THE SPURIOUS SIDE OF DIAGONAL MULTIPOINT PADÉ APPROXIMANTS

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ABSTRACT. We survey at an introductory level, the topic of multipoint Padé approximants, especially the issues of spurious poles and convergence for diagonal rational approximants.

Padé approximation, Multipoint Padé approximants, spurious poles. 41A21, 41A20, 30E10. In memory of Yingkang Hu

1. INTRODUCTION[1]

Given $n+1$ distinct points on the real line and a function defined on those points, the Lagrange interpolation formula provides a simple expression for the unique polynomial of degree $\leq n$ that interpolates to the given function at those points. What is the situation for interpolation by rational functions? Let $m, n \geq 0$, $\{z_j\}_{j=1}^{m+n+1}$ be $m+n+1$ distinct points in the complex plane, and let f be a function defined on $\{z_j\}_{j=1}^{m+n+1}$. In addition, let

$$R_{mn}(z) = P(z)/Q(z)$$

be a rational function of type (m, n), that is P and Q have respective degrees at most m, n, while Q is not identically 0. We look for R_{mn} satisfying

$$R_{mn}(z_j) = f(z_j), \ 1 \leq j \leq m+n+1. \tag{1.1}$$

Why $m+n+1$? Well, P has $m+1$ coefficients, while Q has $n+1$, so there are a total of $m+n+2$ coefficients. However, we lose 1 degree of freedom in dividing, so expect to satisfy $m+n+1$ interpolation conditions.

Unfortunately, the problem (1.1) does not always have a solution. As a simple example, let $m = n = 1$, so $m+n+1 = 3$, and consider interpolation by

$$R_{11}(z) = \frac{az+b}{cz+d},$$

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at 3 points, say

$$R_{11}(0) = 0;\ R_{11}(1) = 0;\ R_{11}(2) = 1.$$

The first two conditions force $a = b = 0$, so that R_{11} is identically 0. Then we cannot satisfy the third condition. If you want to read more about this type of failing, see [58], [69].

Fortunately, there is a way to get around examples like this, by linearizing the interpolation conditions:

Definition 1.1
Let $m, n \geq 0$, $\{z_j\}_{j=1}^{m+n+1}$ *be* $m+n+1$ *distinct points in the plane, and let* f *be a function defined on* $\{z_j\}_{j=1}^{m+n+1}$. *We say that a rational function* $R_{mn} = P/Q$ *of type* (m,n) *is a* ***multipoint Padé approximant*** *to* f *at* $\{z_j\}_{j=1}^{m+n+1}$, *if*

$$(fQ - P)(z_j) = 0,\ 1 \leq j \leq m+n+1. \tag{1.2}$$

R_{mn} is also often called a rational interpolant with free poles.

Proposition 1.2
R_{mn} *exists and is unique.*
Proof
The equations (1.2) constitute $m+n+1$ homogeneous linear equations in the $m+n+2$ coefficients of P, Q. As there are more unknowns than equations, there are non-trivial solutions. Moreover, Q cannot be identically 0 (if it were, P would have $m+n+1$ zeros and would be identically 0). So $R_{mn} = P/Q$ exists.

If P_1/Q_1 were another such interpolant, then

$$P_1 Q - P Q_1 = Q(P_1 - fQ_1) - Q_1(P - fQ)$$

has at least $m+n+1$ distinct zeros. As a polynomial of degree $\leq m+n$, it must be identically zero, so

$$P_1/Q_1 = P/Q.$$

Thus R_{mn} is also unique. ■

One can actually solve the homogeneous linear equations (1.2) above, using Cramer's rule. First one uses finite differences to express f as a Newton-Taylor polynomial of high enough degree, plus a remainder term. Recall how we recursively define finite differences:

$$f[z_1] = f(z_1);$$

$$f[z_1, z_2] = \frac{f(z_2) - f(z_1)}{z_2 - z_1};$$

and more generally for $k \geq 1$,

$$f[z_1, z_2, ..., z_{k+1}] = \frac{f[z_1, z_2, ..., z_{k-1}, z_{k+1}] - f[z_1, z_2, ..., z_{k-1}, z_k]}{z_{k+1} - z_k}.$$

Also, for notational brevity, let

$$f_{j,k} = f[z_{j+1}, z_{j+2}, ..., z_{k+1}]$$

when $k \geq j$, and let $f_{j,k} = 0$ otherwise. Cramer's rule gives:

Theorem 1.3
If the denominator determinant is not identically 0,

$$R_{mn}(z) = \frac{\det \begin{bmatrix} \sum_{j=0}^{m} f_{0,j} \prod_{k=1}^{j} (z - z_k) & f_{0,m+1} & f_{0,m+2} & \cdots & f_{0,m+n} \\ \sum_{j=1}^{m} f_{1,j} \prod_{k=1}^{j} (z - z_k) & f_{1,m+1} & f_{1,m+2} & \cdots & f_{1,m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=n}^{m} f_{n,j} \prod_{k=1}^{j} (z - z_k) & f_{n,m+1} & f_{n,m+2} & \cdots & f_{n,m+n} \end{bmatrix}}{\det \begin{bmatrix} 1 & f_{0,m+1} & f_{0,m+2} & \cdots & f_{0,m+n} \\ z - z_1 & f_{1,m+1} & f_{1,m+2} & \cdots & f_{1,m+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=1}^{n} (z - z_k) & f_{n,m+1} & f_{n,m+2} & \cdots & f_{n,m+n} \end{bmatrix}}.$$

(1.3)

For a proof of this see [9, pp. 338-340]. Of course, in the above formula, empty products are interpreted as 1, and empty sums as 0. What about the case when not all the interpolation points are distinct? This is actually more widely applied than the distinct case. We can modify the definition as follows:

Definition 1.4
Let $m, n \geq 0$, $\{z_j\}_{j=1}^{m+n+1}$ *be* $m + n + 1$ *not necessarily distinct points in the plane, and let* f *be a function analytic in an open set containing* $\{z_j\}_{j=1}^{m+n+1}$. *Let*

$$\omega(z) = \prod_{j=1}^{m+n+1} (z - z_j).$$

We say that a rational function $R_{mn} = P/Q$ of type (m,n) is the ***multipoint Padé approximant*** *to f at $\{z_j\}_{j=1}^{m+n+1}$, if $\frac{fQ-P}{\omega}$ is analytic at $\{z_j\}_{j=1}^{m+n+1}$.*

Observe that the condition of analyticity forces $fQ - P$ to have a zero of multiplicity at z_j, at least equal to the number of times z_j is repeated as an interpolation point. One can prove the existence and uniqueness of this more general interpolant in much the same way as in Proposition 1.2. The formula (1.3) remains true: the finite differences become multiples of derivatives of appropriate order when some z_j are repeated.

The case where all $z_j = 0$ is particularly important, so much so that it has its own special notation: when all $z_j = 0$, we denote R_{mn} by $[m/n]$, and call it the *Padé approximant to f of type (m,n)*. In this case it is appropriate to express f as a (possibly formal or divergent) power series

$$f(z) = \sum_{j=0}^{\infty} a_j z^j. \tag{1.4}$$

Moreover, Theorem 1.3 has a simpler form [9, pp. 4-6]:

Theorem 1.5
If the denominator is not identically 0,

$$[m/n](z) = \frac{\det \begin{bmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ a_{m-n+2} & a_{m-n+3} & \cdots & a_{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \\ \sum_{j=0}^{m-n} a_j z^{n+j} & \sum_{j=0}^{m-n+1} a_j z^{n+j-1} & \cdots & \sum_{j=0}^{m} a_j z^j \end{bmatrix}}{\det \begin{bmatrix} a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ a_{m-n+2} & a_{m-n+3} & \cdots & a_{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m+1} & \cdots & a_{m+n} \\ z^n & z^{n-1} & \cdots & 1 \end{bmatrix}}.$$

(we take $a_j = 0$ if $j < 0$).

(1.5)

One might think that this last determinant formula, and its more general cousin (1.3), are just a curiosity. However, they can be useful, for example, in obtaining explicit Padé approximants to the exponential function and more general hypergeometric functions [9, Chapter 1].

Obviously it is much easier to first focus on the Padé approximant with its single confluent interpolation point than on the general multipoint Padé approximant. There are just two parameters, namely numerator and denominator degrees, and so it is natural to arrange them into a table, called the *Padé table*:

$$\begin{array}{cccccc}
[0/0] & [0/1] & [0/2] & [0/3] & [0/4] & \cdots \\
[1/0] & [1/1] & [1/2] & [1/3] & [1/4] & \cdots \\
[2/0] & [2/1] & [2/2] & [2/3] & [2/4] & \cdots \\
[3/0] & [3/1] & [3/2] & [3/3] & [3/4] & \cdots \\
[4/0] & [4/1] & [4/2] & [4/3] & [4/4] & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

This is rightly named after Henri Eugene Padé, who investigated its structure in his thesis, supervised by Hermite. He showed that the table decomposes into square blocks, such that all approximants in a given block are equal, and no other approximants in the table equal those in the block. It is an open problem, perhaps first posed by Nick Trefethen [68, p. 179] as to what types of square block patterns can arise in Padé tables, from all formal power series. Trefethen noted that not every pattern is possible, and referred to a private communication from A. Magnus.

2. Some Padé History and Connections

The propensity of mathematicians to wrongly attribute theorems and concepts is often called Arnold's principle: "If a notion bears a personal name, then this name is not the name of the discoverer." It certainly applies to the multipoint Padé approximant, which Padé apparently never investigated. Hermite first introduced what are now called Hermite-Padé approximants to prove the transcendence of the number e. Later Lindemann used Hermite's method to resolve the even more famous problem of transcendence of π. Hermite gave the special case of the Padé approximant to Padé to investigate in his thesis.

Somehow, every approximant that has a Padé flavor now bears the name Padé - including the Hermite-Padé approximants that should be called simply Hermite approximants. This is a rather rare case of the less famous student's name being attached to the work of his more famous supervisor. Note too that formulae such as (1.5) were already known to Jacobi back in 1846, while mathematicians such as Bernoulli, Cauchy, Jacobi, and Frobenius developed many of the ideas above long before Hermite or Padé [12].

There is another very interesting historical connection, namely to the development of the Riemann-Stieltjes integral. In the process of trying to give analytic meaning to certain continued fractions, Thomas Jan Stieltjes was compelled to develop a new type of integral, which we now call the Riemann-Stieltjes integral. It appeared in a memoir published only after his death. The convergents of those continued fractions are Padé approximants to power series of the form

$$f(z) = \sum_{j=0}^{\infty} \left(\int_0^{\infty} t^j \, d\mu(t) \right) (-z)^j \tag{2.1}$$

where μ is a monotone increasing function for which all moments are finite, hence the Riemann-Stieltjes integral. If we interchange series and integral, and sum the possibly divergent geometric series, we obtain, at least formally,

$$f(z) = \int_0^{\infty} \frac{d\mu(t)}{1+tz}. \tag{2.2}$$

This is a function analytic in the cut plane $\mathbb{C} \backslash (-\infty, 0]$. The series (2.1) is often called a *Markov* or *Stieltjes* or *Markov-Stieltjes* series. (Markov earlier considered the absolutely continuous case, with $d\mu$ compactly supported). The denominators in its $[n-1/n]$ and $[n/n]$ Padé approximants are orthogonal polynomials, and there are connections with the moment problem.

Padé approximants have a plethora of connections and applications: to numerical solution of partial differential equations, acceleration of convergence of sequences, numerical inversion of Laplace transforms, solution of integral equations, design of electrical circuits... . For all this, see [9].

However, it was probably their application to problems in scattering theory in mathematical physics in the 1960's that brought them real prominence. A model problem would run as follows: we know the first $2n+1$ coefficients of a Maclaurin series, but would like to know something about the location of singularities of the underlying function f. We can use those coefficients to form the $[n/n]$ Padé approximant to f. Under suitable conditions, the poles of $[n/n]$ will predict where are the singularities of f. Multipoint Padé approximants offer still more applications, see for example [30].

3. Convergence

Obviously, a fundamental question is whether

$$R_{mn} \to f$$

or more specially

$$[m/n] \to f$$

as $m+n \to \infty$. This turns out to be a very complex problem, depending on the relative growth of m and n, as well as on the underlying function f.

Since $[m/0]$ is just the mth partial sum of the Maclaurin series of f, the sequence $\{[m/0]\}_{m\geq 1}$ converges only inside the circle of convergence of the power series. The first general theorem that moved beyond this is de Montessus' de Ballore's Theorem [9, p. 282]. It asserts that if f is analytic at 0, but has poles of total multiplicity n in the ball $B_r = \{z : |z| < r\}$, then the $(n+1)$st column of the Padé table, namely $\{[m/n]\}_{m\geq 1}$, converges uniformly inside compact subsets of B_r omitting poles.

There are many generalizations of de Montessus' theorem, for example, to multipoint Padé approximants [53]. The Russian school of Padé approximators under A. Gončar thoroughly investigated the inverse problem, where we assume only knowledge of the asymptotic behavior of the poles of $\{[m/n]\}_{m\geq 1}$ and expect to deduce that the underlying function is meromorphic with poles of total multiplicity n. See [66].

These "column" sequences are special cases of the "non-diagonal" sequences where n is allowed to grow in such a way that

$$n/m \to 0$$

as $m \to \infty$. The "ray" sequences are those for which $m/n \to \lambda$ as $m \to \infty$, for some finite positive λ. Sometimes these are also called diagonal sequences, though the true diagonal sequence is $\{[n/n]\}_{n\geq 1}$. It is on this "main diagonal" sequence, and its multipoint cousin $\{R_{nn}\}_{n\geq 1}$, that we shall focus in the rest of this article. The other sequences could all easily merit a lengthy survey of their own.

Since Markov-Stieltjes series play such a central role in Padé approximation, it is not surprising that they were the first general class of functions for which the main diagonal was shown to converge. For the case where $d\mu$ is absolutely continuous and has support inside a finite interval, this follows from work of Markov published in 1884 [9, p. 228]. Stieltjes dealt with the more general Riemann-Stieltjes measures supported on $[0, \infty)$ [9, p. 240]. We say such a $d\mu$ is *determinate* if it is the unique solution of its moment problem. That is, if ν is a monotone increasing function with

$$\int_0^\infty t^j d\nu(t) = \int_0^\infty t^j d\mu(t) \text{ for all } j \geq 0,$$

then $d\nu = d\mu$. This determinacy is true if for example, the moments of $d\mu$ do not grow "too fast", say the jth moment grows no faster than $(2j)!$ [9, pp. 239-240].

Theorem 3.1 (Markov-Stieltjes Theorem)
Let $d\mu$ be a determinate positive measure on $[0,\infty)$ with $\int_0^\infty t^j d\mu(t)$ finite for all $j \geq 0$. Let f be given by (2.2). Then

$$\lim_{n\to\infty} [n/n](z) = f(z),$$

uniformly in compact subsets of $\mathbb{C}\backslash(-\infty,0]$. Moreover, for each n, all poles of $[n/n]$ lie in $(-\infty,0)$, are simple, and have positive residues.

Note that the underlying Maclaurin series for f could well have radius of convergence 0, so that $\{[n/n]\}$ formed from an everywhere divergent power series converges everywhere in the cut plane. Indeed, this was often the case in applications in mathematical physics.

What about convergence of multipoint Padé approximants? Since Stieltjes series are real on the real axis, it is natural to consider interpolation points that are symmetric about the real axis: that is, if z_j is an interpolation point, the conjugate $\bar{z}_j$ is also amongst the interpolation points. The earliest papers establishing convergence in this case are due to Gončar and Lopez [27], [34], and Gelfgren [24]. They considered symmetric arrays that are a positive distance (independent of n) from the support of the measure, and established convergence of $\{R_{n-1,n}\}$ to the underlying determinate Stieltjes series.

There is now a very extensive literature on this topic, and we cannot hope to survey it here. Many of the relevant references can be found in the survey paper of B. de La Calle Ysern, in the Festschrift for G. Lopez's 60th birthday [19]. To the best of my knowledge, there has not been any work for the case when the interpolation arrays are not symmetric.

We emphasize too that there have been many generalizations of Stieltjes series, and investigations of classic Padé approximants to them. Series of Hamburger involve measures on the whole real line, while the case of complex measures on a segment, or measures on arcs in the plane have also been considered, as have been rational perturbations of Stieltjes functions. See for example [2], [3], [17], [20], [21], [35], [36], [55], [66], [67].

If Stieltjes series provide the natural setting for Padé approximants because of their connections to orthogonal polynomials and the moment problem, there are other classes of functions for which diagonal Padé sequences have been shown to converge. For classical special functions, such as hypergeometric and $q-$hypergeometric series, the fact that Padé denominators appear in convergents to explicitly given continued fractions allow one to establish convergence. In the case of the exponential function, a very detailed analysis of the location of zeros and poles of classical Padé approximants was undertaken by Saff and Varga [54]. See also [22].

Polya frequency series are the largest class of "non-special" functions for which uniform convergence has been established [4]. These have the form

$$f(z) = z^J e^{\gamma z} \prod_{j=1}^{\infty} \frac{1+\alpha_j z}{1-\beta_j z},$$

where J is a nonnegative integer, $\gamma \geq 0$, and all $\alpha_{j,}\beta_j \geq 0$, with $\sum_j \left(\alpha_j + \beta_j\right) < \infty$. Here the key ingredient is the total positivity of Toeplitz matrices whose entries are the Maclaurin coefficients of f.

Because they involve determinants, Padé approximants are very sensitive to perturbations in the series coefficients, so are expected to behave well when the coefficients are "smooth". The author [41] (see also [40]) was able to show that when the coefficients in (1.4) decay smoothly and rapidly in the sense that for some $|q| < 1$,

$$\lim_{j\to\infty} \frac{a_{j-1}a_{j+1}}{a_j^2} = q, \tag{3.1}$$

then the determinants can be estimated, and uniform convergence of $\{[n/n]\}$ in compact sets follows.

All this is for classical Padé approximants. What about multipoint Padé approximants, beyond the Stieltjes case reviewed above? As far as the author is aware, there are very few results for the diagonal multipoint case. Until recently, e^z is the only function for which convergence of $\{R_{nn}\}$ has been established when interpolation points are drawn from a fixed compact set, without symmetry or distribution restrictions on the interpolation points. This required deep Riemann-Hilbert techniques [73]. The case of unbounded interpolation arrays was later studied in [18]. The easier case of symmetric interpolation points was handled earlier [72].

If one allows the interpolation array to depend on the specific function, then one can apply a classical observation of Eli Levin [32], [33]. This asserts that best L_2 rational approximants of type (n,n) on the unit disc, interpolate the function from which they are formed in $2n+1$ points, and thus are multipoint Padé approximants. This immediately gives an array of interpolation points for which $\{R_{nn}\}$ converges uniformly. The Newton-Padé case where one keeps previous interpolation points, so that at the nth stage you add only 2 new interpolation points, is more delicate. Using Nevanlinna theory, the author showed [42] that given a function f meromorphic in the whole plane, one can construct an array depending on the function for which $\{R_{nn}\}$ converges uniformly in compact sets omitting poles. It is an interesting unsolved problem as to whether one can find Newton-Padé arrays whose rational interpolants converge locally uniformly, tailored individually to functions that are analytic only in the unit ball.

What makes it so difficult to prove convergence of $\{[n/n]\}$, or more generally, $\{R_{nn}\}$? It is the location of the poles of the approximants. A rough rule is that there is convergence in "nice" regions away from limit points of poles, even if we assume nothing about the underlying function. Indeed, A. Gončar put this into a precise form in [26] for diagonal Padé sequences. For example, if there are no limit poles of $\{[n/n]\}$ in some open disk center 0, or some open half-plane containing 0, then Gončar proved that the approximants converge locally uniformly, and the underlying function must be analytic there.

Unfortunately, in general, the approximants may have poles that do not reflect the analytic properties of the underlying function. These are called *spurious poles*, and the phenomenon was observed a long time ago, at least as far back as 1908, in the thesis of S. Dumas [23]. There are many examples involving spurious poles, but the most striking is due to Hans Wallin [70]:

Theorem 3.2 (Wallin's Example)
There is an entire function f such that its diagonal sequence $\{[n/n]\}$ diverges everywhere in $\mathbb{C}\setminus\{0\}$. More precisely, for $z \in \mathbb{C}\setminus\{0\}$,

$$\limsup_{n\to\infty} |[n/n](z)| = \infty.$$

Wallin constructed series with large gaps between sections of nonzero coefficients, with the poles of a *subsequence* of approximants chosen to ensure divergence. It is interesting that some other subsequence of $\{[n/n]\}$ in Wallin's example converges uniformly in compact sets. His example was actually a special case of a more general one, establishing sharpness of his results on convergence of diagonal Padé sequences outside sets of $\alpha-$dimensional measure 0, for any $\alpha > 0$. It appears that one can choose the example in Theorem 3.2 to be entire of finite order, perhaps even of order 2.

While Wallin's example showed that pointwise convergence, or even convergence a.e. is not always possible, it was understood somewhat earlier that spurious poles affect the quality of approximation only in a small area. This idea was crystallized in a landmark 1970 paper of John Nuttall [48]. Let *meas* denote planar Lebesgue measure.

Theorem 3.3 (Nuttall's Theorem)
Let f be meromorphic in $\mathbb{C}$, and analytic at 0. Then the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ converges in meas in compact subsets of the plane. That is, given $r, \varepsilon > 0$,

$$meas\,\{z : |z| \leq r \text{ and } |f - [n/n]|\,(z) \geq \varepsilon\} \to 0 \text{ as } n \to \infty.$$

One consequence is that a subsequence converges a.e. As noted above, in his 1974 paper [70] containing his counterexample, Wallin also gave conditions on the size of the power series coefficients for convergence a.e. of the full diagonal sequence. Nuttall's theorem was soon extended by Pommerenke, using the concept of cap (logarithmic capacity). For a compact set K, we define

$$cap\,(K) = \lim_{n\to\infty} \left(\inf\left\{\|P\|_{L_\infty(K)} : P \text{ a monic polynomial of degree } n\right\}\right)^{1/n},$$

and we extend this to arbitrary sets E as inner capacity:

$$cap\,(E) = \sup\{cap\,(K) : K \subset E,\, K \text{ compact}\}.$$

Capacity is a "thinner" set function than planar or linear measure - any set of capacity 0 has Hausdorff dimension 0 [52].

Pommerenke [50] proved:

Theorem 3.4 (Pommerenke's Theorem)
Let f be analytic in $\mathbb{C}\backslash E$, and analytic at 0, where $cap\,(E) = 0$. Then, given $r, \varepsilon > 0$,

$$cap\,\{z : |z| \leq r \text{ and } |f - [n/n]|\,(z) \geq \varepsilon^n\} \to 0 \text{ as } n \to \infty.$$

Since any countable set has capacity 0, Pommerenke's theorem implies Nuttall's. The two are often combined and called the Nuttall-Pommerenke theorem. It is widely considered to be the central convergence theorem for diagonal Padé sequences.

While E above may be uncountable, it cannot include branchpoints. The latter require far deeper techniques, developed primarily by Herbert Stahl in a rigorous form, building on earlier ideas from Nuttall. Stahl showed that one can cut the plane joining the branchpoints in a certain way, yielding a set of minimal capacity, outside which the Padé approximants converge in capacity. This celebrated and deep theory, is expounded in [56], [57], [60], [61].

Theorem 3.5 (Stahl's Theorem)
Let f be analytic at 0, and in $\mathbb{C}$, except for a set of cap 0. Assume also f has branchpoints. There is an extremal domain D such that $\{[n/n]\}_{n\geq 1}$ converges in capacity to f in D and diverges outside of D.

We emphasize that this is a rather imprecise statement of Stahl's great theorem. He considered Padé approximants formed at ∞ (so we replace series in z by series in $1/z$) and showed that the extremal domain D has the form $\mathbb{C}\backslash K_0$, where K_0 minimizes the logarithmic capacity amongst all sets K such that f has a single valued analytic continuation from ∞ to $\mathbb{C}\backslash K$. He also established the precise geometric convergence rate in capacity, and characterized the set $\mathbb{C}\backslash D$ in terms of a certain symmetry property.

What about multipoint Padé approximants? Wallin extended the Nuttall-Pommerenke theorem to this case in 1979 [71], by showing that if all the interpolation points lie in a fixed compact set in which the underlying function f is analytic, then the conclusion remains true. The proof follows much the same lines as Pommerenke's theorem. Stahl extended his Padé theorem to multipoint Pade approximants in 1989 [59], though one has to assume appropriate asymptotic distribution of the interpolation points, in keeping with the structure of the extremal domain. Buslaev recently extended Stahl's theorem to "piecewise analytic" functions [16].

A common feature of these two central theorems, the Nuttall-Pommerenke theorem, and Stahl's theorem, is that the function f has to be analytic in "most" of the plane. What happens if, for example, f is analytic in the unit ball, but has a natural boundary on the unit circle? Unfortunately, very little seems to be true. E.A. Rakhmanov [51] and the author [37] showed independently that if all we know is that the function is analytic inside the unit ball, then $\{[n/n]\}$ may not converge in capacity or in measure even in any open subset, no matter hows small, or close to 0. A similar problem occurs for multipoint Padé approximants [38]. An attempt to say something positive was given in [43].

In all the negative results above, the pathologies occur only for a subsequence of the approximants, something observed back in the 1950's by George A. Baker, Jr., who did so much to develop both the theory of Pade approximation, and its application in physics. Some other subsequence is "good". Accordingly, he and his coworkers posed in 1961 [8]:

Conjecture 3.6 (Baker-Gammel-Wills Conjecture)

Let f be meromorphic in the unit ball, and analytic at 0. There is an infinite subsequence $\{[n/n]\}_{n\in\mathcal{S}}$ of the diagonal sequence $\{[n/n]\}_{n=1}^{\infty}$ that converges uniformly in all compact subsets of the unit ball omitting poles of f.

In the first form of the conjecture, f was required to have a non-polar singularity on the unit circle, but this was subsequently relaxed (cf. [5, p. 188 ff.]). In other forms of the conjecture, f is assumed to be analytic in the unit ball. There is also apparently a cruder form of the conjecture due to Padé himself, dating back to the 1900's, something the author leaned from J. Gilewicz.

While the Baker-Gammel-Wills Conjecture was widely believed to be false by the 1970's, a counterexample remained elusive. It is very difficult to show pathological behavior of a *full sequence* of Padé approximants. After many years of searching, the author found a counterexample in the continued fraction of Rogers-Ramanujan [44]. For q not a root of unity, let

$$G_q(z) := \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2)\dots(1-q^j)} z^j$$

denote the Rogers-Ramanujan function, and

$$H_q(z) = G_q(z)/G_q(qz).$$

Theorem 3.7 (Meromorphic Counterexample)
Let $q := \exp(2\pi i\tau)$ where $\tau := \frac{2}{99+\sqrt{5}}$. Then H_q is meromorphic in the unit ball and analytic at 0. There does not exist any subsequence of $\{[n/n]\}_{n=1}^{\infty}$ that converges uniformly in all compact subsets of $\mathcal{A} := \{z : |z| < 0.46\}$ omitting poles of H_q.

It did not take long for A.P. Buslaev to improve on this, by finding a function analytic in the unit ball, for which the Baker-Gammel-Wills Conjecture, as well as some of Stahl's conjectures [62] for algebraic functions fail [13], [14]. Buslaev considered 3 periodic continued fractions and constructed his example by a very clever choice of parameters. When expressed in closed form it is given in the following:

Theorem 3.8 (Buslaev's Analytic Counterexample)
Let

$$f(z) = \frac{-27 + 6z^2 + 3(9+j)z^3 + \sqrt{81\left(3-(3+j)z^3\right)^2 + 4z^6}}{2z\left(9+9z+(9+j)z^2\right)},$$

where $j = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. *The branch of the* $\sqrt{}$ *is chosen so that* $f(0) = 0$. *Then for some* $R > 1 > r > 0$, f *is analytic in* $\{z : |z| < R\}$, *but for large enough* n, $[n/n]$ *has a pole in* $|z| < r$, *and consequently no subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *converges uniformly in all compact subsets of* $\{z : |z| < 1\}$.

Buslaev later showed [15] that for q a suitable root of unity, the Rogers-Ramanujan function above, is also a counterexample to both Baker-Gammel-Wills and conjectures of Stahl for hypelliptic functions. Although this resolves the conjecture, it raises further questions. In both the above counterexamples, uniform convergence fails due to the persistence of spurious poles in a specific compact subset of the unit ball. Moreover, in both the above examples, given any point of analyticity of f in the unit ball, some subsequence converges in some neighborhood of that point. In fact, just two subsequences are enough to provide uniform convergence throughout the unit ball, as pointed out by Baker in [6]. Accordingly, in 2005, George Baker modified his 1961 conjecture [7]:

Conjecture 3.9 (George Baker's "Patchwork" Conjecture)
Let f *be analytic in the unit ball, except for at most finitely many poles, none at* 0. *Then there exist a finite number of subsequences of* $\{[n/n]\}_{n=1}^{\infty}$ *such that for any given point of analyticity* z *in the ball, at least one of these subsequences converges pointwise to* $f(z)$.

It seems that if true in this form, the convergence would be uniform in some neighborhood of z. Baker also includes poles amongst the permissible z, with the understanding that the corresponding subsequence diverges to ∞. Even solving the following weaker conjecture would be of interest:

Conjecture 3.10 (Conjecture on convergence in capacity of a subsequence)
Let f *be analytic or meromorphic in the unit ball, and analytic at* 0. *There exists a subsequence of* $\{[n/n]\}_{n=1}^{\infty}$ *and* $r > 0$ *such that the subsequence converges in measure or capacity to* f *in* $\{z : |z| < r\}$.

Notice that we are not even asking for convergence in capacity throughout the unit ball, nor for the r to be independent of f. Another obvious point is that all the counterexamples involve a function with finite radius of meromorphy. What about entire functions, or functions meromorphic in the whole plane?

Conjecture 3.11 (Baker-Gammel-Wills Conjecture for entire/ meromorphic functions)
Let f be entire, or meromorphic in $\mathbb{C}$ and analytic at 0. There exists a subsequence of $\{[n/n]\}_{n=1}^{\infty}$ that converges uniformly to f in compact subsets of $\mathbb{C}$.

The author proved [39] that the Baker-Gammel-Wills conjecture is true for most entire functions in the sense of category.

There are a number of important conjectures about the Padé approximants for hyperelliptic functions, due to Nuttall and Stahl, amongst others, that we cannot discuss in detail here. Some of Stahl's Conjectures and a version of the Baker-Gammel-Wills Conjecture were established for a large class of hyperelliptic functions by S.P. Suetin [65]. Some very impressive recent related work, due to Aptekarev, Baratchart, and Yattselev appears in [3], [10]. See the surveys of Aptekarev, Buslaev, Martinez-Finkleshtein, and S.P. Suetin [2], and Martinez-Finkelshtein, Rakhmanov, and Suetin [47]. Deep Riemann-Hilbert techniques play a key role in these papers. Multipoint Padé approximants have also been essential tools in obtaining asymptotics for errors of best rational approximation [1], [29], [64].

4. Spurious Poles and Varying Interpolation Arrays

When Padé approximants have spurious poles, they also tend to have close by spurious zeros, that is zeros that bear no relation to the zeros of the underlying function. These pairs of spurious poles and zeros are often called *Froissart doublets* [11]. The term was apparently first used in the setting of random perturbations of power series [25], but I believe has been more and more used to describe the general phenomenon.

Another accompanying feature of spurious poles is *overinterpolation*, namely that there are more than the expected number of interpolation points. Thus for example, $f-[n/n]$ may have more than $2n+1$ zeros, counting multiplicity, in a fixed ball containing 0 (or, if more appropriate ∞) as $n\to\infty$. In Stahl's work on functions with branchpoints, he typically showed that $[n/n]$ can have at most $o(n)$ extra interpolation points. For more special classes of algebraic and elliptic functions, the $o(n)$ can be replaced by $O(1)$. See [3], [31], [67].

The author recently realized that by considering all possible choices of interpolation points in an open set, one can more precisely relate spurious poles and extra interpolation points. This requires a change in our notation: let $\mathcal{D}$ be an open connected subset of $\mathbb{C}$ and $f:\mathcal{D}\to\mathbb{C}$ be analytic. Given $n\geq 1$ and not necessarily distinct points

$\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1}$ in $\mathcal{D}$, we denote the multipoint Padé approximant to f with interpolation set Λ_n by

$$R_{nn}(\Lambda_n, z) = \frac{p_n(\Lambda_n, z)}{q_n(\Lambda_n, z)},$$

so that

$$e_n(\Lambda_n, z) = \frac{f(z)\, q_n(\Lambda_n, z) - p_n(\Lambda_n, z)}{\prod_{j=1}^{2n+1} (z - z_{jn})}$$

is analytic in $\mathcal{D}$. We can now define an exact interpolation index:

Definition 4.1
Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Let $\mathcal{L} \subset \mathcal{D}$ and $n \geq 1$. We say n is an **exact interpolation index for f and $\mathcal{L}$** *if for every set of $2n+1$ not necessarily distinct interpolation points $\Lambda_n = \{z_{jn}\}_{j=1}^{2n+1}$ in $\mathcal{L}$, and the corresponding interpolant, $e_n(\Lambda_n, z)$ has no zeros in $\mathcal{L}$.*

It is relatively straightforward to show that whenever $R_{n+1,n+1}$ formed from interpolation points in $\mathcal{D}$ has no spurious poles even in some tiny open set $\mathcal{B}$, then necessarily the previous index n is an exact interpolation index [45]:

Proposition 4.2
Let $\mathcal{D} \subset \mathbb{C}$ be a connected open set, and $f : \mathcal{D} \to \mathbb{C}$ be analytic. Let $n \geq 1$ and $\mathcal{L}$ and $\mathcal{B}$ be open subsets of $\mathcal{D}$. Assume that whenever we are given a set of $2n+3$ not necessarily distinct points $\Lambda_{n+1} \subset \mathcal{L} \cup \mathcal{B}$, $R_{n+1,n+1}(\Lambda_{n+1}, z)$ does not have poles in $\mathcal{B}$. Then n is an exact interpolation index for f and $\mathcal{L}$.

As a consequence, when n is not an exact index, that is there are extra interpolation points, some close by set of interpolation points leads to an interpolant with spurious poles for the next degree $n+1$. We proved in [45] a much deeper partial converse, that exact interpolation forces the absence of spurious poles, at least for a subsequence.

Theorem 4.3
Let f be entire. Assume that for every $r > 0$, n is an exact interpolation index for f and $B_r = \{z : |z| < r\}$ for large enough n. Then there exists a subsequence $\mathcal{S}$ of positive integers with the following property: let $r, s > 0$, and for $n \geq 1$, choose interpolation sets Λ_n in B_r. Then

for large enough $n \in \mathcal{S}$, $R_{nn}(\Lambda_n, z)$ is analytic in B_s. Consequently, uniformly for z in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty, n\in\mathcal{S}} R_{nn}(\Lambda_n, z) = f(z).$$

We emphasize that the same subsequence $\mathcal{S}$ works for all sets of interpolation points in B_r, and for all r. In [45], we also considered more general sequences $\{n_k\}$ of exact indices. In addition, under mild regularity of errors of best rational approximation, we established uniform convergence for full sequences, not just subsequences.

The idea that we should consider not just interpolation at a given set of points, but at all possible choices of interpolation points in an open set, is also relevant to the Baker-Gammel Wills Conjecture. In [46], we formulated a generalization of Conjecture 3.11:

Conjecture 4.4
Let f be entire. Then there is an infinite subsequence $\mathcal{S}$ of positive integers with the following property: given any $r > 0$ and for $n \in \mathcal{S}$, multipoint Padé approximants $R_{nn}(\Lambda_n, z)$ to f of type (n, n) formed from interpolation points $\Lambda_n \subset B_r$, we have

$$\lim_{n\to\infty, n\in\mathcal{S}} R_{nn}(z) = f(z)$$

uniformly in compact subsets of the plane.

In [46], we proved that this is true for most entire functions in the sense of category. We also showed that when the Maclaurin series coefficients $\{a_j\}$ satisfy (3.1) for some $|q| < 1$, full sequences $\{R_{nn}(\Lambda_n, z)\}_{n\geq 1}$ of interpolants converge uniformly in compact sets.

5. Concluding Remarks

In writing this brief survey, the author has been struck by the paucity of results on uniform convergence of diagonal multipoint Padé-approximation in classical settings. Yes, the central theorems of Nuttall-Pommerenke and Stahl have been established for this case. Yes, they have been an essential tool in the deep works of Aptekarev, Gončar-Rakhmanov, and Stahl on errors of best rational approximation. Yes for Stieltjes series, there are deep results, especially through relations to varying weights and varying orthogonal polynomials. However, what about classical special functions beyond e^z? What about Polya-frequency functions? What about finding Newton-Padé arrays that would yield uniformly convergent multipoint Padé approximants for functions analytic in the unit ball?

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