Scaling Limits of Polynomials and Entire Functions of Exponential Type

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Abstract The connection between polynomials and entire functions of exponential type is an old one, in some ways harking back to the simple limit

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^z.$$

On the left-hand side, we have $P_n\left(\frac{z}{n}\right)$, where P_n is a polynomial of degree n, and on the right, an entire function of exponential type. We discuss the role of this type of scaling limit in a number of topics: Bernstein's constant for approximation of $|x|$; universality limits for random matrices; asymptotics of L_p Christoffel functions and Nikolskii inequalities; and Marcinkiewicz-Zygmund inequalities. Along the way, we mention a number of unsolved problems.

1 Introduction

The classical limit

$$\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n=e^z, \tag{1.1}$$

plays a role in many areas of mathematics, expressing very simply the scaling limit of a sequence of a polynomials as an entire function of exponential type 1. Recall that an entire function f has exponential type A if for every $\varepsilon>0$,

$$|f(z)|=O\left(e^{(A+\varepsilon)|z|}\right),\quad \text{as } |z|\to\infty,$$

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and A is the smallest number with this property. Within approximation theory, this connection has long been recognized, perhaps most notably in relating asymptotics of errors of polynomial approximation to errors of approximation by entire functions of exponential type. Indeed, the classical monograph of Timan (see [58] for a translation of the Russian original) presents some of this theory, and this topic continues to be explored to this day, notably in the works of Michael Ganzburg [21], [23]. In this survey, we explore this connection in a number of topics in approximation theory.

A less trivial example than (1.1) involves Lagrange interpolation at roots of unity. Given $n \geq 1$, the fundamental polynomials of Lagrange interpolation at the nth roots of unity are

$$\ell_{jn}(z) = \frac{1}{n}\frac{z^n - 1}{ze^{-2\pi ij/n} - 1}, \quad 0 \leq j \leq n-1,$$

satisfying

$$\ell_{jn}\left(e^{2\pi ik/n}\right) = \delta_{jk}.$$

Let us fix t and take the scaling limit: As $n \to \infty$,

$$\begin{aligned}\ell_{jn}\left(e^{2\pi it/n}\right) &= \frac{1}{n}\frac{e^{2\pi it} - 1}{e^{2\pi i(t-j)/n} - 1} \\ &= \frac{1}{n}\frac{e^{\pi it}}{e^{\pi i(t-j)/n}}\frac{\sin \pi t}{\sin \frac{\pi(t-j)}{n}} \\ &\to e^{i\pi t}\frac{\sin \pi t}{\pi (t-j)}.\end{aligned}$$

Thus

$$\lim_{n\to\infty} \ell_{jn}\left(e^{2\pi it/n}\right) = e^{i\pi t}(-1)^j \mathbb{S}(t-j), \tag{1.2}$$

where $\mathbb{S}$ is the classical sinc kernel

$$\mathbb{S}(t) = \begin{cases} \frac{\sin \pi t}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

If

$$\sum_{j=0}^{\infty} |c_j|^2 < \infty,$$

and for $n \geq 1$, we let

$$P_n(z) = \sum_{j=0}^{n-1} (-1)^j c_j \ell_{jn}(z),$$

then one can use (1.2) to show

$$\lim_{n\to\infty} P_n\left(e^{2\pi it/n}\right) = e^{i\pi t}\sum_{j=0}^{\infty} c_j \mathbb{S}\left(t-j\right) = e^{i\pi t} f\left(t\right), \tag{1.3}$$

where

$$f\left(t\right) = \sum_{j=0}^{\infty} c_j \mathbb{S}\left(t-j\right)$$

is entire of exponential type $\leq \pi$, and its restriction to the real line lies in $L_2\left(\mathbb{R}\right)$. The space of all such entire functions with the usual L_2 norm on the real line is L_2 Paley-Wiener space PW_2. Its remarkable reproducing kernel $\mathbb{S}$ plays a role in everything from sampling theory to random matrices, satisfying the orthonormality relation

$$\int_{-\infty}^{\infty} \mathbb{S}\left(t-j\right)\mathbb{S}\left(t-k\right) dt = \delta_{jk}.$$

The reproducing kernel relation is

$$f\left(z\right) = \int_{-\infty}^{\infty} f\left(t\right)\mathbb{S}\left(t-z\right) dt, \quad z\in\mathbb{C}, \quad f\in PW_2.$$

The paper is organized as follows: in the next section, we discuss approximation of $|x|$ by polynomials. Section 3 deals with universality limits for random matrices, Section 4 deals with asymptotics of Christoffel functions, Section 5 with Nikolskii inequalities, and Section 6 deals with Marcinkiewicz-Zygmund inequalities.

2 Bernstein's Constant in approximation of $|x|$

Let $f:[-1,1]\to\mathbb{R}$. For $n\geq 1$, let $E_n\left[f\right]$ denote the error in best uniform approximation of $|x|$ by polynomials of degree $\leq n$, so that

$$E_n\left[f\right] = \inf\left\{\| f-P \|_{L_\infty[-1,1]} : \deg\left(P\right)\leq n\right\}.$$

In a 1913 paper [6], S. N. Bernstein established that the error in approximating $f\left(x\right)=|x|$ decays exactly like $\frac{1}{n}$, that is,

$$\Lambda_1 = \lim_{n\to\infty} nE_n\left[|x|\right]$$

exists, and is finite and positive. Λ_1 is often called the Bernstein constant. The proof that the limit exists is long and difficult, and is unclear in parts. Twenty five years later [7] he presented a much simpler proof, that works for the more general function $|x|^\alpha$ for all $\alpha>0$ that is not an even integer. It involves dilations of the interval, making essential use of the homogeneity of

$|x|^\alpha$, namely that for $\lambda > 0$,

$$|\lambda x|^\alpha = \lambda^\alpha |x|^\alpha .$$

This enabled Bernstein to relate the error in approximation on $[-\lambda, \lambda]$ to that on $[-1, 1]$. It also yielded a formulation of the limit as the error in approximation on the whole real axis by entire functions of exponential type, namely

$$\begin{aligned} \Lambda_\alpha &:= \lim_{n\to\infty} n^\alpha E_n [|x|^\alpha] \\ &= \inf \left\{ \| \, |x|^\alpha - f(x) \, \|_{L_\infty(\mathbb{R})} : f \text{ is entire of exponential type } \le 1 \right\} . \end{aligned} \tag{2.1}$$

You might ask: what is the connection of all this to scaling limits? Well, Bernstein related the errors of polynomial approximation by polynomials of degree $\le n$ on $[-1, 1]$ and $[-n, n]$. Thus if we let P_n^* denote the unique polynomial of degree $\le n$ that best approximates $|x|^\alpha$ in the uniform norm on $[-n, n]$, Bernstein's proof essentially involved scaling $P_n^*(x)$ to $P_n^*(x/n)$. Since 0 is the place where $|x|^\alpha$ is least smooth, it's not surprising that we scale about 0. One of the classical unsolved problems of approximation theory is

Problem 2.1. Give an explicit representation for Λ_1.

Of course, this is a little imprecise, but something such as Λ_1 is a root of an explicit equation, or given by some explicit series, would be a real achievement. Bernstein obtained upper and lower bounds for Λ_1, and using these, speculated that possibly

$$\Lambda_1 = \frac{1}{2\sqrt{\pi}} = 0.28209\,47917 \ldots .$$

Some 70 years later, this was disproved by Varga and Carpenter [64], [65] using high precision scientific computation. They showed that

$$\Lambda_1 = 0.28016\,94990 \ldots$$

They also showed numerically that the normalized error $2nE_{2n}[|x|]$ should admit an asymptotic expansion in negative powers of n. Further numerical explorations for approximation of $|x|^\alpha$ have been provided by Carpenter and Varga [9].

Bernstein also showed that for $\alpha > 1$, [7], [9, p. 194]

$$\frac{\left|\sin \frac{\alpha\pi}{2}\right|}{\pi} \Gamma(\alpha) \left(1 - \frac{1}{\alpha - 1}\right) < \Lambda_\alpha < \frac{\left|\sin \frac{\alpha\pi}{2}\right|}{\pi} \Gamma(\alpha) .$$

Surprisingly, the much deeper analogous problem of rational approximation has already been solved, by the great Herbert Stahl [54]. He proved, using sophisticated methods of potential theory and other complex analytic tools, that

$$\lim_{n\to\infty} e^{\pi\sqrt{n}} R_n\left[|x|\right] = 8,$$

where $R_n\left[|x|\right]$ denotes the error in best L_∞ approximation of $|x|$ on $[-1,1]$ by rational functions with numerator and denominator degree $\leq n$. Later [55], he extended this to $|x|^\alpha$, establishing

$$\lim_{n\to\infty} e^{\pi\sqrt{\alpha n}} R_n\left[|x|^\alpha\right] = 4^{1+\alpha/2}\left|\sin\frac{\pi\alpha}{2}\right|.$$

Although Λ_α is not known explicitly, the ideas of Bernstein have been refined, and greatly extended. They are covered in the monograph of Timan [58, p. 48 ff.]. M. Ganzburg has shown limit relations of this type for large classes of functions, in one and several variables, even when weighted norms are involved [21], [22], [23], [24]. Nikolskii [46] and Raitsin [48], [49] considered not only uniform, but also L_p norms. They and later Ganzburg [22] showed that for $1 \leq p \leq \infty$, there exists

$$\Lambda_{p,\alpha} = \lim_{n\to\infty} n^{\alpha+\frac{1}{p}} \inf\left\{\|\,|x|^\alpha - P(x)\,\|_{L_p[-1,1]} \colon \deg(P) \leq n\right\}.$$

More explicitly, Nikolskii [46] proved that at least for odd integers α,

$$\Lambda_{1,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 8\Gamma(\alpha+1)\sum_{k=0}^{\infty}(-1)^k(2k+1)^{-\alpha-2}.$$

He also established an integral representation valid for all $\alpha > -1$, and Bernstein later noted that this implies the above series representation for all $\alpha > -1$. Raitsin [49] proved that for $\alpha > -\frac{1}{2}$,

$$\Lambda_{2,\alpha} = \frac{\left|\sin\frac{\alpha\pi}{2}\right|}{\pi} 2\Gamma(\alpha+1)\sqrt{\pi/(2\alpha+1)}.$$

These are the only known explicit values of $\Lambda_{p,\alpha}$. The extremal entire functions associated with these constants were given explicit form in [25].

Vasiliev [63] extended Bernstein's results in another direction, replacing the interval $[-1,1]$ by fairly general compact sets E. Totik [62] has put Vasiliev's results in final form, using sophisticated estimates for harmonic measures. In this more general setting, Λ_α still appears, multiplied by a quantity involving the equilibrium density of potential theory for E. The Bernstein constant was discussed in the recent book of Finch on mathematical constants [17, p. 257 ff.] in different areas of mathematics.

Another recent mode of attack on the Bernstein problem involves sophisticated properties of conformal maps of comb domains: instead of approximating directly on $[-1,1]$, one solves the asymptotic problem on the symmetric interval $[-1,-a]\cup[a,1]$. Reknowned complex analysts such as Eremenko, Nazarov, Peherstorfer, Volberg, and Yuditskii have been involved in this effort [16], [43].

The author's own attempts at this problem directly involve scaling limits: as above, let P_n^* denote the unique polynomial of degree $\leq n$ that best approximates $|x|^\alpha$ in the uniform norm. In [35] we proved a pointwise limit, namely that uniformly in compact subsets of $\mathbb{C}$,

$$\lim_{n\to\infty} n^\alpha P_n^*(z/n) = H_\alpha^*(z), \tag{2.2}$$

where H_α^* is the unique entire function of exponential type 1 attaining the inf in (2.1). That paper also contained L_p analogues, and implicit integral representations of H_α^*. Closely related to Problem 2.1 is:

Problem 2.2. Give an explicit description of the function H_α^*.

3 Universality Limits in Random Matrices

Although they have much older roots, the theory of random matrices rose to prominence in the 1950's, when the physicist Eugene Wigner found them an indispensable tool in analysing scattering theory for neutrons off heavy nuclei. The mathematical context of the unitary case may be briefly described as follows. Let $\mathcal{M}(n)$ denote the space of n by n Hermitian matrices $M = (m_{ij})_{1\leq i,j\leq n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$\begin{aligned} P^{(n)}(M) &= cw(M)\,dM \\ &= cw(M)\left(\prod\nolimits_{j=1}^{n} dm_{jj}\right)\left(\prod\nolimits_{j<k} d(\mathrm{Re}\ m_{jk})\,d(\mathrm{Im}\ m_{jk})\right). \end{aligned}$$

Here $w(M)$ is a function defined on $\mathcal{M}(n)$, and c is a normalizing constant. The most important case is

$$w(M) = \exp(-2n\ \mathrm{tr}\ Q(M)),$$

involving the trace tr, for appropriate functions Q defined on $\mathcal{M}(n)$. In particular, the choice

$$Q(M) = M^2,$$

leads to the Gaussian unitary ensemble (apart from scaling) that was considered by Wigner. One may identify $P^{(n)}$ above with a probability density on the eigenvalues $x_1 \leq x_2 \leq \cdots \leq x_n$ of M,

$$P^{(n)}(x_1, x_2, \ldots, x_n) = c\left(\prod_{j=1}^{m} w(x_j)\right)\left(\prod\nolimits_{i<j}(x_i - x_j)^2\right).$$

See [10, p. 102 ff.]. Again, c is a normalizing constant.

Orthogonal polynomials enable one to explicitly represent $P^{(n)}$ and a number of other statistical quantities. Let μ be a finite positive Borel measure with support in the real line, with infinitely many points in the support, and all finite power moments. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$, satisfying the orthonormality conditions

$$\int p_j p_k d\mu = \delta_{jk}. \tag{3.1}$$

The nth reproducing kernel for μ is

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y), \tag{3.2}$$

and the normalized kernel is

$$\widetilde{K}_n(x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y), \tag{3.3}$$

where μ' denotes the Radon-Nikodym derivative of μ.

There is the basic formula for the probability distribution $P^{(n)}$ [10, p. 112]:

$$P^{(n)}(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le n}.$$

One may use this to compute a host of statistical quantities – for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One particularly important quantity is the m-point correlation function for $M(n)$ [10, p. 112]:

$$\begin{aligned} R_m(x_1, x_2, \ldots, x_m) &= \frac{n!}{(n-m)!} \int \cdots \\ &\quad \int P^{(n)}(x_1, x_2, \ldots, x_n)\, dx_{m+1}\, dx_{m+2} \ldots dx_n \\ &= \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le m}. \end{aligned}$$

This last remarkable identity is due to Freeman Dyson.

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, ξ in the interior of the support of μ, and real $a_1, a_2, \ldots, a_m$, we have

$$\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\xi,\xi)^m} R_m\left(\xi + \frac{a_1}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi,\xi)}, \dots, \xi + \frac{a_m}{\tilde{K}_n(\xi,\xi)}\right)$$
$$= \det\left(\mathbb{S}(a_i - a_j)\right)_{1\le i,j\le m}.$$

Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \mathbb{S}(a-b), \tag{3.4}$$

for real a, b. Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a scaling limit involving orthogonal polynomials. The term universal is quite justified: the limit on the right-hand side of (3.4) is independent of ξ, but more importantly is independent of the underlying measure. Since in many cases

$$\lim_{n\to\infty} \frac{1}{n}\tilde{K}_n(\xi,\xi) = \omega(\xi),$$

where ω is an appropriate "equilibrium density", we can also often recast (3.4) as

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{n\omega(\xi)}, \xi + \frac{b}{n\omega(\xi)}\right)}{n\omega(\xi)} = \mathbb{S}(a-b). \tag{3.5}$$

For example, if $\mu' > 0$ is positive a.e. in $(-1, 1)$,

$$\omega(x) = \frac{1}{\pi\sqrt{1-x^2}},$$

the ubiquitous *arcsine distribution*.

Typically, the limit (3.4) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n, we can also establish its validity for complex a, b, that is,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \mathbb{S}(a-b). \tag{3.6}$$

There are a variety of methods to establish (3.4–6). Perhaps the deepest methods are the Riemann-Hilbert methods, which yield far more than universality. See [5], [10], [11], [12], [27], [28], [29], [42] for Riemann-Hilbert references.

Inspired by the 60th birthday conference for Percy Deift, the author came up with a new comparison method to establish universality. Let μ be a measure supported on $(-1, 1)$, and assume for example that $\mu' > 0$ a.e. in $(-1, 1)$. Let μ be absolutely continuous in a neighborhood of some given $\xi \in (-1, 1)$ and assume that μ' is positive and continuous at ξ. Then in [38] we established (3.5). This result was soon extended to a far more general setting by Findley, Simon and Totik [18], [51], [52], [60], [61]. In particular, when μ is a measure with compact support that is regular, and $\log \mu'$ is integrable in a subinterval of the support (c, d), then Totik established that the universality (3.5) holds a.e. in (c, d). Totik used the method of polynomial pullbacks to go first from one to finitely many intervals, and then used the latter to approximate general compact sets. In contrast, Simon used the theory of Jost functions.

The drawback of this comparison method is that it requires regularity of the measure μ. Although the latter is a weak global condition, it is nevertheless most probably an unnecessary restriction. To circumvent this, the author developed a different method, based on tools of classical complex analysis, such as normal families, and the theory of entire functions of exponential type. In [39], this was used to show that universality holds in linear Lebesgue measure, *meas*, without any local or global conditions, in the set

$$\{\mu' > 0\} := \{\xi : \mu'(\xi) > 0\}.$$

Theorem 3.1. *Let μ be a measure with compact support and with infinitely many points in the support. Let $\varepsilon > 0$ and $r > 0$. Then*

$$meas\Bigg\{\xi \in \{\mu' > 0\} : \sup_{|u|,|v|\le r}\left|\frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \mathbb{S}(u-v)\right| \ge \varepsilon\Bigg\} \to 0 \text{ as } n \to \infty. \tag{3.7}$$

The method of proof of this result is instructive, because it contains ideas often used in establishing scaling limits:

Step 1. Let

$$f_n(u, v) := \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)},$$

a polynomial in u, v. For $R > 0$ and "most" ξ, one can use tools such as the Bernstein-Walsh growth lemma to show that for $n \ge n_0(R)$ and $|u|, |v| \le R$,

$$|f_n(u, v)| \le C_1 e^{C_2(|u|+|v|)}. \tag{3.8}$$

Here C_1 and C_2 are independent of n, u, v.

Step 2. The uniform boundedness in (3.8) allows us to choose a subsequence $\{f_n\}_{n\in S}$ that converges uniformly for u, v in compact subsets of the plane to an entire function f satisfying the bound

$$|f(u, v)| \le C_1 e^{C_2(|u|+|v|)}, \quad u, v \in \mathbb{C}.$$

Thus f is of exponential type in each variable.
Step 3. Inasmuch as each K_n is a reproducing kernel for polynomials of degree $\leq n-1$, one expects its subsequential limit f to be a reproducing kernel for some space of functions. Indeed, it is, and this is the hard part: to show that f is a reproducing kernel for Paley-Wiener space PW. As reproducing kernels are unique, it follows that

$$f(u,v) = \mathbb{S}(u-v).$$

Since the limit is independent of the subsequence, we have the result for the full sequence.

It is unlikely that convergence in measure in (3.7) can be replaced by convergence a.e., but nevertheless we pose:

Problem 3.1. Does universality hold a.e. in $\{\mu' > 0\}$?

We emphasize that this is a tiny slice of a major topic. At the endpoints of compactly supported μ (the "edge of the spectrum") one scales not with x/n but with x/n^2 and the limiting kernel is a Bessel kernel. For moving boundaries, one scales with $x/n^{2/3}$ and the limiting kernel is an Airy kernel. Other kernels arise when there are jump or other discontinuities, and there are several other more complex scaling limits associated with other universality limits. See, for example, [1], [2], [3], [4], [15], [19], [20], [27], [31], [37], [57].

4 L_p Christoffel Functions

Let μ denote a finite positive Borel measure on $[-1,1]$. Its L_2 Christoffel function is

$$\lambda_n(\mu,x) = \inf_{\deg(P)\leq n-1} \frac{\int_{-1}^{1} |P(t)|^2 \, d\mu(t)}{|P(x)|^2}. \tag{4.1}$$

If $\{p_j\}$ are the orthonormal polynomials for μ, then it follows from Cauchy-Schwarz' inequality and orthogonality that

$$\lambda_n(\mu,x) = 1/\sum_{j=0}^{n-1} p_j^2(x), \tag{4.2}$$

while a minimizing polynomial for a given x is

$$P(t) = K_n(x,t).$$

From these formulas, it's fairly clear why $\lambda_n(\mu,x)$ is useful: bounds on λ_n are essentially bounds on averages of the orthonormal polynomials. Moreover, the extremal or variational property (4.1) allows comparison of λ_n for different measures.

As is often the case with orthogonal polynomials, it is easier first to start on the unit circle and then later map to $[-1,1]$. Accordingly, let ω denote a finite positive Borel measure on the unit circle, or equivalently $[-\pi,\pi]$. Its L_2 Christoffel function is

$$\lambda_n(\omega,z)=\inf_{\deg(P)\leq n-1}\frac{\int_{-\pi}^{\pi}\left|P\left(e^{i\theta}\right)\right|^2 d\omega(\theta)}{|P(z)|^2},$$

and yes again, there is a connection to orthogonal polynomials on the unit circle.

Asymptotics of Christoffel functions have been studied for a very long time. Paul Nevai's 1986 ode to them [45] is still very relevant, while Barry Simon's books [50], [53] contain the most recent research. In a breakthrough 1991 paper, Maté, Nevai and Totik [41] proved that when ω is *regular*, and satisfies in some subinterval I of $[-\pi,\pi]$

$$\int_I \log\omega'\left(e^{i\theta}\right)d\theta>-\infty,$$

then for a.e. $\theta\in I$,

$$\lim_{n\to\infty} n\lambda_n\left(\omega,e^{i\theta}\right)=\omega'(\theta).$$

Here ω is *regular* if

$$\lim_{n\to\infty}\left(\inf_{\deg(P)\leq n}\frac{\int_{-\pi}^{\pi}|P|^2 d\omega}{\|P\|^2_{L_\infty(|z|=1)}}\right)^{1/n}=1.$$

A sufficient condition for regularity, the so-called Erdős-Turán condition, is that $\omega'>0$ a.e. in $[-\pi,\pi]$. However, there are pure jump measures, and pure singularly continuous measures that are regular [56].

That 1991 paper of Maté, Nevai and Totik also addresses measures on $[-1,1]$. If μ is regular on $[-1,1]$ and satisfies on some subinterval I,

$$\int_I \log\mu'>-\infty,$$

then

$$\lim_{n\to\infty} n\lambda_n(\mu,x)=\pi\sqrt{1-x^2}\mu'(x), \tag{4.3}$$

for a.e. $x\in I$. Totik subsequently extended this to measures μ with arbitrary compact support [59].

The extension of Christoffel functions from L_2 to L_p also goes back a long way – in some contexts, back to Szegő. For ω as above, define its L_p Christoffel function

$$\lambda_{n,p}(\omega, z) := \inf_{\deg(P)\leq n-1} \frac{\int_{-\pi}^{\pi} \left|P\left(e^{i\theta}\right)\right|^p d\omega(\theta)}{|P(z)|^p}. \tag{4.4}$$

By a compactness argument, there is a polynomial $P^*_{n,p,z}$ of degree $\leq n-1$ with $P^*_{n,p,z}(z) = 1$ and

$$\lambda_{n,p}(\omega, z) = \int_{-\pi}^{\pi} \left|P^*_{n,p,z}\left(e^{i\theta}\right)\right|^p d\omega(\theta). \tag{4.5}$$

When $p \geq 1$, this polynomial is unique.

The classical Szegő theory provides asymptotics for $\lambda_{n,p}(\omega, z)$ when $|z| < 1$. For example, if ω is absolutely continuous, then [50, p. 153] for $|z| < 1$,

$$\lim_{n\to\infty} \lambda_{n,p}(\omega, z) = \inf\left\{\int |f|^p d\omega : f \in H^\infty \text{ and } f(z) = 1\right\}.$$

Here H^∞ is the usual Hardy space for the unit disc. Moreover, for general measures, there is an alternative expression involving the Poisson kernel for the unit disc [50, p. 154].

On the unit circle, and for measures on $[-1, 1]$, bounds for L_p Christoffel functions have been known for a long time, notably those in Paul Nevai's landmark memoir [44]. However, limits for L_p Christoffel functions on the circle or interval, were first established by Eli Levin and the author [33]. The asymptotic involves an extremal problem for the L_p Paley-Wiener space PW_p. This is the set of all entire functions f of exponential type at most π, whose restriction to the real lies in $L_p(\mathbb{R})$. We define

$$\mathcal{E}_p = \inf\left\{\int_{-\infty}^{\infty} |f(t)|^p dt : f \in PW_p \text{ and } f(0) = 1\right\}. \tag{4.6}$$

Moreover, we let $f^*_p \in PW_p$ be a function attaining the infimum in (4.6), so that $f^*_p(0) = 1$ and

$$\mathcal{E}_p = \int_{-\infty}^{\infty} \left|f^*_p(t)\right|^p dt.$$

When $p \geq 1$, f^*_p is unique. For $p < 1$, uniqueness is apparently unresolved. For $p > 1$, we may give an alternate formulation:

$$\mathcal{E}_p = \inf \int_{-\infty}^{\infty} \left|\mathbb{S}(t) - \sum_{j=-\infty, j\neq 0}^{\infty} c_j \mathbb{S}(t-j)\right|^p dt, \tag{4.7}$$

where the inf is taken over all $\{c_j\} \in \ell_p$, that is over all $\{c_j\}$ satisfying

$$\sum_{j=-\infty, j\neq 0}^{\infty} |c_j|^p < \infty. \tag{4.8}$$

When $p = 2$, the orthonormality of the integer translates $\{\mathbb{S}(t-j)\}$ shows that $f_2^* = \mathbb{S}$, and

$$\mathcal{E}_2 = \int_{-\infty}^{\infty} \mathbb{S}(t)^2 \, dt = 1.$$

The precise value of $\mathcal{E}_p$ is apparently not known for $p \neq 2$. The estimate

$$\mathcal{E}_p > p^{-1}$$

goes back to 1949, to Korevaar's thesis [8, p. 102], [26].

We proved [33]:

Theorem 4.1. *Let $p > 0$, let ω be a finite positive measure supported on the unit circle, and assume that ω is regular. Let $|z_0| = 1$, and assume that z_0 is a Lebesgue point of ω, while the derivative ω' of the absolutely continuous part of ω is lower semi-continuous at z_0.*

(a) *Then*

$$\lim_{n\to\infty} n\lambda_{n,p}(\omega, z_0) = 2\pi\mathcal{E}_p\omega'(z_0). \tag{4.9}$$

(b) *If also $\omega'(z_0) > 0$ and $p > 1$, we have*

$$\lim_{n\to\infty} P^*_{n,p,z_0}\left(z_0 e^{2\pi i z/n}\right) = e^{i\pi z} f_p^*(z), \tag{4.10}$$

uniformly for z in compact subsets of the plane.

The proof of this theorem very heavily depends on scaling limits. Here are some ideas when $p > 1$, when ω is Lebesgue measure on the unit circle and when $z_0 = 1$: fix any $f \in PW_p$ with $f(0) = 1$. It admits the expansion

$$f(z) = \sum_{j=-\infty}^{\infty} f(j)\,\mathbb{S}(z-j), \tag{4.11}$$

that converges locally uniformly in the plane. This allows us to construct polynomials along the lines in the introduction: fix $m \geq 1$ and let

$$S_n(z) = \sum_{|j|\leq m} f(j)(-1)^j \ell_{jn}(z).$$

Here $\{\ell_{jn}\}$ are the fundamental polynomials of Lagrange interpolation at the roots of unity, as in Section 1. Since $S_n(1) = f(0) = 1$, we have

$$\lambda_{n,p}(\omega, 1) \leq \int_{-\pi}^{\pi} |S_n(z)|^p \, d\theta.$$

Here, for each $r > 0$, the limit (1.3) shows

$$\lim_{n\to\infty} n \int_{-2\pi r/n}^{2\pi r/n} |S_n(z)|^p \, d\theta = 2\pi \lim_{n\to\infty} \int_{-r}^{r} \left| S_n\left(e^{2\pi it/n}\right)\right|^p dt$$

$$= 2\pi \int_{-r}^{r} \left| \sum_{|j|\le m} f(j)\, \mathbb{S}(t-j) \right|^p dt.$$

We estimate the tails of both sides, let $r \to \infty$, and deduce that

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \le \limsup_{n\to\infty} n \int_{-\pi}^{\pi} |S_n(z)|^p \, d\theta$$

$$\le 2\pi \int_{-\infty}^{\infty} \left| \sum_{|j|\le m} f(j)\, \mathbb{S}(t-j) \right|^p dt.$$

Next, we let $m \to \infty$, and obtain

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \le 2\pi \int_{-\infty}^{\infty} |f(t)|^p \, dt.$$

As we may choose any $f \in L_\pi^p$, with $f(0) = 1$, we obtain

$$\limsup_{n\to\infty} n\lambda_{n,p}(\omega,1) \le 2\pi \mathcal{E}_p.$$

The converse inequality is more difficult, but also involves scaling limits.

There are also analogous results on $[-1,1]$. Let μ be a finite positive measure with support $[-1,1]$. It was probably Paul Nevai, who first systematically studied for measures on $[-1,1]$, the general L_p Christoffel function

$$\lambda_{n,p}(\mu,x) = \inf_{\deg(P)\le n-1} \frac{\int_{-1}^{1} |P(t)|^p \, d\mu(t)}{|P(x)|^p}, \tag{4.12}$$

in his 1979 memoir [44]. It was useful in establishing Bernstein and Nikolskii inequalities, in estimating quadrature sums, and in studying convergence of Lagrange interpolation and orthogonal expansions. Let $P_{n,p,\xi}^*$ denote a polynomial of degree $\le n-1$ with $P_{n,p,\xi}^*(\xi) = 1$, that attains the inf in (4.12).

Let us say that μ is *regular on* $[-1,1]$, or just *regular*, if

$$\lim_{n\to\infty} \left(\inf_{\deg(P)\le n} \frac{\int_{-1}^{1} P^2 d\mu}{\|P\|^2_{L_\infty[-1,1]}} \right)^{1/n} = 1.$$

As for the unit circle, a simple sufficient condition for regularity is that $\mu' > 0$ a.e. in $[-1,1]$, although it is far from necessary. We proved [33]:

Theorem 4.2. *Let $p > 0$, and let μ be a finite positive measure supported on $[-1,1]$, and assume that $\mu$is regular. Let $\xi \in (-1,1)$ be a Lebesgue point* *of μ, and let the derivative of its absolutely continuous part μ' be lower semi-continuous at ξ.*

(a) *Then*

$$\lim_{n\to\infty} n\lambda_{n,p}(\mu,\xi) = \pi\sqrt{1-\xi^2}\mathcal{E}_p\mu'(\xi).$$

(b) *If also $\mu'(\xi) > 0$ and $p > 1$, we have*

$$\lim_{n\to\infty} P^*_{n,p,\xi}\left(\xi + \frac{\pi\sqrt{1-\xi^2}z}{n}\right) = f^*_p(z), \tag{4.13}$$

uniformly for z in compact subsets of the plane, where f^*_p is the function attaining the inf in (4.6)*.*

Observe that for both the unit circle and $[-1,1]$, the only difference between the L_2 and L_p asymptotics is the constant $\mathcal{E}_p$. This suggests:

Problem 4.1. Evaluate $\mathcal{E}_p$, or at least estimate it, for $p \neq 2$.

Problem 4.2. Characterize the entire function f^*_p attaining the inf in (4.6).

5 Nikolskii Inequalities

Nikolskii inequalities compare the norms of polynomials in different L_p spaces. Accordingly, define

$$\|P\|_p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|P\left(e^{i\theta}\right)\right|^p d\theta\right)^{1/p}, \quad \text{if } p < \infty$$

and

$$\|P\|_\infty = \sup_{|z|=1}|P(z)|.$$

The simplest Nikolskii inequalities assert that given $q > p > 0$, there exists C depending on p, q, such that for $n \geq 1$ and polynomials P of degree $\leq n$,

$$\frac{\|P\|_p}{\|P\|_q} \geq Cn^{\frac{1}{q}-\frac{1}{p}}. \tag{5.1}$$

They are useful in studying convergence of orthonormal expansions and Lagrange interpolation, and in analyzing quadrature and discretization of integrals. A proof for trigonometric polynomials, which includes this case, appears in [13, Theorem 2.6, p. 102]. The converse sharp inequality, namely

$$\frac{\|P\|_p}{\|P\|_q} \leq 1$$

follows from Hölder's inequality. It is a longstanding problem to determine the sharp constant in (5.1). Accordingly define

$$\Lambda_{n,p,q} = \inf_{\deg(P)\leq n-1} \frac{\|P\|_p}{\|P\|_q}. \tag{5.2}$$

Our results from the previous section resolve the case $q=\infty$: as $n\to\infty$,

$$\lim_{n\to\infty} \Lambda_{n,p,\infty} n^{1/p} = \inf \left\{ \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_\infty(\mathbb{R})}} : f \in PW_p \right\}.$$

This suggests a generalization that might work for Nikolskii inequalities. Define

$$\mathcal{E}_{p,q} = \inf \left\{ \frac{\|f\|_{L_p(\mathbb{R})}}{\|f\|_{L_q(\mathbb{R})}} : f \in PW_p \right\}. \tag{5.3}$$

Using precisely the sort of scaling limits discussed in the introduction, and in the previous section, Eli Levin and I proved [32] that if $q>p>0$, then

$$\limsup_{n\to\infty} \Lambda_{n,p,q} n^{\frac{1}{p}-\frac{1}{q}} \leq \mathcal{E}_{p,q}. \tag{5.4}$$

Despite repeated attempts, we were unable to prove the limit. Accordingly we pose:

Problem 5.1. Prove

$$\lim_{n\to\infty} \Lambda_{n,p,q} n^{\frac{1}{p}-\frac{1}{q}} = \mathcal{E}_{p,q}. \tag{5.5}$$

Problem 5.2. Characterize, or describe, the entire functions attaining the inf in $\mathcal{E}_{p,q}$.

6 Marcinkiewicz-Zygmund Inequalities

The Plancherel-Polya inequalities [30, p. 152] assert that for $1<p<\infty$, and entire functions f of exponential type at most π,

$$A_p \sum_{j=-\infty}^{\infty} |f(j)|^p \leq \int_{-\infty}^{\infty} |f|^p \leq B_p \sum_{j=-\infty}^{\infty} |f(j)|^p, \tag{6.1}$$

provided the integral in the middle is finite. For $0<p\leq 1$, the left-hand inequality is still true, but the right-hand inequality requires additional restrictions [8],[47]. Here A_p and B_p are independent of f. The Marcinkiewicz-Zygmund inequalities assert [66, Vol. II, p. 30] that for $p>1, n\geq 1$, and polynomials P of degree $\leq n-1$,

$$\frac{A'_p}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p \le \int_0^1 \left|P\left(e^{2\pi it}\right)\right|^p dt \tag{6.2}$$
$$\le \frac{B'_p}{n}\sum_{j=1}^{n}\left|P\left(e^{2\pi ij/n}\right)\right|^p.$$

Here too, A'_p and B'_p are independent of n and P, and the left-hand inequality is also true for $0 < p \le 1$ [34]. We assume that A_p, B_p, A'_p, B'_p are the sharp constants, so that A_p and A'_p are as large as possible, while B_p and B'_p are as small as possible. These inequalities are useful in studying convergence of Fourier series, Lagrange interpolation, in number theory, and weighted approximation. See [14], [34], [35]. Of course if $p = 2$, then $A_2 = B_2 = A'_2 = B'_2 = 1$.

In [40], I proved that the sharp constants in (6.1) and (6.2) are the same:

Theorem 6.1. *For* $0 < p < \infty$,

$$A_p = A'_p$$

and for $1 < p < \infty$,

$$B_p = B'_p.$$

Moreover if $p \ne 2$, *then* $A_p < 1 < B_p$.

In [8, p. 101, Thm. 6.7.15], it is proven that $A_p \ge \frac{\pi}{4e^{\pi\pi/2}}$ (this was recorded incorrectly in [40]).

Problem 6.1. Evaluate or estimate A_p and B_p.

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