n(x,y) \\
&\quad -\frac{A_0}{\lambda}\sum_{k=0}^{n}\mathbf{K}^{\mu,k+\delta}_{\left[\frac{n-k}{2}\right]}(2r^2-1,1)^t\,\Lambda^{(\mu,k+\delta;M_k)}_{\left[\frac{n-k}{2}\right]}\,\mathbf{K}^{\mu,k+\delta}_{\left[\frac{n-k}{2}\right]}(2s^2-1,1) \\
&\quad \times 2^k\,(r\,s)^k\,\frac{k+\delta}{\delta}\,C^{\delta}_k(\langle\xi,\varrho\rangle),
\end{aligned}$$

where $x = r\,\xi$, $y = s\,\varrho$, $r = \|x\|$, $s = \|y\|$, $\xi, \varrho \in \mathbb{S}^{d-1}$.

Proof. Using (17), (20), and (29), we get

$$\begin{aligned}
&Q^m_{j,\nu}(x)Q^m_{j,\nu}(y)(\tilde{H}^m_{j,\nu})^{-1} \\
&\quad = q_j^{(\mu,\beta_j^m)}(2\,r^2-1)Y^{m-2j}_\nu(x)q_j^{(\mu,\beta_j^m)}(2\,s^2-1)Y^{m-2j}_\nu(y) \\
&\qquad \times 2^{\beta_j^m+\mu+2}\,\frac{\omega_\mu}{\sigma_d}\,(\tilde{h}_j^{(\mu,\beta_j^m)})^{-1} \\
&\quad = \Big(P_j^{(\mu,\beta_j^m)}(2\,r^2-1)P_j^{(\mu,\beta_j^m)}(2\,s^2-1)\Big(h_j^{(\mu,\beta_j^m)}\Big)^{-1} \\
&\qquad -\mathbf{K}_j^{\mu,\beta_j^m}(2r^2-1,1)^t\,\Lambda_j^{(\mu,\beta_j^m;M_{m-2j})}\,\mathbf{K}_j^{\mu,\beta_j^m}(2s^2-1,1) \\
&\qquad +\mathbf{K}_{j-1}^{\mu,\beta_j^m}(2r^2-1,1)^t\,\Lambda_{j-1}^{(\mu,\beta_j^m;M_{m-2j})}\,\mathbf{K}_{j-1}^{\mu,\beta_j^m}(2s^2-1,1)\Big) \\
&\qquad\quad \times 2^{m-2j}\,\frac{A_0}{\lambda}\,Y^{m-2j}_\nu(x)\,Y^{m-2j}_\nu(y).
\end{aligned}$$

Then, summing above expressions for m, j, and ν we obtain

$$\begin{aligned}
&\tilde{\mathbb{L}}_n(x,y) = \mathbb{L}_n(x,y) \\
&\quad -\sum_{m=0}^{n}\sum_{j=0}^{[m/2]}\sum_{\nu=1}^{a^d_{m-2j}}\mathbf{K}_j^{\mu,\beta_j^m}(2r^2-1,1)^t\,\Lambda_j^{(\mu,\beta_j^m;M_{m-2j})}\,\mathbf{K}_j^{\mu,\beta_j^m}(2s^2-1,1) \\
&\qquad \times 2^{m-2j}\,\frac{A_0}{\lambda}\,(r\,s)^{m-2j}\,Y^{m-2j}_\nu(\xi)\,Y^{m-2j}_\nu(\varrho) \\
&\quad +\sum_{m=2}^{n}\sum_{j=1}^{[m/2]}\sum_{\nu=1}^{a^d_{m-2j}}\mathbf{K}_{j-1}^{\mu,\beta_j^m}(2r^2-1,1)^t\,\Lambda_{j-1}^{(\mu,\beta_j^m;M_{m-2j})}\,\mathbf{K}_{j-1}^{\mu,\beta_j^m}(2s^2-1,1) \\
&\qquad \times 2^{m-2j}\,\frac{A_0}{\lambda}\,(r\,s)^{m-2j}\,Y^{m-2j}_\nu(\xi)\,Y^{m-2j}_\nu(\varrho),
\end{aligned}$$

where we have used that $\mathbf{K}_{-1}^{\mu,\beta_j^m}(2r^2-1,1) = 0$. Taking into account the addition formula of spherical harmonics for $d \geqslant 3$ (see [**?** , p. 9])

$$\sum_{\nu=1}^{a_k^d} Y_\nu^k(\xi)\, Y_\nu^k(\varrho) = \frac{k+\delta}{\delta}\, C_k^\delta(\langle \xi, \varrho\rangle),$$

we deduce

$$\begin{aligned}
\tilde{\mathbb{L}}_n(x,y) &= \mathbb{L}_n(x,y) \\
&- \sum_{m=0}^{n} \sum_{j=0}^{[m/2]} \mathbf{K}_j^{\mu,\beta_j^m}(2r^2-1,1)^t\, \Lambda_j^{(\mu,\beta_j^m;M_{m-2j})}\, \mathbf{K}_j^{\mu,\beta_j^m}(2s^2-1,1) \\
&\times 2^{m-2j}\frac{A_0}{\lambda}\,(r\,s)^{m-2j}\,\frac{m-2j+\delta}{\delta}\, C_{m-2j}^\delta(\langle\xi,\varrho\rangle) \\
&+ \sum_{m=2}^{n} \sum_{j=1}^{[m/2]} \mathbf{K}_{j-1}^{\mu,\beta_j^m}(2r^2-1,1)^t\, \Lambda_{j-1}^{(\mu,\beta_j^m;M_{m-2j})}\, \mathbf{K}_{j-1}^{\mu,\beta_j^m}(2s^2-1,1) \\
&\times 2^{m-2j}\frac{A_0}{\lambda}\,(r\,s)^{m-2j}\,\frac{m-2j+\delta}{\delta}\, C_{m-2j}^\delta(\langle\xi,\varrho\rangle).
\end{aligned}$$

Therefore, since $\beta_{j+1}^{m+2} = \beta_j^m$, a change in the indexes in the last term gives

$$\tilde{\mathbb{L}}_n(x,y) = \mathbb{L}_n(x,y) - F(n) - F(n-1),$$

where

$$\begin{aligned}
F(n) &= \frac{A_0}{\lambda}\sum_{j=0}^{[\frac{n}{2}]} \mathbf{K}_j^{\mu,\beta_j^n}(2r^2-1,1)^t\, \Lambda_j^{(\mu,\beta_j^n;M_{m-2j})}\, \mathbf{K}_j^{\mu,\beta_j^n}(2s^2-1,1) \\
&\times 2^{n-2j}\,\frac{\omega_\mu}{\sigma_d}\,(r\,s)^{n-2j}\,\frac{n-2j+\delta}{\delta}\, C_{n-2j}^\delta(\langle\xi,\varrho\rangle),
\end{aligned}$$

190 for $n \geqslant 0$, and $F(-1) = 0$.

Finally, taking the change of indexes $n - 2j = k$ in both expressions $F(n)$ and $F(n-1)$, and summing, we get

$$\begin{aligned}
&F(n) + F(n-1) \\
&= \frac{A_0}{\lambda}\sum_{k=0}^{n} \mathbf{K}_{[\frac{n-k}{2}]}^{\mu,k+\delta}(2r^2-1,1)^t\, \Lambda_{[\frac{n-k}{2}]}^{(\mu,k+\delta;M_k)}\, \mathbf{K}_{[\frac{n-k}{2}]}^{\mu,k+\delta}(2s^2-1,1) \\
&\times 2^k\,(r\,s)^k\,\frac{k+\delta}{\delta}\, C_k^\delta(\langle\xi,\varrho\rangle),
\end{aligned}$$

and the result follows.

■

6. Asymptotics for Christoffel functions

For the boundary of the ball, we shall prove:

Theorem 3. *Assume that* $\mu \geqslant -\frac{1}{2}$. *For* $\|x\| = 1$,

$$\lim_{n\to\infty} \frac{\mathbb{L}_n(x,x) - \tilde{\mathbb{L}}_n(x,x)}{n^{2\mu+d+1}} = \frac{2}{\Gamma\left(2\mu+d+2\right)} \frac{(\mu+1)(\mu+3)}{(\mu+2)^2}. \tag{34}$$

Moreover,

$$\lim_{n\to\infty} \frac{\tilde{\mathbb{L}}_n(x,x)}{n^{2\mu+d+1}} = \frac{2}{\Gamma(2\mu+d+2)(\mu+2)^2}.$$

We note that the restriction $\mu \geqslant -\frac{1}{2}$ arises because existing asymptotics for Christoffel functions in the non–Sobolev case have only been established for this range of μ. Asymptotics for the interior of the ball have been obtained as well.

Theorem 4. *For* $r = \|x\| < 1$, *we have*

$$\begin{aligned} 0 &< \mathbb{L}_n(x,x) - \tilde{\mathbb{L}}_n(x,x) \\ &\leqslant Cn^{d-1} \log n \Big(2(1-r^2) + \frac{4}{n^2}\Big)^{-\mu-\frac{1}{2}} \Big(2r^2 + \frac{4}{n^2}\Big)^{-\delta-\frac{1}{2}}. \end{aligned}$$

Here C *is independent of* n *and* x. *Consequently if* $\mu \geqslant -\frac{1}{2}$, *uniformly for* x *in compact subsets of* $\{x : 0 < \|x\| < 1\}$,

$$\lim_{n\to\infty} \tilde{\mathbb{L}}_n(x,x) \Big/ \binom{n+d}{d} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\mu+1)\Gamma(\frac{d+1}{2})}{\Gamma(\mu+\frac{d}{2}+1)} \left(1 - \|x\|^2\right)^{-\frac{1}{2}-\mu}. \tag{35}$$

This last limit also holds for $x = 0$.

In this section, we shall use the abbreviation that for $m = [\frac{n-k}{2}]$,

$$K_m(x,y) = K_m(x,y;\mu,k+\delta).$$

Thus k, m and n are linked. We now turn to the Christoffel function.

Lemma 8. *For* $d \geqslant 3$ *and* $n \geqslant 0$, *we get*

$$\tilde{\mathbb{L}}_n(x,x) = \mathbb{L}_n(x,x) - \Psi_n(x),$$

where

$$\Psi_n(x) = \frac{A_0^2}{\lambda\,\delta} \sum_{k=0}^{n} 2^{2k} (k+\delta) \binom{k+d-3}{k} r^{2k} F_{k,m}(t). \tag{36}$$

Here $x = r\xi,\ r = \|x\|\,,\ m = [\frac{n-k}{2}],\ t = 2r^2 - 1,$ *and*

$$F_{k,m}(t) = \frac{k^2 K_m(t,1)^2 + 8k K_m(t,1) K_m^{(0,1)}(t,1) + 16 K_m^{(0,1)}(t,1)^2}{1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 8k K_m^{(1,0)}(1,1) + 16 K_m^{(1,1)}(1,1) \right\}}. \tag{37}$$

Proof. From Proposition 2,

$$\begin{aligned} \Psi_n(x) &= \frac{A_0}{\lambda} \sum_{k=0}^{n} \mathbf{K}^{\mu,k+\delta}_{[\frac{n-k}{2}]}(2r^2-1,1)^t \Lambda^{(\mu,k+\delta;M_k)}_{[\frac{n-k}{2}]} \mathbf{K}^{\mu,k+\delta}_{[\frac{n-k}{2}]}(2r^2-1,1) \\ &\quad \times 2^k r^{2k} \frac{k+\delta}{\delta} C_k^{\delta}(1). \end{aligned} \tag{38}$$

Here C_k^{δ} is an ultraspherical polynomial, so that [**?** , p. 80, (4.7.3)]

$$C_k^{\delta}(1) = \binom{k+2\delta-1}{k} = \binom{k+d-3}{k}. \tag{39}$$

Using Lemma 7 and (28), a straightforward computation shows that

$$\begin{aligned} &\mathbf{K}^{\mu,k+\delta}_{[\frac{n-k}{2}]}(2r^2-1,1)^t \Lambda^{(\mu,k+\delta;M_k)}_{[\frac{n-k}{2}]} \mathbf{K}^{\mu,k+\delta}_{[\frac{n-k}{2}]}(2r^2-1,1) \\ &= \frac{2^k A_0}{\Delta_{k,m}} \left\{ k^2 K_m(t,1)^2 + 8k K_m(t,1) K_m^{(0,1)}(t,1) + 16 K_m^{(0,1)}(t,1)^2 \right\}. \end{aligned}$$

Substituting this, (31) and (39) into (38), gives the result.

■

In particular, for $r = \|x\| = 1$, we see that $t = 1$ and

$$\Psi_n(x) = \frac{A_0^2}{\lambda\,\delta} \sum_{k=0}^{n} 2^{2k} (k+\delta) \binom{k+d-3}{k} F_{k,m}(1)\,,$$

where

$$F_{k,m}(1) = \frac{k^2 K_m(1,1)^2 + 8k K_m(1,1) K_m^{(0,1)}(1,1) + 16 K_m^{(0,1)}(1,1)^2}{1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 8k K_m^{(1,0)}(1,1) + 16 K_m^{(1,1)}(1,1) \right\}}. \tag{40}$$

Next, we obtain asymptotics involving the reproducing kernel as $m \to \infty$:

Lemma 9. *As $m \to \infty$, uniformly for $k \geqslant 0$, the following asymptotics hold*

(i)

$$K_m(1,1) = 2^{-k}(m+k)^{\mu+1} m^{\mu+1} B_0(1+o(1)), \tag{41}$$

where

$$B_0 = \frac{2^{-\mu-\delta-1}}{\Gamma(\mu+1)\Gamma(\mu+2)}. \tag{42}$$

(ii)

$$\frac{K_m^{(0,1)}(1,1)}{K_m(1,1)} = \frac{(m+\mu+k+\delta+2)m}{2(\mu+2)} = \frac{(m+k)m}{2(\mu+2)}(1+o(1)). \tag{43}$$

(iii)

$$\begin{aligned} \frac{K_m^{(1,1)}(1,1)}{K_m(1,1)} &= \frac{(m+\mu+k+\delta+2)m}{4(\mu+1)(\mu+2)(\mu+3)} \\ &\quad \times ((\mu+2)m(m+\mu+k+\delta+2)+k+\delta) \\ &= \frac{(m+k)^2 m^2}{4(\mu+1)(\mu+3)}(1+o(1)). \end{aligned} \tag{44}$$

Proof. From Lemma 1 with $\alpha = \mu$ and $\beta = k+\delta$, the formulas follow using $\Gamma(x+1) = x\Gamma(x)$ and the following consequence of Stirling's formula: for fixed a, b, as $x \to \infty$,

$$\frac{\Gamma(x+b)}{\Gamma(x+a)} = x^{b-a}(1+o(1)).$$

■

Lemma 10. (i) *Let*

$$D_0 = B_0 \frac{(\mu+1)(\mu+3)}{A_0(\mu+2)^2}. \tag{45}$$

Then as $m \to \infty$, uniformly in $k \geqslant 0$,

$$2^{2k} F_{k,m}(1) = D_0 (m+k)^{\mu+1} m^{\mu+1}(1+o(1)).$$

(ii) *For $\|x\| = 1$, let*

$$\Psi_{n,1}(x) = \frac{A_0^2}{\lambda\,\delta} \sum_{k=0}^{n-[\log n]} 2^{2k}(k+\delta)\binom{k+d-3}{k} F_{k,m}(1).$$

Then

$$\lim_{n\to\infty} \frac{\Psi_{n,1}(x)}{n^{2\mu+d+1}} = E_0, \tag{46}$$

where

$$E_0 = \frac{2}{\Gamma(2\mu+d+2)} \frac{(\mu+1)(\mu+3)}{(\mu+2)^2}. \tag{47}$$

PROOF. (i) Assuming $n-k \to \infty$, so that $m = [\frac{n-k}{2}] \to \infty$, the previous lemma gives for the term in the numerator in (40),

$$\begin{aligned} & k^2 K_m(1,1)^2 + 8kK_m(1,1)K_m^{(0,1)}(1,1) + 16K_m^{(0,1)}(1,1)^2 \\ &= K_m(1,1)^2 \\ &\times \left\{ k^2 + 8k\frac{(m+k)m}{2(\mu+2)}(1+o(1)) + 4\frac{(m+k)^2m^2}{(\mu+2)^2}(1+o(1)) \right\} \\ &= K_m(1,1)^2 4\frac{(m+k)^2m^2}{(\mu+2)^2}(1+o(1)). \end{aligned}$$

Also the term in the denominator in (40) has the form

$$\begin{aligned} & 1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 8kK_m^{(1,0)}(1,1) + 16K_m^{(1,1)}(1,1) \right\} \\ &\quad = \quad 1 + 2^k A_0 K_m(1,1) \left\{ k^2 + 4k\frac{(m+k)m}{(\mu+2)}(1+o(1)) \right. \\ &\qquad\quad \left. +4\frac{(m+k)^2m^2}{(\mu+1)(\mu+3)}(1+o(1)) \right\} \\ &\quad = \quad 1 + 2^{k+2} A_0 K_m(1,1)\frac{(m+k)^2m^2}{(\mu+1)(\mu+3)}(1+o(1)). \end{aligned}$$

Thus

$$F_{k,m}(1) = \frac{K_m(1,1)^2 4\frac{(m+k)^2m^2}{(\mu+2)^2}(1+o(1))}{1 + 2^{k+2}A_0K_m(1,1)\frac{(m+k)^2m^2}{(\mu+1)(\mu+3)}(1+o(1))}.$$

Here

$$2^k K_m(1,1) = (m+k)^{\mu+1} m^{\mu+1} B_0(1+o(1)) \to \infty \text{ as } m \to \infty,$$

so

$$\begin{aligned} 2^{2k} F_{k,m}(1) &= 2^k K_m(1,1)\frac{(\mu+1)(\mu+3)}{A_0(\mu+2)^2}(1+o(1)) \\ &= D_0(m+k)^{\mu+1}m^{\mu+1}(1+o(1)). \end{aligned}$$

(ii) From (i), and as $\frac{m}{n} = \frac{1}{2}(1 - \frac{k}{n}) + O(\frac{1}{n})$,

$$\begin{aligned}
\frac{\Psi_{n,1}(x)}{n^{2\mu+2}} &= \frac{A_0^2 D_0}{\lambda\,\delta} \\
&\quad \times \sum_{k=0}^{n-[\log n]} (k+\delta)\binom{k+d-3}{k}\Big(\frac{m+k}{n}\Big)^{\mu+1}\Big(\frac{m}{n}\Big)^{\mu+1}(1+o(1)) \\
&= \frac{A_0^2 D_0 n^{d-2}}{\lambda\,\delta 2^{2\mu+2}(d-3)!}(1+o(1)) \\
&\quad \times \sum_{k=0}^{n-[\log n]} \Big(\Big(\frac{k}{n}\Big)^{d-2}\Big(1-\Big(\frac{k}{n}\Big)^2\Big)^{\mu+1} + O\Big(\frac{1}{n}\Big)\Big) \\
&= \frac{A_0^2 D_0 n^{d-1}}{\lambda\,\delta 2^{2\mu+2}\,(d-3)!}(1+o(1))\Big(\int_0^1 x^{d-2}\left(1-x^2\right)^{\mu+1} dx + o\left(1\right)\Big).
\end{aligned}$$

Here, setting $x = s^{1/2}$,

$$\int_0^1 x^{d-2}\left(1-x^2\right)^{\mu+1} dx = \frac{1}{2}\int_0^1 s^{d/2-3/2}\left(1-s\right)^{\mu+1} ds = \frac{\Gamma(\frac{d-1}{2})\Gamma(\mu+2)}{2\Gamma(\frac{d+2\mu+3}{2})}.$$

Then

$$\frac{\Psi_{n,1}(x)}{n^{2\mu+d+1}} = \frac{A_0^2 D_0}{\lambda\,\delta\, 2^{2\mu+2}(d-3)!}\frac{\Gamma(\frac{d-1}{2})\Gamma(\mu+2)}{2\Gamma(\frac{d+2\mu+3}{2})}(1+o(1)).$$

Now we simplify the constant. Using (11), (13), (18), (42), and (45),

$$\begin{aligned}
&\frac{A_0^2 D_0}{\lambda\,\delta(d-3)!\,2^{2\mu+2}}\frac{\Gamma(\frac{d-1}{2})\Gamma(\mu+2)}{2\Gamma(\frac{d+2\mu+3}{2})} \\
&= \frac{1}{\lambda\,(d-2)!\,2^{2\mu+1}} A_0 B_0 \frac{(\mu+1)(\mu+3)}{(\mu+2)^2}\frac{\Gamma(\frac{d-1}{2})\Gamma(\mu+2)}{2\Gamma(\frac{d+2\mu+3}{2})} \\
&= \frac{1}{(d-2)!\,2^{2\mu+2}}\frac{\Gamma(\frac{d}{2})}{\Gamma(\mu+\frac{d}{2}+1)}\frac{(\mu+1)(\mu+3)}{(\mu+2)^2}\frac{\Gamma(\frac{d-1}{2})}{\Gamma(\frac{d+2\mu+3}{2})} = E_0,
\end{aligned}$$

say. Using Legendre's duplication formula

$$\Gamma(2a) = \frac{2^{2a-1}}{\sqrt{\pi}}\Gamma(a)\Gamma\Big(a+\frac{1}{2}\Big),$$

with $a = \mu + \frac{d}{2} + 1$, we see that

$$\Gamma\Big(\mu+\frac{d}{2}+1\Big)\,\Gamma\Big(\mu+\frac{d}{2}+\frac{3}{2}\Big) = 2^{-2\mu-d-1}\sqrt{\pi}\,\Gamma(2\mu+d+2),$$

and with $a = \frac{d-1}{2}$,

$$\Gamma\Big(\frac{d-1}{2}\Big)\,\Gamma\Big(\frac{d}{2}\Big) = 2^{-d+2}\sqrt{\pi}\,\Gamma(d-1).$$

So, finally we obtain

$$E_0 = \frac{2}{\Gamma(2\mu + d + 2)} \frac{(\mu+1)(\mu+3)}{(\mu+2)^2}.$$

■

We shall need an estimate on the reproducing kernels that is uniform in k:

Lemma 11. *Fix $\mu > -1, \delta \geqslant 0$. For $m = \left[\frac{n-k}{2}\right] \geqslant 1$, $k \geqslant 0$, and $t \in [-1,1]$,*

$$\begin{aligned} K_m(t,t) &= K_m(t,t;\mu,k+\delta) \\ &\leqslant \; C(1+t)^{-k}\Big(m + \Big[\frac{k}{2}\Big] + 1\Big) \\ &\quad \times\Big(1 - t + \frac{1}{(m+[\frac{k}{2}]+1)^2}\Big)^{-\mu-\frac{1}{2}}\Big(1 + t + \frac{1}{(m+[\frac{k}{2}]+1)^2}\Big)^{-\delta-\frac{1}{2}}. \end{aligned}$$

Here C depends on μ and δ but not on k, n, t.

Proof. Suppose first k is even, say $k = 2\ell$. Then from the extremal properties for Christoffel functions,

$$\begin{aligned} K_m(t,t;\mu,k+\delta) &= \sup_{\deg(P)\leqslant m} \frac{P^2(t)}{\int_{-1}^1 P^2(s)\,(1-s)^\mu\,(1+s)^{k+\delta}\,ds} \\ &= (1+t)^{-k} \sup_{\deg(P)\leqslant m} \frac{\Big(P(t)\,(1+t)^\ell\Big)^2}{\int_{-1}^1 \Big(P(s)\,(1+s)^\ell\Big)^2\,(1-s)^\mu\,(1+s)^\delta\,ds} \\ &\leqslant (1+t)^{-k} \sup_{\deg(R)\leqslant m+\ell} \frac{R(t)^2}{\int_{-1}^1 R(s)^2\,(1-s)^\mu\,(1+s)^\delta\,ds} \\ &= (1+t)^{-k} K_{m+[\frac{k}{2}]}(t,t;\mu,\delta). \end{aligned}$$

We now use a result from Nevai's 1979 Memoir [**?** , p. 108, Lemma 5], that for $m + [\frac{k}{2}] \geqslant 1$ and $t \in [-1,1]$,

$$\begin{aligned} K_{m+[\frac{k}{2}]}(t,t;\mu,\delta) &\leqslant C\Big(m + \Big[\frac{k}{2}\Big] + 1\Big) \\ &\quad \times\Big(1 - t + \frac{1}{(m+[\frac{k}{2}]+1)^2}\Big)^{-\mu-\frac{1}{2}}\Big(1 + t + \frac{1}{(m+[\frac{k}{2}]+1)^2}\Big)^{-\delta-\frac{1}{2}}. \end{aligned}$$

The case $k = 2\ell + 1$ is similar.

■

Now, we have the necessary tools in order to prove the main theorems of this section.

Proof of Theorem 3. We already have a limit for $\Psi_{n,1}$, and must now estimate the remaining part of Ψ_n, namely, for $\|x\| = 1$,

$$\Psi_{n,2}(x) = \frac{A_0^2}{\lambda\,\delta} \sum_{k=n-[\log n]+1}^{n} 2^{2k}(k+\delta)\binom{k+d-3}{k} F_{k,m}(1).$$

We shall show that $\Psi_{n,2}(x) = o(n^{2\mu+d+1})$ which, together with (46), will give the result. Now if $m \geqslant 1$,

$$\begin{aligned}2^k A_0 F_{k,m}(1) &= 2^k A_0 \frac{k^2 K_m(1,1)^2 + 8k K_m(1,1) K_m^{(0,1)}(1,1) + 16 K_m^{(0,1)}(1,1)^2}{1 + 2^k A_0 \left\{ k^2 K_m(1,1) + 8k K_m^{(1,0)}(1,1) + 16 K_m^{(1,1)}(1,1) \right\}} \\ &\leqslant K_m(1,1) + K_m(1,1) + \frac{K_m^{(0,1)}(1,1)^2}{K_m^{(1,1)}(1,1)}.\end{aligned}$$

Here as k is close to n, and $m = O(\log n)$, (43) and (44) give

$$\left(\frac{K_m^{(0,1)}(1,1)}{K_m(1,1)}\right)^2 \leqslant C(nm)^2,$$

while if $m \geqslant 1$,

$$\frac{K_m(1,1)}{K_m^{(1,1)}(1,1)} \leqslant C(n^2 m^2)^{-1}.$$

When $m = 0$, the estimation is simpler as $K_m^{(0,1)} = 0 = K_m^{(1,1)}$. Thus

$$2^k A_0 F_{k,m}(1) \leqslant C K_m(1,1),$$

where C is a constant independent of m and n, so for some possibly different C,

$$\Psi_{n,2}(x) \leqslant C \sum_{k=n-[\log n]+1}^{n} 2^k (k+\delta)\binom{k+d-3}{k} K_m(1,1).$$

Here, using Lemma 9,

$$2^k K_m(1,1) \leqslant C n^{2\mu+2},$$

while

$$(k+\delta)\binom{k+d-3}{k} \leqslant C n^{d-2}.$$

Thus

$$\Psi_{n,2}(x) \leqslant C\, n^{2\mu+d} \log n = o(n^{2\mu+d+1}).$$

Then (34) follows from Lemma 10. Finally, we note that if $\mu \geqslant -\frac{1}{2}$, (1.10) of Theorem 1.1 in [**?** , p. 120] gives

$$\lim_{n\to\infty} \mathbb{L}_n(x,x)/n^{2\mu+d+1} = \frac{2}{\Gamma(2\mu+d+2)}.$$

Take there $\rho = \mu + \frac{1}{2}$, and note that the normalization constant ω_ρ is incorporated in [**?** , p. 119] in a different way to that here. Then

$$\begin{aligned}
\lim_{n\to\infty} \tilde{\mathbb{L}}_n(x,x)/n^{2\mu+d+1} &= \frac{2}{\Gamma(2\mu+d+2)} - \frac{2}{\Gamma(2\mu+d+2)} \frac{(\mu+1)(\mu+3)}{(\mu+2)^2} \\
&= \frac{2}{\Gamma(2\mu+d+2)} \left\{ 1 - \frac{(\mu+1)(\mu+3)}{(\mu+2)^2} \right\} \\
&= \frac{2}{\Gamma(2\mu+d+2)(\mu+2)^2}.
\end{aligned}$$

■

Next we deal with $\|x\| < 1$.

Proof of Theorem 4. We must estimate $F_{k,m}(t)$ defined in (37), with $t = 2r^2 - 1$ and $r = \|x\|$.

Let us assume that $t \leqslant 1 - \eta$ for some $\eta > 0$. Then, with the convention $p_j = p_j^{(\mu,k+\delta)}$ for orthonormal Jacobi polynomials, $j = m, m+1$,

$$\begin{aligned}
|K_m(t,1)| &= \frac{\gamma_m}{\gamma_{m+1}} \left| \frac{p_{m+1}(t)p_m(1) - p_m(t)p_{m+1}(1)}{t-1} \right| \\
&\leqslant \frac{C}{2} \frac{\sqrt{p_m^2(t) + p_{m+1}^2(t)}\sqrt{p_m^2(1) + p_{m+1}^2(1)}}{\eta} \\
&\leqslant \frac{C}{2\eta} K_{m+1}(t,t)^{1/2} \sqrt{p_m^2(1) + p_{m+1}^2(1)}, \qquad (48)
\end{aligned}$$

where $\gamma_m = k_m/\sqrt{h_m}$ is the leading coefficient of p_m.

Also,

$$
\begin{aligned}
\left|K_m^{(0,1)}(t,1)\right| &= \frac{\gamma_m}{\gamma_{m+1}} \\
&\times \left|\frac{(p_{m+1}(t)p'_m(1) - p_m(t)p'_{m+1}(1))(t-1) + p_{m+1}(t)p_m(1) - p_m(t)p_{m+1}(1)}{(t-1)^2}\right| \\
&\leqslant \frac{C}{2\eta^2}\sqrt{p_m^2(t) + p_{m+1}^2(t)}\left\{2\sqrt{(p'_m(1))^2 + (p'_{m+1}(1))^2} + \sqrt{p_m^2(1) + p_{m+1}^2(1)}\right\} \\
&\leqslant \frac{C}{2\eta^2} K_{m+1}(t,t)^{1/2}\left\{2\sqrt{(p'_m(1))^2 + (p'_{m+1}(1))^2} + \sqrt{p_m^2(1) + p_{m+1}^2(1)}\right\}. \quad (49)
\end{aligned}
$$

Next, we note that given any real number a, there exists $C_a > 1$ such that for all x with $\min(x, x+a) \geqslant 1$,

$$C_a^{-1} x^a \leqslant \frac{\Gamma(x+a)}{\Gamma(x)} \leqslant C_a x^a.$$

This follows from Stirling's formula and the positivity and continuity of $\frac{\Gamma(x+a)}{\Gamma(x)}$ for this range of x. Then from (7), if $m \geqslant 1$, $\alpha = \mu$, $\beta = k + \delta$,

$$|p'_m(1)| \leqslant C\frac{(m+k)^{3/2+\mu/2}\, m^{1+\mu/2}}{2^{k/2}}, \tag{50}$$

and from (6),

$$|p_m(1)| \leqslant C\frac{(m+k)^{1/2+\mu/2}\, m^{\mu/2}}{2^{k/2}}.$$

Substituting these into (48) and (49) gives for $m \geqslant 1$,

$$|K_m(t,1)| \leqslant C\Big(\frac{K_{m+1}(t,t)}{2^k}\Big)^{1/2}(m+k)^{1/2+\mu/2}\, m^{\mu/2} \tag{51}$$

and

$$\left|K_m^{(0,1)}(t,1)\right| \leqslant C\Big(\frac{K_{m+1}(t,t)}{2^k}\Big)^{1/2}(m+k)^{3/2+\mu/2}\, m^{1+\mu/2}. \tag{52}$$

Next, by (41) and (44),

$$2^k K_m^{(1,1)}(1,1) \geqslant C(m+k)^{\mu+3}\, m^{\mu+3}, \tag{53}$$

so, inserting (51), (52) and (53) into (37),

$$
\begin{aligned}
2^k F_{k,m}(t) &\leqslant C K_{m+1}(t,t) \\
&\quad \times \left\{\frac{k^2\,(m+k)^{1+\mu}\, m^{\mu} + k\,(m+k)^{2+\mu}\, m^{1+\mu} + (m+k)^{3+\mu}\, m^{2+\mu}}{(m+k)^{\mu+3}\, m^{\mu+3}}\right\} \\
&\leqslant \ C K_{m+1}(t,t)\,/\,(m+1)\,.
\end{aligned}
$$

This bound holds also for $m = 0$, thought it is obtained in a simpler way since K_0 is a constant,

$$F_{k,0}(t) = \frac{k^2 K_0(t,1)^2}{1 + 2^k A_0 k^2 K_0(1,1)} \leqslant C\frac{K_0(t,t)}{2^k} \leqslant C\frac{K_1(t,t)}{2^k}.$$

Then,

$$\Psi_n(x) \leqslant C\sum_{k=0}^{n} 2^k (k+\delta)\binom{k+d-3}{k} r^{2k}\frac{K_{m+1}(t,t)}{m+1}.$$

Using Lemma 11, and that $1+t = 2r^2$ and $m = \left[\frac{n-k}{2}\right]$, we continue this as

$$\begin{aligned}
&\Psi_n(x)\\
&\leqslant \; C\sum_{k=0}^{n}(k+1)^{d-2}\Big(\frac{m+[\frac{k}{2}]+2}{[\frac{n-k}{2}]+1}\Big)\\
&\qquad \times\Big(1-t+\frac{1}{(m+[\frac{k}{2}]+2)^2}\Big)^{-\mu-\frac{1}{2}}\Big(1+t+\frac{1}{(n+[\frac{k}{2}]+2)^2}\Big)^{-\delta-\frac{1}{2}}\\
&\leqslant \; Cn^{d-1}\Big(1-t+\frac{4}{n^2}\Big)^{-\mu-\frac{1}{2}}\Big(1+t+\frac{4}{n^2}\Big)^{-\delta-\frac{1}{2}}\sum_{k=0}^{n}\frac{1}{[\frac{n-k}{2}]+1}\\
&\leqslant \; Cn^{d-1}\log n\Big(2(1-r^2)+\frac{4}{n^2}\Big)^{-\mu-\frac{1}{2}}\Big(2r^2+\frac{4}{n^2}\Big)^{-\delta-\frac{1}{2}}.
\end{aligned}$$

Finally, [**?** , Theorem 1.3] gives

$$\begin{aligned}
\lim_{n\to\infty}\mathbb{L}_n(x,x)\Big/\binom{n+d}{d} &= \frac{\omega_\mu W_0(x)}{\big(1-\|x\|^2\big)^{\mu}}\\
&= \frac{1}{\sqrt{\pi}}\frac{\Gamma(\mu+1)\Gamma(\frac{d+1}{2})}{\Gamma(\mu+\frac{d}{2}+1)}\Big(1-\|x\|^2\Big)^{-\frac{1}{2}-\mu},
\end{aligned}$$

uniformly for x in compact subsets of the unit ball. Thus $\mathbb{L}_n(x,x)$ grows like $n^d >> n^{d-1}\log n$, so (35) follows.

It remains to deal with the case $x = 0$, that is $r = 0$. In this case all terms in $\Psi_n(x)$ in (36) vanish except for $k = 0$. We see that

$$\Psi_n(0) = A_0^2 F_{0,[\frac{n}{2}]}(-1) = \frac{A_0^2 16 K^{(0,1)}_{[\frac{n}{2}]}(-1,1)^2}{1+16A_0 K^{(1,1)}_{[\frac{n}{2}]}(1,1)}. \tag{54}$$

With $m = [\frac{n}{2}]$, $k = 0$, we see as above that

$$\begin{aligned}
\Big|K_m^{(0,1)}(-1,1)\Big| \;\leqslant\; & C\Big(|p_{m+1}(-1)|\,|p'_m(1)| + |p_m(-1)|\,\big|p'_{m+1}(1)\big|\\
&+|p_{m+1}(-1)|\,|p_m(1)| + |p_m(-1)|\,|p_{m+1}(1)|\Big).
\end{aligned} \tag{55}$$

We shall need the classic bound [**?** , p. 36, eqn. (20–21)]

$$|p_m(t)| \leqslant C\Big(1-t+\frac{1}{m^2}\Big)^{-\frac{\mu}{2}-\frac{1}{4}}\Big(1+t+\frac{1}{m^2}\Big)^{-\frac{\delta}{2}-\frac{1}{4}}, \quad t\in[-1,1].$$

Here C depends only on μ and δ. Then

$$|p_m(-1)| \leqslant Cm^{\delta+\frac{1}{2}}, \qquad |p_m(1)| \leqslant Cm^{\mu+\frac{1}{2}}.$$

Moreover, (50) gives (recall $k=0$),

$$|p_m'(1)| \leqslant Cm^{\mu+\frac{5}{2}}.$$

Substituting all these bounds in (55) yields

$$\Big|K_m^{(0,1)}(-1,1)\Big| \leqslant Cm^{\delta+\mu+3}.$$

In addition, (53) leads to

$$\Big|K_m^{(1,1)}(1,1)\Big| \geqslant Cm^{2\mu+6}.$$

Substituting the last two bounds in (54) gives

$$|\Psi_n(0)| \leqslant Cn^{2\delta} = Cn^{d-2}.$$

Then (35) follows also for this case.

■

7. The two dimensional case

In the case $d=2$ results are somewhat different, but Theorems 3 and 4 also hold. In this case $\delta=0$, $\omega_\mu=\pi$, $\sigma_d=2\pi$, then $\omega_\mu/\sigma_d=1/2$ and $A_0=\lambda\,2^{\mu+1}$. Moreover the reproducing kernel of spherical harmonics is obtained in a different way.

Proposition 3. *For $n\geqslant 0$ and $d=2$, we get*

$$\begin{aligned}\tilde{\mathbb{L}}_n(x,y) &= \mathbb{L}_n(x,y)\\ &\quad -\frac{A_0}{\lambda}\sum_{k=0}^{n}\mathbf{K}^{\mu,k}_{[\frac{n-k}{2}]}(2r^2-1,1)^t\,\Lambda^{(\mu,k;M_k)}_{[\frac{n-k}{2}]}\,\mathbf{K}^{\mu,k}_{[\frac{n-k}{2}]}(2s^2-1,1)\\ &\quad \times 2^k\,(r\,s)^k\,\cos(n(\theta-\widehat{\theta})),\end{aligned}$$

where $x = r\,(\cos\theta, \sin\theta)$, $y = s\,(\cos\widehat{\theta}, \sin\widehat{\theta})$, $r = \|x\|$, $s = \|y\|$, $\theta, \widehat{\theta} \in [0, 2\pi]$.

Proof. The proof is the same as in Proposition 2 taking into account that in this case $a_k^d = 2$, for $k \geqslant 0$, and the addition formula of spherical harmonics for $d = 2$ reduces to the addition formula for the cosines (see [**?** , p. 20]), then

$$\sum_{\nu=1}^{a_k^d} Y_\nu^k(\cos\theta, \sin\theta)\, Y_\nu^k(\cos\widehat{\theta}, \sin\widehat{\theta}) = \cos(n(\theta - \widehat{\theta})).$$

■

Lemma 8 can be rewritten for the case $d = 2$ as

Lemma 12. *For $d = 2$ and $n \geqslant 0$, we get*

$$\tilde{\mathbb{L}}_n(x, x) = \mathbb{L}_n(x, x) - \Psi_n(x),$$

where

$$\Psi_n(x) = \frac{A_0^2}{\lambda} \sum_{k=0}^{n} 2^{2k} r^{2k} F_{k,m}(t).$$

Here $r = \|x\|$, $m = [\frac{n-k}{2}]$, $t = 2r^2 - 1$, *and $F_{k,m}(t)$ is given as in (37).*

Lemma 9 and part (i) of Lemma 10 are true for $d = 2$, with $\delta = 0$, and part (ii) of Lemma 10 turns out

Lemma 13. *For $\|x\| = 1$, let*

$$\Psi_{n,1}(x) = \frac{A_0^2}{\lambda} \sum_{k=0}^{n-[\log n]} 2^{2k}\, F_{k,m}(1).$$

Then

$$\lim_{n\to\infty} \frac{\Psi_{n,1}(x)}{n^{2\mu+3}} = E_0,$$

where E_0 is given in (47) with $d = 2$.

Moreover, Lemma 11 works for $d = 2$, so finally Theorem 3 and Theorem 4 also hold in this case.

Acknowledgements

First, second, fourth and fifth authors (A.M.D., L.F., T.E.P., M.A.P.) thank MINECO of Spain and the European Regional Development Fund (ERDF) through grants MTM2011–28952–C02–02 and MTM2014–53171–P, and Junta de Andalucía grant P11–FQM–7276 and research group FQM–384.

The work of the third author (D.S.L.) was supported in part by NSF grant DMS136208.

This work was finished during a visit of second author (L.F.) to School of Mathematics, Georgia Institute of Technology.

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