BULK UNIVERSALITY HOLDS IN MEASURE FOR COMPACTLY SUPPORTED MEASURES

DORON S. LUBINSKY

ABSTRACT. Let μ be a measure with compact support, with orthonormal polynomials $\{p_n\}$, and associated reproducing kernels $\{K_n\}$. We show that bulk universality holds in measure in $\{\xi : \mu'(\xi) > 0\}$. More precisely, given $\varepsilon, r > 0$, the linear Lebesgue measure of the set of ξ with $\mu'(\xi) > 0$ and for which

$$\sup_{|u|,|v|\leq r} \left| \frac{K_n\left(\xi + \frac{u}{K_n(\xi,\xi)}, \xi + \frac{v}{K_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi (u-v)}{\pi (u-v)} \right| \geq \varepsilon$$

approaches 0 as $n \to \infty$. There are no local or global regularity conditions on the measure μ.

1. INTRODUCTION

It was the physicist Eugene Wigner who in the 1950's first used eigenvalues of random matrices to model the interactions of neutrons for heavy nuclei. Random matrices have since become a major research area with connections to mathematical physics, probability theory, number theory, and orthogonal polynomials. One form of the mathematical setting can be described as follows: let $\mathcal{M}(n)$ denote the space of n by n Hermitian matrices $M = (m_{ij})_{1\leq i,j\leq n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$P^{(n)}(M) = cw(M)\, dM$$
$$= cw(M) \left(\prod_{j=1}^{n} dm_{jj} \right) \left(\prod_{j<k} d(\operatorname{Re} m_{jk})\, d(\operatorname{Im} m_{jk}) \right).$$

Here $w(M)$ is a function defined on $\mathcal{M}(n)$, and c is a normalizing constant. One important case is

$$w(M) = \exp(-2n \operatorname{tr} Q(M)),$$

involving the trace tr, for appropriate functions Q defined on $\mathcal{M}(n)$. In particular, the choice

$$Q(M) = M^2,$$

leads to the Gaussian unitary ensemble, apart from scaling, that was considered by Wigner. One may identify $P^{(n)}$ above with a probability density on the eigenvalues $x_1 \leq x_2 \leq \cdots \leq x_n$ of M,

$$P^{(n)}(x_1, x_2, \ldots, x_n) = c \left(\prod_{j=1}^{m} w(x_j) \right) \left(\prod_{i<j} (x_i - x_j)^2 \right).$$

See [7, p. 102 ff.]. Again, c is a normalizing constant.

It is at this stage that orthogonal polynomials arise [7], [35]. Let μ be a finite positive Borel measure with compact support and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,$$

$n = 0, 1, 2, \ldots$, satisfying the orthonormality conditions

$$\int p_j p_k d\mu = \delta_{jk}.$$

Throughout we use μ' to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

(1.1)
$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

and the normalized kernel is

(1.2)
$$\widetilde{K}_n(x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y).$$

When

$$\mu'(x) = e^{-2nQ(x)} dx,$$

there is the basic formula for the probability distribution $P^{(n)}$ [7, p. 112]:

$$P^{(n)}(x_1, x_2, \ldots, x_n) = \frac{1}{n!} \det \left(\tilde{K}_n(x_i, x_j) \right)_{1 \leq i, j \leq n}.$$

One may use this to compute a host of statistical quantities — for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One important quantity is the m-point correlation function for $\mathcal{M}(n)$ [7, p. 112]:

$$R_m(x_1, x_2, \ldots, x_m)$$
$$= \frac{n!}{(n-m)!} \int \cdots \int P^{(n)}(x_1, x_2, \ldots, x_n) \, dx_{m+1} dx_{m+2} \ldots dx_n$$
$$= \det \left(\tilde{K}_n(x_i, x_j) \right)_{1 \leq i, j \leq m}.$$

This can be used to describe the number of m-tuples of eigenvalues lying in a given set. For example, if $B \subset \mathbb{R}$ is measurable, then

$$\int_B \int_B R_2(x_1, x_2) \, dx_1 \, dx_2$$

is the expected number of pairs (λ_1, λ_2) of eigenvalues λ_1, λ_2, both lying in B.

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, and ξ in the interior of the support of μ, and real $a_1, a_2, \ldots, a_m$, we have

$$\lim_{n \to \infty} \frac{1}{\tilde{K}_n(\xi, \xi)^m} R_m \left(\xi + \frac{a_1}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi, \xi)}, \ldots, \xi + \frac{a_m}{\tilde{K}_n(\xi, \xi)} \right)$$
$$= \det \left(\frac{\sin \pi (a_i - a_j)}{\pi (a_i - a_j)} \right)_{1 \leq i,j \leq m}.$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin \pi (a_i - a_j)}{\pi (a_i - a_j)}$ as 1. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$(1.3) \qquad \lim_{n \to \infty} \frac{\tilde{K}_n \left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{b}{\tilde{K}_n(\xi, \xi)} \right)}{\tilde{K}_n(\xi, \xi)} = \frac{\sin \pi (a - b)}{\pi (a - b)}.$$

Thus, an assertion about the distribution of eigenvalues of random matrices reduces to a technical limit involving orthogonal polynomials. The adjective universal is justified: the limit on the right-hand side of (1.3) is independent of ξ, but more importantly is independent of the underlying measure.

Typically, the limit (1.3) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n, we can also establish its validity for complex a, b, that is,

$$(1.4) \qquad \lim_{n \to \infty} \frac{K_n \left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{b}{\tilde{K}_n(\xi, \xi)} \right)}{K_n(\xi, \xi)} = \frac{\sin \pi (a - b)}{\pi (a - b)}.$$

There are a variety of methods to establish (1.4). Many involve, essentially, substitution of asymptotics for p_{n-1} and p_n as $n \to \infty$ into the Christoffel-Darboux formula. Perhaps the deepest methods are the Riemann-Hilbert methods, which yield far more than universality. See [2], [3], [7], [8], [9], [12], [14], [18], [23], [24], [27], [28], [33], [34], [37], [42], [43], [50] for various methods and results.

There are several settings, other than that described above, for universality limits for random matrices. One involves $n \times n$ Hermitian matrices whose entries are independently distributed random variables, and whose off diagonal entries are identically distributed, see the recent survey [47] and [10], [11], [44], [48]. Others involve distributions on the space of $n \times n$ orthogonal or symplectic matrices — see [8]. It is noteworthy that the same sinc kernel arises in the orthogonal and symplectic cases, and in the case of independently distributed entries. This paper deals exclusively with the unitary case.

Inspired by the 60th birthday conference for Percy Deift, the author came up with a new comparison method to establish universality in the unitary case. The basic tool is the following inequality: assume that μ^* is a measure with $\mu \leq \mu^*$. Let K_n^* denote the nth reproducing kernel for μ^*. Then for

all real x, y,

$$\left|K_n\left(x, y\right) - K_n^*\left(x, y\right)\right| / K_n\left(x, x\right) \le \left(\frac{K_n\left(y, y\right)}{K_n\left(x, x\right)}\right)^{1/2}\left[1 - \frac{K_n^*\left(x, x\right)}{K_n\left(x, x\right)}\right]^{1/2}.$$

The crux is that the right-hand side involves only the diagonal $K_n\left(x, x\right)$, which admits an extremal property, and is hence easier to handle. This allows one to use universality limits for μ^* to establish them for μ. Here is a typical result: let μ be a measure supported on $(-1, 1)$, that is regular in the sense of Stahl, Ullmann and Totik [45], so that

$$\lim_{n \to \infty} \gamma_n^{1/n} = 2.$$

Let μ be absolutely continuous in a neighhborhood of some given $\xi \in (-1, 1)$ and assume that μ' is positive and continuous at ξ. Then [28] we established (1.4). Regularity is a weak global condition, that is satisfied if $\mu' > 0$ a.e. in the support of μ.

This result was soon extended to far more general settings by Findley, Simon and Totik [12], [42], [50]. In particular, when μ is a measure with compact support that is regular, and $\log\mu'$ is integrable in a subinterval (c, d) of the support, then Totik established that the universality (1.4) holds a.e. in (c, d). Totik used the method of polynomial pullbacks and other ingenious arguments to go first from one to finitely many intervals, and then used the latter to approximate general compact sets. In contrast, Simon used the theory of Jost functions to obtain equally impressive results.

The drawback of the comparison method is that it requires regularity of the measure μ. Although the latter is a weak global condition, it is nevertheless probably an unnecessary restriction. To circumvent this, the author developed a different method, based on classical complex analysis such as normal families, and the theory of entire functions of exponential type [27]. Let μ be a measure with compact support, and assume that μ' is absolutely continuous near ξ, while μ' is bounded above and below by positive constants in that neighborhood. Then the universality (1.4) is equivalent to *universality along the diagonal*, that is, for all real a,

$$(1.5) \qquad \lim_{n \to \infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{a}{\tilde{K}_n(\xi, \xi)}\right)}{K_n\left(\xi, \xi\right)} = 1.$$

Unfortunately, so far this equivalence has not led to an explicit extension of the results of Simon, Totik and Findley. Primarily, this is because there is no known method to estimate the ratio in the left-hand side of (1.5) that does not first give limits for the Christoffel functions, and all known methods for the latter require regularity of the measure. However, the method has been useful in other contexts [2], [22], [43].

In [30], it was shown that for compactly supported measures, which are not assumed to be regular, we can choose a sequence $\{\xi_n\}$ close to a given

ξ at which μ' is continuous, such that the universality (1.4) holds when ξ is replaced by ξ_n.

In this paper, we show that universality holds in linear Lebesgue measure, *meas*, without any local or global conditions, in the set

$$\{\mu' > 0\} = \{\xi : \mu'(\xi) > 0\}.$$

Our result is:

Theorem 1.1. *Let μ be a measure with compact support and with infinitely many points in the support. Let $\varepsilon > 0$ and $r > 0$. Then as $n \to \infty$,*

$$meas\left\{\xi \in \{\mu' > 0\} : \right.$$

$$(1.6) \qquad \sup_{|u|,|v| \leq r} \left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{\sin \pi(u-v)}{\pi(u-v)} \right| \geq \varepsilon \right\} \to 0.$$

Note that in the supremum, u, v are complex variables. Because convergence in measure implies convergence a.e. of subsequences, we deduce:

Corollary 1.2. *Assume the hypotheses of Theorem 1.1. Let $\mathcal{S}$ be an infinite sequence of positive integers. There is a subsequence $\mathcal{T}$ of $\mathcal{S}$ such that for a.e. $\xi \in \{\mu' > 0\}$,*

$$\lim_{n \to \infty, n \in \mathcal{T}} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi(u-v)}{\pi(u-v)},$$

uniformly for u, v in compact subsets of $\mathbb{C}$.

Remarks 1. (a) One cannot expect universality with the sinc kernel at points ξ where $\mu'(\xi) = 0$: indeed even, the normalized kernel $\tilde{K}_n(\xi,\xi)$ reduces to 0. When such a ξ lies at an endpoint of an interval in the support of μ, then effects associated with the edge of the spectrum occur, and these have been studied in great detail, leading to Airy and Bessel kernels [9], [14], [18], [19], [24], [25], [26], [34], [44], [52]. However, if for example, μ has support $[-2, 2]$, and μ has no absolutely continuous component in $[-2, 2] \setminus [-1, 1]$, Breuer, Simon and Last [6] constructed an example that effectively implies universality with a sinc kernel does not occur in $[-2, 2] \setminus [-1, 1]$.

(b) When μ' has a jump discontinuity at ξ, in the interior of the support, there is a different limiting kernel. This has been established with an explicit kernel by Folquie Moreno, Martinez-Finkelshtein and Sousa [13] using Riemann-Hilbert methods. The limiting kernels that arise in this and similar cases turn out to be reproducing kernels of de Brange spaces that equal classical Paley-Wiener spaces as sets [29].

(c) If $J \subset \{\mu' > 0\}$, and for a.e. ξ in J, there exists

$$\lim_{n \to \infty} \frac{K_n \left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)} \right)}{K_n (\xi, \xi)}$$

for some given u, v, then Corollary 1.2 shows that necessarily the limit is $\frac{\sin \pi(u-v)}{\pi(u-v)}$. Thus the sinc kernel is the only limiting kernel for a.e. ξ.

When the left-hand side of (1.4) is uniformly bounded in ξ, the convergence in measure yields convergence in L_p norms:

Corollary 1.3. *Let μ be a measure with compact support and with infinitely many points in the support. Let O be an open set in which μ is absolutely continuous, and such that for some $C_1 > 1$,*

$$(1.7) \qquad C_1^{-1} \leq \mu' \leq C_1 \ a.e. \ in \ O.$$

Let J be a compact subset of O. Let $p > 0$ and $r > 0$. Then
(1.8)

$$\lim_{n \to \infty} \int_J \sup_{|u|, |v| \leq r} \left| \frac{K_n \left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)} \right)}{K_n (\xi, \xi)} - \frac{\sin \pi (u - v)}{\pi (u - v)} \right|^p d\xi = 0.$$

Corollary 1.4. *Assume the hypotheses of Corollary 1.3. Let j, k be non-negative integers and*

$$(1.9) \qquad \tilde{K}_n^{(j,k)} (x, y) = \mu' (x)^{1/2} \mu' (y)^{1/2} \sum_{m=0}^{n-1} p_m^{(j)} (x) p_m^{(k)} (y).$$

Let

$$(1.10) \qquad \tau_{j,k} = \begin{cases} 0, & j + k \ odd \\ \frac{(-1)^{(j-k)/2}}{j+k+1}, & j + k \ even \end{cases}.$$

Then

$$(1.11) \qquad \lim_{n \to \infty} \int_J \left| \frac{\tilde{K}_n^{(j,k)} (\xi, \xi)}{\tilde{K}_n (\xi, \xi)^{j+k+1}} - \pi^{j+k} \tau_{j,k} \right|^p d\xi = 0.$$

This paper is structured as follows: in Section 2, we present the ideas of proof. In Section 3, we establish a lower bound for K_n, and in Section 4, an upper bound for K_n. In Section 5, we deduce normality of the normalized reproducing kernels. In Section 6, we prove an identity theorem for the sinc kernel. In Section 7, we estimate some tail integrals using maximal functions. Finally in Section 8, we prove Theorem 1.1 and its corollaries.

We close this section with some notation. Recall the Christoffel function

$$(1.12) \qquad \lambda_n (x) = \frac{1}{K_n (x, x)} = \inf_{\deg(P) \leq n-1} \int \frac{P^2 (t)}{P^2 (x)} d\mu (t).$$

For $r > 0$, we define the tail integrals

$$(1.13) \qquad \Psi_n(x, r) = \frac{\int_{|t-x| \geq \frac{r}{K_n(x,x)}} K_n(x,t)^2 \, d\mu(t)}{K_n(x,x)}$$

and

$$(1.14) \qquad \Phi_n(x, r) = \frac{\int_{|t-x| \geq \frac{r}{n}} K_n(x,t)^2 \, d\mu(t)}{K_n(x,x)}.$$

When $\mu'(x) = 0$, we set $\Psi_n(x, r) = \Phi_n(x, r) = 0$. Let

$$(1.15) \qquad A_n(x) = p_{n-1}^2(x) + p_n^2(x).$$

For a finite positive measure ν on the real line, define the maximal function

$$(1.16) \qquad \mathcal{M}[d\nu](x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} d\nu$$

and the maximal Hilbert transform

$$(1.17) \qquad H^*[d\nu](x) = \sup_{\varepsilon>0} \left| \int_{|t-x| \geq \varepsilon} \frac{1}{t-x} d\nu(t) \right|.$$

Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, t, and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. We shall use calligraphic symbols such as $\mathcal{E}_n, \mathcal{F}_n, \mathcal{G}_n, \mathcal{H}_n, \ldots$ to denote sets that typically have small measure.

For complex u, v, real ξ, and $r > 0$, we let

$$(1.18) \qquad f_n(u, v, \xi) = \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi, \xi)};$$

$$\Gamma_n(u, v, \xi, r) = \sup_{s \geq r \frac{\tilde{K}_n(\xi,\xi)}{n}} \left| f_n(u, v, \xi) \right.$$

$$(1.19) \qquad \left. - \int_{-s}^{s} f_n(u, t, \xi) f_n(v, t, \xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right|.$$

In the integral in the right-hand side, t is the variable of integration.

$$(1.20)$$
$$I_n(\xi, r) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Gamma_n(u, v, \xi, r)^{1/2} \left(f_n(u, u, \xi) f_n(v, v, \xi) \right)^{-1/4} du \, dv.$$

For $\sigma > 0$, PW_σ denotes the *Paley-Wiener space*, consisting of entire functions of exponential type at most σ that are square integrable on the real

axis, with the usual $L_2(\mathbb{R})$ norm. The reproducing kernel for PW_σ is $\frac{\sin \sigma(u-v)}{\pi(u-v)}$. Thus for $g \in PW_\sigma$, and all complex z [46, p. 95],

$$g(z) = \int_{-\infty}^{\infty} g(t) \frac{\sin \sigma(t-z)}{\pi(t-z)} dt.$$

The *Cartwright class* consists of all entire functions g of exponential type such that

$$(1.21) \qquad \int_{-\infty}^{\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty,$$

where $\log^+ x = \max\{0, \log x\}$. We shall assume that $\{\mu' > 0\}$ has positive measure, for otherwise there is nothing to prove.

2. IDEAS OF PROOF

Recall our notation

$$f_n(u,v,\xi) = \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}.$$

The basic idea is to show that for most ξ, and appropriate subsequences $\mathcal{T}$ of positive ntegers, $\{f_n\}_{n \in \mathcal{T}}$ is uniformly bounded in compact subsets of $\mathbb{C}^2$, and hence forms a normal family. If $f(\cdot, \cdot, \xi)$ is the limit of some subsequence, then f is entire in each variable. We use a uniqueness theorem for the sinc kernel to show that

$$(2.1) \qquad f(u,v,\xi) = \frac{\sin \pi(u-v)}{\pi(u-v)}.$$

Let us now flesh out some of the details. In Section 3, we use standard methods involving the extremal property in (1.12) to obtain upper bounds for Christoffel functions, and hence lower bounds for $K_n(\xi,\xi)$. With the help of Egorov's theorem, this yields the following: given $\varepsilon > 0$, there exists $C > 0$, and a set $\mathcal{E}$ of measure $\leq \varepsilon$ such that for any $T > 0$, large enough n, and all $\xi \in \{\mu' > 0\} \backslash \mathcal{E}$,

$$(2.2) \qquad \inf_{s \in [-T,T]} K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) \geq Cn.$$

The corresponding upper bound for K_n, established in Section 4, is more difficult, and involves an exceptional set that depends on n. From the identity

$$\int K_n(t,t) \, d\mu(t) = n,$$

it follows that for "most" t, and some large enough Λ,

$$\tilde{K}_n(t,t) = K_n(t,t) \, \mu'(t) \leq \Lambda n.$$

To extend this to an estimate in the complex plane, we need the Green's function $g_{\mathbb{C} \backslash E_n}$ for the complement of

$$E_n = \{t : K_n(t,t) \leq \Lambda n\}.$$

E_n consists of at most finitely many closed intervals (some of which may reduce to a point), so the equilibrium measure ν_{E_n} for E_n is absolutely continuous. Using elementary estimation, we show that for $\xi \in E_n$ and all complex u,

$$g_{\mathbb{C}\backslash E_n}(\xi + u) \leq 26\,|u|\,\mathcal{M}\,[d\nu_{E_n}](\xi) + |\mathrm{Re}\ u|\,|H^*\,[d\nu_{E_n}](\xi)|\,,$$

where $\mathcal{M}$ and $\mathcal{H}^*$ denote respectively the maximal function and maximal Hilbert transform. Using the classical weak type $(1,1)$ inequalities for both $\mathcal{M}$ and $\mathcal{H}^*$, we show that for "most" $\xi \in \mathrm{supp}\,[\mu]$, and all complex u,

$$g_{\mathbb{C}\backslash E_n}(\xi + u) \leq C\,|u|\,.$$

The maximum principle for subharmonic functions, applied essentially to $\log K_n(z, \bar{z}) - (2n - 2)\,g_{\mathbb{C}\backslash E_n}(z)$, and Cauchy-Schwarz, gives for most ξ and all complex u, vi

$$\left| K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right) \right| \leq C_1 n e^{C_2(|u|+|v|)}.$$

Here C_1, C_2 depend on ε, but are independent of u, v, n, ξ.

Together, this and (2.2) yield the desired uniform boundedness: let $\varepsilon > 0$. There exist $C_1, C_2 > 0$ and a set $\mathcal{G}_n$ having meas $\leq \varepsilon$, such that for $\xi \in \{\mu' > 0\} \backslash \mathcal{G}_n$ and all complex u, v,

$$|f_n(u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)}.$$

This is established in Theorem 5.1. We also show there that if $f(\cdot, \cdot, \xi)$ is a subsequential limit, it is of exponential type in each variable. Moreover, there exists $\sigma > 0$ such that for all real a, $f(a, \cdot, \xi)$ is of exponential type σ, and lies in Cartwright's class. Some assertions about the zeros of $f(0, \cdot, \xi)$ are also proved.

The most difficult step is to show that (2.1) holds. One natural approach is to show that f is a reproducing kernel for a Paley-Wiener space, and then to use the uniqueness of the reproducing kernel. Thus one starts with the reproducing kernel relation

$$(2.3) \qquad\qquad P(x) = \int P(t)\,K_n(x, t)\,d\mu(t)\,,$$

valid for all polynomials P of degree $\leq n - 1$. Using the substitution $t = \xi + \frac{s}{K_n(\xi,\xi)}$, where ξ is a given Lebesgue point of μ, and appropriate polynomials P, and letting $n \to \infty$, one hopes to show that

$$(2.4) \qquad\qquad g(a) = \int_{-\infty}^{\infty} g(t)\,f(t, a, \xi)\,dt,$$

for all g in a suitable Paley-Wiener space. This approach failed because, even if one can estimate the tail of the integral in (2.3), it seems impossible to establish (2.4) for all g of exponential type as large as that of $f(0, \cdot, \xi)$.

So instead we adopt an indirect approach, based on the uniqueness theorem, Theorem 6.1. The essential feature there, is that the relation

$$(2.5) \qquad f(a,b,\xi) = \int_{-\infty}^{\infty} f(a,t,\xi) \, f(b,t,\xi) \, dt,$$

for all complex a, b, together with $f(0,0,\xi) = 1$, and some other restrictions on zeros of $f(0,\cdot)$, yields (2.1).

To establish (2.5), we first use maximal functions, in Section 7, to estimate the tail integral

$$\Psi_n(x,r) = \frac{\int_{|t-x| \geq \frac{r}{K_n(x,x)}} K_n(x,t)^2 \, d\mu(t)}{K_n(x,x)},$$

obtaining for a.e. x,

$$\Psi_n(x,r) \leq \frac{8}{r} \left(\frac{\gamma_{n-1}}{\gamma_n} \mathcal{M}[A_n d\mu](x) \right)^2.$$

Since $\int A_n d\mu = 2$, we can use weak type $(1,1)$ inequalities for maximal functions to estimate $\Psi_n(x,r)$ for most x. It is really this estimate that is so crucial, and allows us to dispense with the global hypothesis that μ is regular. The latter condition has always been used to estimate tail integrals.

We next use the reproducing kernel relation

$$K_n(a,b) = \int K_n(a,t) \, K_n(b,t) \, d\mu(t),$$

and the substitutions $t = \xi + \frac{s}{K_n(\xi,\xi)}$, $a = \xi + \frac{u}{K_n(\xi,\xi)}$, $b = \xi + \frac{v}{K_n(\xi,\xi)}$ giving

$$f_n(u,v,\xi) = \int_{-r}^{r} f_n(u,t,\xi) \, f_n(v,t,\xi) \, \frac{d\mu\left(\xi + \frac{s}{K_n(\xi,\xi)}\right)}{\mu'(\xi)} + T,$$

where T is a tail term that can be estimated using Cauchy-Schwarz and our estimates for Ψ_n. If ξ is a Lebesgue point of μ, and we let $n \to \infty$ through a suitable subsequence, we obtain

$$f(u,v,\xi) = \int_{-r}^{r} f(u,t,\xi) \, f(v,t,\xi) \, dt + T,$$

where T is a small tail term. To actually show this, we essentially estimate, in Section 7,

$$\int_{-1}^{1} \int_{-1}^{1} \left| f_n(u,v,\xi) - \int_{-r}^{r} f_n(u,t,\xi) \, f_n(v,t,\xi) \, \frac{d\mu\left(\xi + \frac{s}{K_n(\xi,\xi)}\right)}{\mu'(\xi)} \right| du \, dv.$$

This leads to (2.5) in appropriate setting, and then we can prove Theorem 1.1, and deduce Corollaries 1.2, 1.3, and 1.4.

3. A Lower Bound for K_n

We prove:

Lemma 3.1. *Let μ be a measure with compact support, and with infinitely many points in its support. For each Lebesgue point ξ of μ, there exists $C = C(\xi)$ with the following property: let $T > 0$. Then there exists n_0 such that for $n \geq n_0$,*

$$(3.1) \qquad \inf_{s \in [-T,T]} K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) \geq Cn.$$

Moreover, this also holds at every point $\xi \notin \operatorname{supp}[\mu]$.

We emphasize that C is independent of T, although n_0 does depend on T.

Corollary 3.2. *Let $\varepsilon > 0$. There exists a set $\mathcal{E}$ of measure $\leq \varepsilon$, and $\delta > 0$ with the following property: let $T > 0$. Then there exists $n_0 = n_0(T)$ such that for $n \geq n_0$ and all $\xi \notin \mathcal{E}$,*

$$(3.2) \qquad \inf_{s \in [-T,T]} K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) \geq \delta n.$$

Remarks 2.   (a) The essential feature is that C and δ don't depend on T.

   (b) There are far more sophisticated forms of this estimate. Namely, Maté, Nevai, and Totik [32], Totik [49] essentially showed that

$$\liminf_{n \to \infty} \left[\frac{1}{n} \inf_{s \in [-T,T]} K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) \right] \geq \frac{\nu'(\xi)}{\mu'(\xi)},$$

where ν is the equilibrium measure of the support of μ. See also a result of Simon [43, Thm. 14.1]. However, we feel it is best to give a self contained proof of the form we need.

Proof of Lemma 3.1. After a translation and dilation, we may assume that $\operatorname{supp}[\mu] \subset \left[-\frac{1}{2}, \frac{1}{2}\right]$. We shall also assume that $\xi \in \left[-\frac{3}{4}, \frac{3}{4}\right]$ (note that $K_n(t,t)$ increases as t recedes from the smallest interval containing $\operatorname{supp}[\mu]$). We let K_n^T denote the kernel for the Chebyshev weight $\frac{1}{\sqrt{1-x^2}}$ on $[-1,1]$. It is well known [36, p. 78, p. 92] that

$$K_n^T(x,x) \geq C_1 n, \quad x \in [-1,1],$$

and

$$\left| K_n^T(x,y) \right| \leq \frac{C_2}{\frac{1}{n} + |x-y|}, \quad x, y \in [-1,1].$$

Let $r \geq 1$ and $|s| \leq r$. The parameter r can be identified with T in the statement of the lemma. Let $\zeta = \xi + \frac{s}{n}$. Then from the extremal property of Christoffel functions,

$$(3.3) \qquad \lambda_n(d\mu, \zeta) \leq \int \frac{K_n^T(\zeta,t)^2}{K_n^T(\zeta,\zeta)^2} d\mu(t) \leq C \int \left(1 + n|\zeta - t|\right)^{-2} d\mu(t).$$

Now for $|\xi - t| \geq \frac{2r}{n}$, we have

$$|\zeta - t| \geq |\xi - t| - \frac{r}{n} \geq \frac{1}{2} |\xi - t|,$$

so

$$\int_{|\xi-t|\geq\frac{2r}{n}} (1 + n |\zeta - t|)^{-2} d\mu(t) \leq 4 \int_{|\xi-t|\geq\frac{2r}{n}} (n |\xi - t|)^{-2} d\mu(t)$$

$$\leq 4 \sum_{k=0}^{\infty} \int_{2^{k+1}\frac{2r}{n}\geq|\xi-t|\geq 2^k\frac{2r}{n}} \left(2^k \cdot 2r\right)^{-2} d\mu(t)$$

$$\leq \frac{8}{n} \sum_{k=0}^{\infty} \left(2^k \cdot r\right)^{-1} \mathcal{M}[d\mu](\xi)$$

$$\text{(3.4)} \hspace{3cm} = \frac{16}{nr} \mathcal{M}[d\mu](\xi).$$

Note that as μ has compact support, and either $\xi \notin \mathrm{supp}\,[\mu]$, or ξ is a Lebesgue point, $\mathcal{M}[d\mu](\xi)$ is finite. In the second last line, we used the definition of the maximal function to obtain

$$\frac{1}{2\left(2^{k+1}\frac{2r}{n}\right)} \int_{2^{k+1}\frac{2r}{n}\geq|\xi-t|} d\mu(t) \leq \mathcal{M}[d\mu](\xi).$$

Next, if μ_s denotes the singular part of μ,

$$\int_{|\xi-t|<\frac{2r}{n}} (1 + n |\zeta - t|)^{-2} d\mu(t) \leq \mu'(\xi) \int_{|\xi-t|<\frac{2r}{n}} (1 + n |\zeta - t|)^{-2} dt$$

$$+ \int_{|\xi-t|<\frac{2r}{n}} (1 + n |\zeta - t|)^{-2} |\mu'(t) - \mu'(\xi)| dt + \int_{|\xi-t|<\frac{2r}{n}} d\mu_s(t)$$

$$\leq \mu'(\xi) \int_{-\infty}^{\infty} (1 + n |\zeta - t|)^{-2} dt$$

$$+ \frac{4r}{n} \left(\frac{1}{2}\frac{n}{2r}\right) \left[\int_{|\xi-t|<\frac{2r}{n}} |\mu'(t) - \mu'(\xi)| dt + \int_{|\xi-t|<\frac{2r}{n}} d\mu_s(t)\right].$$

Combining this, (3.3), and (3.4), and using the fact that ξ is a Lebesgue point of μ, or that $\xi \notin \mathrm{supp}\,[\mu]$, so that

$$\lim_{h\to 0+} \frac{1}{2h} \left[\int_{|\xi-t|<h} |\mu'(t) - \mu'(\xi)| dt + \int_{|\xi-t|<h} d\mu_s(t)\right] = 0,$$

we obtain

$$\limsup_{n\to\infty} \left(\sup_{|s|\leq r} n\lambda_n\left(\xi + \frac{s}{n}\right)\right) \leq \frac{16}{r}\mathcal{M}[d\mu](\xi) + \pi\mu'(\xi).$$

As $r \geq 1$ and $\mathcal{M}[d\mu](\xi)$ is finite, while $K_n\left(\xi + \frac{s}{n}, \xi + \frac{s}{n}\right) = \lambda_n^{-1}\left(\xi + \frac{s}{n}\right)$, we have the result. $\hspace{1cm}\square$

Proof of Corollary 3.2. We use Egorov's Theorem. As a.e. $\xi \in \mathrm{supp}\,[\mu]$ is a Lebesgue point of μ, it follows from Lemma 3.1, that there is a set $\mathcal{E}_1$ of measure $\leq \varepsilon/2$ and a fixed $\delta > 0$, such that for $\xi \notin \mathcal{E}_1$ and all $T > 0$,

$$\liminf_{n \to \infty} \left[\frac{1}{n} \inf_{s \in [-T,T]} K_n \left(\xi + \frac{s}{n}, \xi + \frac{s}{n} \right) \right] \geq 2\delta.$$

Fix $k \geq 2$. By Egorov's theorem, applied to the functions

$$g_n(\xi) = \min \left\{ 2\delta, \frac{1}{n} \inf_{s \in [-2^k, 2^k]} K_n \left(\xi + \frac{s}{n}, \xi + \frac{s}{n} \right) \right\},$$

we obtain a set $\mathcal{E}_k$ of measure $\leq \varepsilon/2^k$ such that for $n \geq n_k = n_k(k, \varepsilon)$, and all $\xi \notin \mathcal{E}_1 \cup \mathcal{E}_k$,

$$\frac{1}{n} \inf_{s \in [-2^k, 2^k]} K_n \left(\xi + \frac{s}{n}, \xi + \frac{s}{n} \right) \geq \delta.$$

Let

$$\mathcal{E} = \bigcup_{k=1}^{\infty} \mathcal{E}_k.$$

This is a set of measure $\leq \varepsilon$. Given $T > 0$, choose $k \geq 2$ such that $T \leq 2^k$. Then for $n \geq n_k$, and $\xi \notin \mathcal{E}$, we have (3.2). $\qquad\square$

4. Green's Function Estimates and Upper Bounds for K_n

In this section, we establish an elementary estimate for Green's functions $g_{\mathbb{C} \backslash E}(z)$, namely that they grow linearly in $|z|$ most of the time. Recall that if E is a compact set in the plane, with positive logarithmic capacity, it has an equilibrium measure ν_E. This is a probability measure such that the equilibrium potential

$$V^{\nu_E}(z) = \int \log \frac{1}{|t-z|} d\nu_E(t)$$

satisfies

$$V^{\nu}(z) = -\log\, cap(E)$$

q.e. on E, that is, except possibly in a set of logarithmic capacity zero. Here $cap(E)$ denotes the logarithmic capacity of E. Moreover, this equation holds precisely at every point of E that is regular for the Dirichlet problem for $\mathbb{C} \backslash E$ — the so-called *regular points*. The Green's function for $\mathbb{C} \backslash E$ is

$$g_{\mathbb{C} \backslash E}(z) = -\log\, cap(E) - V^{\nu_E}(z).$$

It is a function harmonic and non-negative in $\mathbb{C} \backslash E$, that behaves like $\log |z| + O(1)$ as $z \to \infty$, and has boundary value 0 q.e. on E. For further orientation, see [20], [38]. The result we need involves the maximal function $\mathcal{M}[d\nu_E](\xi)$, and maximal Hilbert transform $\mathcal{H}^*[d\nu_E](\xi)$, of ν_E. Recall that the latter was defined at (1.17).

Lemma 4.1. *Let $E \subset \mathbb{R}$ be a compact set with positive logarithmic capacity, and equilibrium measure ν_E. Then for q.e. ξ in the support of ν_E, and all complex u,*

$$(4.1) \qquad g_{\mathbb{C} \backslash E}\left(\xi + u\right) \leq 26\left|u\right| \mathcal{M}\left[d\nu_E\right]\left(\xi\right) + \left|\operatorname{Re} u\right| \mathcal{H}^*\left[d\nu_E\right]\left(\xi\right).$$

In particular this holds at regular points of E.

Corollary 4.2. *Let $n \geq 1$, and P be a function defined on the complex plane, such that $\log|P|$ is subharmonic there, and as $|z| \to \infty$,*

$$(4.2) \qquad \log\left|P\left(z\right)\right| = n\log\left|z\right| + C + o\left(1\right).$$

Let $A > 0$, $\varepsilon > 0$, and

$$E = \left\{x \in \mathbb{R} : \left|P\left(x\right)\right| \leq A\right\}.$$

Assume that the equilibrium measure ν_E of E is absolutely continuous. There exists a set $\mathcal{E}_\varepsilon$ of linear Lebesgue measure at most ε, such that for $\xi \in E \backslash \mathcal{E}_\varepsilon$ and all complex u,

$$(4.3) \qquad \left|P\left(\xi + u\right)\right| \leq A e^{nC_1|u|/\varepsilon}.$$

Here C_1 is a constant that is independent of n, P, ε, A. In particular, this estimate holds if P is a polynomial of degree n.

Remark 3. We formulated the corollary in a more general form than the usual Bernstein-Walsh inequality for polynomials, since we shall need it for $K_n\left(z, \bar{z}\right)$, which is not a polynomial.

Corollary 4.3. *Let μ be a measure with compact support, and with infinitely many points in its support. Let $\varepsilon > 0$. There exists a set $\mathcal{F}_n$ of measure at most ε such that for $\xi \in \left\{\mu' > 0\right\} \backslash \mathcal{F}_n$, and all complex u, v,*

$$(4.4) \qquad \left|K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)\right| \leq C_1 n e^{C_2(|u|+|v|)/\varepsilon}.$$

Both C_1 and C_2 are independent of n, ξ, u, v, while C_2 is also independent of ε.

Proof of Lemma 4.1. Write $u = x + iy$. Now for regular $\xi \in E$, we have $g_{\mathbb{C} \backslash E}\left(\xi\right) = 0$ [38, p. 111], so for such ξ,

$$\begin{aligned} g_{\mathbb{C} \backslash E}\left(\xi + u\right) &= g_{\mathbb{C} \backslash E}\left(\xi + u\right) - g_{\mathbb{C} \backslash E}\left(\xi\right) \\ &= V^{\nu_E}\left(\xi\right) - V^{\nu_E}\left(\xi + u\right) \\ &= \frac{1}{2} \int \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{\left(\xi - t\right)^2}\right] d\nu_E\left(t\right). \end{aligned}$$

Let $\mathcal{S}_1$ denote the set of t for which

$$\left|\frac{2x}{\xi - t}\right| \leq \frac{2\left|u\right|^2}{\left(\xi - t\right)^2} \iff \left|\xi - t\right| \leq \frac{|u|^2}{|x|}.$$

Let $\mathcal{S}_2$ denote the complementary range. In the case where $x = 0$, of course $\mathcal{S}_2$ is empty. Let us assume $x \neq 0$, the case $x = 0$ is easier. We see that

$$\int_{\mathcal{S}_1} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\nu_E(t)$$

$$\leq \int_{|\xi - t| \leq \frac{|u|^2}{|x|}} \log\left[1 + \frac{3|u|^2}{(\xi - t)^2}\right] d\nu_E(t)$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{-k-1}\frac{|u|^2}{|x|} \leq |\xi - t| \leq 2^{-k}\frac{|u|^2}{|x|}} \log\left[1 + \frac{12x^2}{|u|^2}2^{2k}\right] d\nu_E(t)$$

$$\leq \sum_{k=0}^{\infty} \log\left[1 + \frac{12x^2}{|u|^2}2^{2k}\right] 2^{-k+1}\frac{|u|^2}{|x|}\mathcal{M}[d\nu_E](\xi)$$

$$\leq \frac{|u|^2}{|x|}\mathcal{M}[d\nu_E](\xi)\, 4 \int_0^{\infty} \log\left[1 + \frac{12x^2}{|u|^2 t^2}\right] dt$$

$$(4.5) \qquad = |u|\,\mathcal{M}[d\nu_E](\xi)\, 8\sqrt{3}\pi,$$

cf. [17, p. 525, no. 4.222.1]. Next, in $\mathcal{S}_2$, we have $|\xi - t| \geq |u|^2/|x|$, so using the inequality $\log(1 + t) \leq t$, $t \geq -1$, we obtain

$$\int_{\mathcal{S}_2} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\nu_E(t)$$

$$\leq \int_{|\xi - t| \geq |u|^2/|x|} \left[\frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\nu_E(t)$$

$$(4.6) \qquad \leq 2|x|\,|\mathcal{H}^*[d\nu_E](\xi)| + |u|^2 \int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2} d\nu_E(t).$$

Here,

$$\int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2} d\nu_E(t)$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{k+1}|u|^2/|x| \geq |\xi - t| \geq 2^k|u|^2/|x|} \frac{1}{\left(2^k|u|^2/|x|\right)^2} d\nu_E(t)$$

$$\leq \sum_{k=0}^{\infty} \frac{x^2}{|u|^4}2^{-2k} \cdot 2^{k+2}\frac{|u|^2}{|x|}\mathcal{M}[d\nu_E](\xi)$$

$$= \frac{|x|}{|u|^2}8\mathcal{M}[d\nu_E](\xi).$$

Combining this with (4.5) and (4.6) gives

$$g_{\mathbb{C}\backslash E}(\xi + u) \leq 4\sqrt{3}\pi\,|u|\,\mathcal{M}[d\nu_E](\xi) + |x|\,|\mathcal{H}^*[d\nu_E](\xi)| + |x|\,4\mathcal{M}[d\nu_E](\xi).$$

Estimating the constants gives the result. $\qquad\square$

Proof of Corollary 4.2. Consider

$$h(z) = \log |P(z)| - n g_{\mathbb{C}\setminus E}(z) - \log A.$$

This function is subharmonic in $\mathbb{C}\setminus E$, and has a finite limit at ∞, because of our hypothesis (4.2). Moreover, h has boundary value at most 0 q.e. on E. By the extended maximum principle for subharmonic functions [38, p. 70],

$$h \leq 0 \text{ in } \mathbb{C},$$

so for $z \in \mathbb{C}\setminus E$,

$$|P(z)| \leq A e^{n g_{\mathbb{C}\setminus E}(z)}.$$

Then from Lemma 4.1, for regular $\xi \in E$, and for all complex u,

$$|P(\xi + u)| \leq A e^{26n|u|(\mathcal{M}[d\nu_E](\xi) + \mathcal{H}^*[d\nu_E](\xi))}.$$

Next, we use the fact that both the maximal function and the maximal Hilbert transform are weak type (1,1). That is, for $\lambda > 0$, [39, p. 137, Thm. 7.4]

$$meas\{\xi : \mathcal{M}[\nu_E](\xi) > \lambda\} \leq \frac{3}{\lambda} \int d\nu_E = \frac{3}{\lambda};$$

and provided ν_E is absolutely continuous, [4, p. 130], [16, p. 128 ff.]

$$meas\{\xi : \mathcal{H}^*[d\nu_E](\xi) > \lambda\} \leq \frac{C_0}{\lambda} \int d\nu_E = \frac{C_0}{\lambda}.$$

Here C_0 is independent of ν_E and λ. Choosing $\lambda = \frac{2}{\varepsilon} \max\{3, C_0\}$, we obtain a set $\mathcal{E}_\varepsilon$ of measure $\leq \varepsilon$ such that for $\xi \in supp[\mu] \setminus \mathcal{E}_\varepsilon$, and all complex u,

$$|P(\xi + u)| \leq A e^{104n|u|\max\{3,C_0\}/\varepsilon}.$$

Note that a set of capacity zero has measure zero, so the set of non-regular ξ is of measure 0. $\square$

Proof of Corollary 4.3. Let $\Lambda > 0$, and consider

$$E_n = E_n(\Lambda) = \{t : K_n(t,t) \leq \Lambda n\}.$$

This is a set that consists of at most finitely many compact intervals, some of which may reduce to a single point. It is well known that the equilibrium measure ν_{E_n} is absolutely continuous, and its density is even analytic in the interior of each interval in E_n [40, p. 412]. In particular, each interior point is a regular point. Next, we note that

$$\log K_n(z, \bar{z}) = \log \left(\sum_{k=0}^{n-1} |p_k(z)|^2 \right)$$

is subharmonic in the plane. Indeed, if we fix z, and $r > 0$, we can choose unimodular constants $\{\alpha_k\}$ such that

$$\log K_n(z, \bar{z}) = \log \left| \sum_{k=0}^{n-1} \alpha_k p_k(z)^2 \right|$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \sum_{k=0}^{n-1} \alpha_k p_k \left(z + re^{i\theta} \right)^2 \right| d\theta$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left(\sum_{k=0}^{n-1} \left| p_k \left(z + re^{i\theta} \right) \right|^2 \right) d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log K_n \left(z + re^{i\theta}, \overline{z + re^{i\theta}} \right) d\theta.$$

In the second line, we used subharmonicity of logs of absolute values of analytic functions. Moreover, we see that as $|z| \to \infty$,

$$\log K_n \left(z, \bar{z} \right) = (2n - 2) \log |z| + 2 \log \gamma_{n-1} + o(1).$$

Then Corollary 4.2 shows that there is a set $\mathcal{E}_{\varepsilon/2}$ of measure at most $\frac{\varepsilon}{2}$ such that for $\xi \in E_n \backslash \mathcal{E}_{\varepsilon/2}$, and all complex u,

$$(4.7) \qquad\qquad K_n \left(\xi + u, \overline{\xi + u} \right) \leq \Lambda n e^{nC_1 |u|/\varepsilon}.$$

Here C_1 is independent of $n, u, \varepsilon, \Lambda, \xi$. Next, we claim that if Λ is large enough,

$$(4.8) \qquad\qquad meas \left(\{\mu' > 0\} \backslash E_n \right) < \frac{\varepsilon}{2},$$

with a threshold on Λ that is independent of n. Indeed,

$$\int_{\{\mu' > 0\}} K_n(t, t) \mu'(t) \, dt \leq n,$$

so

$$meas \left\{ t : K_n(t, t) \mu'(t) \geq \sqrt{\Lambda} n \right\} \leq \frac{1}{\sqrt{\Lambda}}.$$

Moreover, for sufficently large Λ, say for $\Lambda \geq \Lambda_0$,

$$meas \left\{ t \in \{\mu' > 0\} : \mu'(t) \leq \frac{1}{\sqrt{\Lambda}} \right\} < \frac{\varepsilon}{4}.$$

Note that the threshold Λ_0 depends on μ', but not on n. Then choosing $\Lambda = \max \left\{ \Lambda_0, \left(\frac{4}{\varepsilon} \right)^2 \right\}$, we have

$$meas \left\{ t \in \{\mu' > 0\} : K_n(t, t) > \Lambda n \right\} \leq \frac{\varepsilon}{2}.$$

With such a choice of Λ, we obtain (4.8). Combining this with (4.7), gives for $\xi \in \{\mu' > 0\} \backslash \mathcal{F}_n$, where $meas(\mathcal{F}_n) < \varepsilon$, and for all complex u,

$$K_n \left(\xi + u, \xi + \bar{u} \right) \leq \Lambda n e^{nC_1 |u|/\varepsilon} \leq C_2 n e^{nC_1 |u|/\varepsilon},$$

where C_1 and C_2 are independent of ξ, n, u. Moreover, C_1 is independent of ε. Now replace u by $\frac{u}{n}$ and use Cauchy-Schwarz to obtain (4.4). $\qquad \square$

5. Normal Family Estimates

Recall the definition (1.18) of f_n. In this section, we prove:

Theorem 5.1. *Let μ be a measure with compact support, and with infinitely many points in its support. Let $\varepsilon > 0$. There exist $C_1, C_2 > 0$ and for $n \geq 1$, sets $\mathcal{G}_n$ of measure $\leq \varepsilon$ with the following properties:*

(a) *For $\xi \in \{\mu' > 0\} \backslash \mathcal{G}_n$, and for all complex u, v,*

$$(5.1) \qquad\qquad |f_n(u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)}.$$

(b) *Assume that $\xi \in \{\mu' > 0\} \backslash \mathcal{G}_n$ for n belonging to some sequence $\mathcal{T}$ of positive integers. Let $f(\cdot, \cdot, \xi)$ be the limit of some subsequence of $\{f_n\}_{n \in \mathcal{T}}$. Then*

   (i) *$f(\cdot, \cdot, \xi)$ is entire in each variable, and with C_1, C_2 as in (a), for all complex u, v,*

$$(5.2) \qquad\qquad |f(u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)}.$$

   (ii) *For each complex u,*

$$(5.3) \qquad\qquad \int_{-\infty}^{\infty} |f(u, s, \xi)|^2 \, ds \leq f(u, \bar{u}, \xi) < \infty.$$

   (iii) *$f(0, \cdot, \xi)$ has infinitely many real simple zeros $\{\rho_j\}_{j \neq 0}$ where*

$$\cdots < \rho_{-2} < \rho_{-1} < 0 < \rho_1 < \rho_2 < \cdots$$

   and no other zeros. Let $\rho_0 = 0$. For $j \neq 0$, $f(\rho_j, \cdot, \xi)$ has zeros $\{\rho_k\}_{k \in \mathbb{Z} \backslash \{j\}}$ and no other zeros.

   (iv) *There exists $C_0 > 0$ such that for all real t,*

$$(5.4) \qquad\qquad f(t, t, \xi) \geq C_0,$$

   and $f(0, 0) = 1$.

   (v) *There exists $\sigma > 0$ such that for each real a, $f(a, \cdot, \xi)$ is an entire function of exponential type σ.*

Remark 4. We emphasize that C_0, C_1, and C_2 are independent of n, ξ, and the particular subsequential limit f. However, they do depend on ε.

Proof of Theorem 5.1(a). From Corollaries 3.2 and 4.3, we deduce that there is a set $\mathcal{G}_n$ of measure $\leq \varepsilon$, such that for $\xi \in \{\mu' > 0\} \backslash \mathcal{G}_n$, and all complex u, v, we have

$$\left| \frac{K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)}{K_n(\xi, \xi)} \right| \leq C_1 e^{C_2(|u|+|v|)}.$$

Here C_1, C_2 depend on ε but not on n, u, v, ξ. Since also

$$\frac{n}{\tilde{K}_n(\xi, \xi)} \leq C$$

in $\{\mu' > 0\}$, except on a set of measure $\leq \varepsilon$, and some C (from Corollary 3.2, and the fact that for some small δ, $\mu' > \delta$ outside a set of small measure), we obtain

$$\left| \frac{K_n \left(\xi + \frac{u}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{v}{\widetilde{K}_n(\xi,\xi)} \right)}{K_n (\xi, \xi)} \right| \leq C_3 e^{C_4(|u|+|v|)}.$$

$\square$

Note that as $\mathcal{G}_n$ includes the exceptional set in Corollary 4.3,

(5.5) $$K_n (\xi, \xi) \leq C_3 n \text{ for } \xi \in \{\mu' > 0\} \backslash \mathcal{G}_n.$$

Here C_3 is independent of n, ξ.

Proof of Theorem 5.1(b). (i) From (i), $\{f_n\}_{n \in \mathcal{T}}$ is a normal family in compact subsets of $\mathbb{C}^2$. If f denotes some subsequential limit, then (a) gives the bound

$$|f (u, v, \xi)| \leq C_1 e^{C_2(|u|+|v|)},$$

for all complex u, v.

(ii) Next, let $u \in \mathbb{C}$, and $U = \xi + \frac{u}{\widetilde{K}_n(\xi,\xi)}$, and use the reproducing kernel relation

$$1 = \int \frac{\left| K_n^2 (U, t) \right|}{K_n (U, \bar{U})} d\mu (t) .$$

We drop most of the integral and make the substitution $t = \xi + \frac{s}{\widetilde{K}_n(\xi,\xi)}$:

$$1 \geq \int_{\xi - \frac{r}{\widetilde{K}_n(\xi,\xi)}}^{\xi + \frac{r}{\widetilde{K}_n(\xi,\xi)}} \frac{\left| K_n^2 (U, t) \right|}{K_n (U, \bar{U})} d\mu (t)$$

$$= \int_{-r}^{r} \frac{|f_n (u, s, \xi)|^2}{f_n (u, \bar{u}, \xi)} \frac{d\mu \left(\xi + \frac{s}{\widetilde{K}_n(\xi,\xi)} \right)}{\mu' (\xi)}.$$

As we may assume that ξ is a Lebesgue point of μ (for a.e. ξ is), and we may assume that as $n \to \infty$ through $\mathcal{T}$, $f_n \to f$ locally uniformly, we obtain

$$1 \geq \int_{-r}^{r} \frac{|f (u, s, \xi)|^2}{f (u, \bar{u}, \xi)} ds.$$

Now let $r \to \infty$.

(iii) Now for each fixed real ξ, with $(p_{n-1} p_n) (\xi) \neq 0$, the function

$$L_n (t, \xi) = (t - \xi) K_n (t, \xi)$$

$$= \frac{\gamma_{n-1}}{\gamma_n} (p_n (t) p_{n-1} (\xi) - p_{n-1} (t) p_n (\xi))$$

has simple zeros that interlace those of p_n. See, for example [15, p. 19 ff.]. More precisely $L_n (\cdot, \xi)$ has a simple zero in $(x_{jn}, x_{j-1,n})$ for $2 \leq j \leq n$, and one zero outside (x_{nn}, x_{1n}). When $(p_{n-1} p_n) (\xi) = 0$, then L_n is a multiple of p_{n-1} or p_n. It follows that in all cases $L_n (\cdot, \xi)$ has a zero in $[x_{jn}, x_{j-1,n})$, $2 \leq j \leq n$, and at most one other zero, outside $[x_{nn}, x_{1n})$. Let $\{t_{jn}\}_{j \neq 0} =$

$\{t_{jn}(\xi)\}_{j\neq 0}$ denote these zeros of $K_n(\xi, t)$, and $t_{0n}(\xi) = \xi$. We order the zeros as

$$\cdots < t_{-1n}(\xi) < t_{0n}(\xi) < t_{1n}(\xi) < t_{2n}(\xi) < \cdots$$

Then $f_n(0, \cdot, \xi)$ has simple zeros

$$\rho_{jn} = \tilde{K}_n(\xi, \xi)(t_{jn} - \xi), \quad j \neq 0,$$

and no other zeros. Let $\rho_{0n} = 0$. Note that

$$\cdots < \rho_{-1,n} < \rho_{0n} = 0 < \rho_{1n} < \rho_{2n} < \cdots$$

Now as $n \to \infty$ through $\mathcal{T}$, we have

$$\lim_{n\to\infty, n\in\mathcal{T}} f_n(0, u, \xi) = f(0, u, \xi)$$

uniformly for u in compact subsets of the plane. Moreover, $f(0, 0, \xi) = 1$, so f is not identically 0. By Hurwitz' theorem, each zero of $f(0, \cdot, \xi)$ is a limit of zeros of $f_n(0, \cdot, \xi)$.

Next, (i) shows that $f(0, \cdot, \xi)$ is of exponential type at most type C_2, while from (ii), $\int_{-\infty}^{\infty} f(0, s, \xi)^2 \, ds < \infty$. A well known bound [21, p. 149] asserts that

$$(5.6) \qquad |f(0, x + iy, \xi)|^2 \leq \frac{2}{\pi} e^{2C_2(|y|+1)} \int_{-\infty}^{\infty} f(0, s, \xi)^2 \, ds$$

for all complex $x + iy$. In particular, then $f(0, \cdot, \xi)$ is bounded on the real axis and so satisfies (1.21) and lies in the Cartwright class. It is also real valued on the real axis. Then [21, p. 130], if $\{\rho_j\}$ are the zeros of $f(0, \cdot, \xi)$,

$$f(0, z, \xi) = \lim_{R\to\infty} \prod_{|\rho_j|<R} \left(1 - \frac{z}{\rho_j}\right).$$

It follows that f has infinitely many zeros $\{\rho_j\}$, and these are then necessarily the limits of the zeros $\{\rho_{j,n}\}$ of $f_n(0, \cdot, \xi)$. Since each $\rho_{j,n}$ is a simple zero of f_n, ρ_j is a simple zero of $f(0, \cdot, \xi)$ unless $\rho_j = \rho_{j-1}$ or ρ_{j+1}.

Next, we note that for $j \neq k$,

$$K_n(t_{jn}, t_{kn}) = 0.$$

Indeed, it follows from the Christoffel-Darboux formula that both t_{jn} and t_{kn} are roots of the equation

$$p_n(t) p_{n-1}(\xi) - p_{n-1}(t) p_n(\xi) = 0.$$

Then for $j \neq k$,

$$f_n(\rho_{jn}, \rho_{kn}, \xi) = 0$$

and because of the locally uniform convergence,

$$f(\rho_j, \rho_k, \xi) = 0.$$

Moreover, because of Hurwitz' theorem, $f(\rho_j, \cdot, \xi)$ has no other zeros. We still have to show the simplicity of the zeros.

(iv) We know from Lemma 3.1, that there exists $C > 0$, such that given $T > 0$, there exists $n_0 = n_0(T)$ such that for $n \geq n_0$,

$$\inf_{s \in [-T,T]} K_n \left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{s}{\tilde{K}_n(\xi,\xi)} \right) \geq Cn,$$

where C is independent of T. Note that $\frac{s}{\tilde{K}_n(\xi,\xi)} \leq C\frac{s}{n}$ for ξ outside our set $\mathcal{G}_n$. Also, we have the upper bound (5.5) for $K_n(\xi,\xi)$. Thus

$$\inf_{s \in [-T,T]} f_n(s,s,\xi) \geq C.$$

As C is independent of T, we obtain

$$\inf_{t \in \mathbb{R}} f(t,t,\xi) \geq C.$$

This also shows that $f(\rho_j, \rho_j, \xi) > 0$, so necessarily $\rho_{j\pm 1} \neq \rho_j$, and all zeros of $f(0,\cdot,\xi)$ are simple.

(v) As above, the zeros of $L_n(t,\xi) = (t - \xi) K_n(t,\xi)$ interlace those of p_n. Let $m > k$. It follows that whatever is ξ, the number j of zeros of $K_n(t,\xi)$ in $[x_{mn}, x_{kn}]$ satisfies

$$|j - (m - k)| \leq 1.$$

Now let $N(g,r)$ denote the number of zeros of a function g in $[-r, r]$. It follows from this last estimate that for any real a, b, and $r > 0$, and $n \geq 1$, we have

$$|N(f_n(a,\cdot,\xi),r) - N(f_n(b,\cdot,\xi),r)| \leq 4.$$

Letting $n \to \infty$ through the appropriate subsequence of integers gives for each $r > 0$,

$$(5.7) \qquad |N(f(a,\cdot,\xi),r) - N(f(b,\cdot,\xi),r)| \leq 4.$$

Since $f(a,\cdot,\xi)$ has only real zeros, and lies in Cartwright's class, as follows from (i) and (ii), so,

$$\lim_{r \to \infty} \frac{N(f(a,\cdot),r)}{2\pi r} = \sigma_a,$$

where σ_a is the exponential type of $f(a,\cdot,\xi)$, see [21, p. 127, eqn. (5)]. It follows from (5.7) that $\sigma_a = \sigma$ is independent of a. We must still show that $\sigma > 0$. To do this, we use the bound (5.6) with $C_2 = \sigma$:

$$|f(0, x + iy, \xi)|^2 \leq \frac{2}{\pi} e^{2\sigma(|y|+1)} \int_{-\infty}^{\infty} |f(0,t,\xi)|^2 \, dt.$$

If $\sigma = 0$, this implies that $f(0,\cdot,\xi)$ is bounded and hence constant, contradicting its square integrability over the real line. $\qquad \square$

6. An Identity Theorem for the Sinc Kernel

In this section, we prove a uniqueness theorem for the sinc kernel:

Theorem 6.1. *Let $\sigma > 0$ and $F : \mathbb{C}^2 \to \mathbb{C}$ be an entire function in each variable with the following properties:*

(i) *For each real a, $F(a, \cdot)$ is an entire function of exponential type σ, that is real on the real axis, with*

$$(6.1) \qquad \int_{-\infty}^{\infty} |F(a,s)|^2 \, ds < \infty.$$

(ii) *Let $\rho_0 = 0$ and $F(0, \cdot)$ have distinct simple zeros $\{\rho_j\}_{j \in \mathbb{Z} \setminus \{0\}}$, ordered in increasing size, and no other zeros. Assume that for $j \neq 0$, $F(\rho_j, \cdot)$ has zeros $\{\rho_k\}_{k \in \mathbb{Z} \setminus \{j\}}$ and no other zeros.*

(iii) *There exists $C_0 > 0$ such that for all real t,*

$$(6.2) \qquad F(t,t) \geq C_0,$$

and $F(0,0) = 1$.

(iv) *For all complex a, b,*

$$(6.3) \qquad F(a,b) = \int_{-\infty}^{\infty} F(a,s) F(b,s) \, ds.$$

Then for all complex u, v,

$$F(u,v) = \frac{\sin \pi (u-v)}{\pi (u-v)}.$$

Proof. First note that the symmetry in the right-hand side of (6.3) forces F to be symmetric. That is, for all a, b,

$$F(b,a) = F(a,b).$$

We now break the proof into several steps.

Step 1: The $\{\rho_j\}$ are "well-spaced". First note that for $j \in \mathbb{Z}$,

$$F(\rho_j, \rho_j) = F(\rho_j, \rho_j) - F(\rho_j, \rho_{j-1})$$

$$= \int_{\rho_{j-1}}^{\rho_j} \frac{\partial F(\rho_j, t)}{\partial t} dt$$

$$\leq (\rho_j - \rho_{j-1})^{1/2} \left(\int_{-\infty}^{\infty} \left(\frac{\partial F(\rho_j, t)}{\partial t} \right)^2 dt \right)^{1/2}$$

$$\leq (\rho_j - \rho_{j-1})^{1/2} \sigma \left(\int_{-\infty}^{\infty} (F(\rho_j, t))^2 \, dt \right)^{1/2}$$

$$= (\rho_j - \rho_{j-1})^{1/2} \sigma (F(\rho_j, \rho_j))^{1/2},$$

by Bernstein's inequality for derivatives of entire functions of exponential type $\leq \sigma$ [1, Thm. 3, p. 144], and by (6.3). We deduce that

$$(6.4) \qquad \inf_{j \in \mathbb{Z}} (\rho_j - \rho_{j-1}) \geq \sigma^{-2} \inf_{t \in \mathbb{R}} F(t,t) > 0.$$

Step 2: The operator G. The spacing condition (6.4) ensures that for any $g \in PW_\sigma$, we have [21, p. 150]

$$(6.5) \qquad \sum_{j=-\infty}^{\infty} |g(\rho_j)|^2 < \infty.$$

Next, from our hypotheses,

$$\int_{-\infty}^{\infty} F(\rho_j, t) F(\rho_k, t)\, dt = \delta_{jk} F(\rho_j, \rho_j),$$

so $\left\{ \dfrac{F(\rho_j, t)}{\sqrt{F(\rho_j, \rho_j)}} \right\}_j$ is an orthonormal sequence in $L_2(\mathbb{R})$. Then it follows using also (6.2) that for any $g \in PW_\sigma$,

$$G[g](\cdot) = \sum_{j=-\infty}^{\infty} g(\rho_j) \frac{F(\rho_j, \cdot)}{F(\rho_j, \rho_j)} \in L_2(\mathbb{R})$$

and

$$(6.6) \qquad \int_{-\infty}^{\infty} |G[g]|^2 = \sum_{j=-\infty}^{\infty} \frac{|g(\rho_j)|^2}{F(\rho_j, \rho_j)}.$$

Step 3: A Lagrange interpolation series for $G[g]$. Let

$$L(z) = zF(0, z).$$

We see that for all j,

$$G[g](\rho_j) = g(\rho_j),$$

so

$$H(z) = \frac{G[g](z) - g(z)}{L(z)}$$

is entire. We shall show that $H \equiv 0$. Now both g and L are of exponential type $\leq \sigma$. We claim that $G[g]$ is also of type $\leq \sigma$. Indeed, letting

$$(6.7) \qquad G_n[g](\cdot) = \sum_{|j| \leq n} g(\rho_j) \frac{F(\rho_j, \cdot)}{F(\rho_j, \rho_j)},$$

we see that $G_n[g]$ is of exponential type $\leq \sigma$, as each $F(\rho_j, \cdot)$ is, and is square integrable on the real axis, so [21, p. 149] for all real x, y,

$$|G_n[g](x + iy)|^2 \leq \frac{2}{\pi} e^{2\sigma(|y|+1)} \int |G_n[g]|^2$$

$$= \frac{2}{\pi} e^{2\sigma(|y|+1)} \sum_{|j| \leq n} \frac{|g(\rho_j)|^2}{F(\rho_j, \rho_j)}.$$

Letting $n \to \infty$ gives that $G[g]$ is entire of exponential type $\leq \sigma$. Then being the entire ratio of functions of type $\leq \sigma$, we see that H is of exponential type $\leq \sigma$ [21, p. 13, Thm. 1].

Next, we claim that for each j,

(6.8) $$\frac{F(\rho_j, z)}{F(\rho_j, \rho_j)} = \frac{L(z)}{L'(\rho_j)(z - \rho_j)}.$$

Indeed, by our hypotheses, both sides have the simple zeros $\{\rho_k\}_{k \neq j}$, and no other zeros. Moreover, both take the value 1 at ρ_j. Also, both $F(\rho_j, \cdot)$ and $F(0, \cdot)$ lie in Cartwright's class, in view of the hypothesis (i), so [21, p. 130] for some constant C,

$$\frac{F(\rho_j, z)}{F(\rho_j, \rho_j)} = C \lim_{R \to \infty} \prod_{|\rho_k| \leq R, k \neq j} \left(1 - \frac{z}{\rho_k}\right),$$

with a similar expression for $\frac{L(z)}{L'(\rho_j)(z - \rho_j)}$. Then the ratio of these two functions is constant, so (6.8) follows. Thus we can write $G[g]$ as the Lagrange interpolation series

$$G[g](z) = L(z) \sum_{j=-\infty}^{\infty} \frac{g(\rho_j)}{L'(\rho_j)(z - \rho_j)}.$$

Step 4: $G[g] = g$. Now

$$\left|\frac{G[g](z)}{L(z)}\right| \leq \left(\sum_{j=-\infty}^{\infty} |g(\rho_j)|^2\right)^{1/2} \left(\sum_{j=-\infty}^{\infty} \left|\frac{1}{L'(\rho_j)(z - \rho_j)}\right|^2\right)^{1/2}.$$

Let $\varepsilon \in \left(0, \frac{\pi}{2}\right)$ and $\mathcal{A}_\varepsilon = \{z : \varepsilon \leq |\arg(z)| \leq \pi - \varepsilon\}$. There exists C_ε independent of z and j, such that for $|z| \geq 2$, and $z \in \mathcal{A}_\varepsilon$, we have for all j,

$$|z - \rho_j| \geq C_\varepsilon |i - \rho_j|.$$

It follows that for each fixed positive integer m,

$$\limsup_{|z| \to \infty, z \in \mathcal{A}_\varepsilon} \left|\frac{G[g](z)}{L(z)}\right|$$

(6.9) $$\leq \left(\sum_{j=-\infty}^{\infty} |g(\rho_j)|^2\right)^{1/2} \left(C_\varepsilon^{-2} \sum_{|j| \geq m} \left|\frac{1}{L'(\rho_j)(i - \rho_j)}\right|^2\right)^{1/2}.$$

Next,

$$\sum_{j=-\infty}^{\infty} \left|\frac{L(i)}{L'(\rho_j)(i - \rho_j)}\right|^2 = \sum_{j=-\infty}^{\infty} \left|\frac{F(i, \rho_j)}{F(\rho_j, \rho_j)}\right|^2$$

$$\leq \frac{1}{C_0} \sum_{j=-\infty}^{\infty} \frac{|F(i, \rho_j)|^2}{F(\rho_j, \rho_j)} = \frac{1}{C_0} \int_{-\infty}^{\infty} |G[F(i, \cdot)]|^2 < \infty,$$

by (6.6) with $g\left(\cdot\right) = F\left(i, \cdot\right)$, and (6.1). We also use (6.2) here, and (i). Thus we can let $m \to \infty$ in (6.9) to deduce that

$$(6.10) \qquad \limsup_{z \to \infty, z \in \mathcal{A}_\varepsilon} \left| \frac{G\left[g\right]\left(z\right)}{L\left(z\right)} \right| = 0.$$

Next, let us assume first that g is of exponential type $\tau < \sigma$. As also g is square integrable on the real axis, it lies in the Cartwright class, so for all $\theta \in [-\pi, \pi]$, (see [21, p. 118] and use properties of the indicator function)

$$\limsup_{r \to \infty} \frac{\log \left| g\left(re^{i\theta}\right) \right|}{r} \le \tau \left| \sin \theta \right|.$$

Also, $L\left(z\right) = zF\left(0, z\right)$ lies in the Cartwright class, and has all real zeros, and by hypothesis has type σ, so for $\theta \in \left(-\pi, \pi\right) \setminus \{0\}$,

$$\lim_{r \to \infty} \frac{\log \left| L\left(re^{i\theta}\right) \right|}{r} = \sigma \left| \sin \theta \right|.$$

Then

$$\limsup_{r \to \infty} \left| \frac{g}{L}\left(re^{i\theta}\right) \right| \le \limsup_{r \to \infty} \exp\left(\left(\tau - \sigma\right) r \left| \sin \theta \right| + o\left(r\right)\right) = 0.$$

Thus from this and (6.10), for all $\theta \in \left(-\pi, \pi\right) \setminus \{0\}$,

$$(6.11) \qquad \lim_{r \to \infty} \left| H\left(re^{i\theta}\right) \right| = 0.$$

Inasmuch as H is entire of order ≤ 1, we can use the Phragmen-Lindelöf principle on sectors of width $< \pi$, to deduce that H is bounded in each such sector [21, p. 37], and hence constant in the plane. As it has limit 0 at ∞, we deduce that $H \equiv 0$, so

$$(6.12) \qquad G\left[g\right] = g.$$

Finally, if g is of exponential type σ, we can let $\varepsilon \in \left(0, 1\right)$ and define $g_\varepsilon\left(z\right) = g\left(\varepsilon z\right)$, which has type $\varepsilon\sigma$. So

$$G\left[g_\varepsilon\right] = g_\varepsilon.$$

It is easily see that we can let $\varepsilon \to 1-$ to obtain (6.12).

Step 5: Deduce F is the sinc kernel. We first show that F is a reproducing kernel for PW_σ. Let $g \in PW_\sigma$ and $G_n\left[g\right]$ be given by (6.7). It follows from our hypothesis (6.3) that

$$G_n[g]\left(a\right) = \int_{-\infty}^{\infty} G_n\left[g\right]\left(t\right) F\left(t, a\right) dt.$$

We can let $n \to \infty$ in this to deduce that

$$g\left(a\right) = \int_{-\infty}^{\infty} g\left(t\right) F\left(t, a\right) dt.$$

The limiting process is justified as $G_n[g] \to g$ in $L_2(\mathbb{R})$ and $F(\cdot, a)$ is square integrable. So indeed, F is a reproducing kernel for PW_σ, and by uniqueness,

$$F(u, v) = \frac{\sin \sigma (u - v)}{\pi (u - v)}.$$

Indeed, as both are reproducing kernels, cf. [46, p. 95]

$$\frac{\sin \sigma (s - t)}{\pi (s - t)} = \int_{-\infty}^{\infty} \frac{\sin \sigma (s - y)}{\pi (s - y)} F(t, y) \, dy = F(t, s).$$

Then for all t,

$$F(t, t) = \frac{\sigma}{\pi}.$$

As $F(0, 0) = 1$, so $\sigma = \pi$. $\hfill \square$

7. Estimates for Tail Integrals

In this section, we estimate the tail integrals Ψ_n and Φ_n, using maximal functions. It is really these estimates that allow us to skip the usual hypothesis that μ is regular. Recall our notation (1.13)–(1.17) and note that

$$\Psi_n(x, r) = \Phi_n\left(x, r \frac{n}{\tilde{K}_n(x, x)}\right) = \frac{\int_{|t-x| \geq \frac{r}{\tilde{K}_n(x,x)}} K_n(x, t)^2 \, d\mu(t)}{K_n(x, x)}.$$

We need both as Ψ_n is easier to estimate in terms of maximal functions, while Φ_n is easier to handle when varying x. Recall too that we set $\Psi_n(x, r) = \Phi_n(x, r) = 0$ when $\mu'(x) = 0$. We note that Lemma 7.1 was already proved and used in [31].

Lemma 7.1. *Let μ be a measure on the real line with infinitely many points in its support. Let $r > 0$. For every Lebesgue point x of $A_n d\mu$, and in particular for a.e. $x \in \mathrm{supp}\,[\mu]$,*

$$(7.1) \qquad\qquad \Psi_n(x, r) \leq \frac{8}{r} \left(\frac{\gamma_{n-1}}{\gamma_n} \mathcal{M}[A_n d\mu](x)\right)^2.$$

Moreover, this holds for all $x \notin \mathrm{supp}\,[\mu]$.

Remark 5. The set of exceptional x, for which (7.1) fails, is independent of r.

Proof. Observe that

$$|K_n(x, t)| = \frac{\gamma_{n-1}}{\gamma_n} \left| \frac{p_n(x) p_{n-1}(t) - p_{n-1}(x) p_n(t)}{x - t} \right|$$

$$\leq \frac{\gamma_{n-1}}{\gamma_n} \frac{A_n(x)^{1/2} A_n^{1/2}(t)}{|x - t|}.$$

Let

$$\beta = \frac{r}{\tilde{K}_n(x, x)}.$$

Then we see that

$$\Psi_n(x,r) \leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \sum_{j=0}^{\infty} \int_{2^j \beta \leq |x-t| \leq 2^{j+1}\beta} \frac{A_n(t)}{|x-t|^2} d\mu(t)$$

$$\leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \sum_{j=0}^{\infty} 2^{-2j}\beta^{-2} \int_{2^j \beta \leq |x-t| \leq 2^{j+1}\beta} A_n(t)\, d\mu(t).$$

Here

$$\frac{1}{2^{j+2}\beta} \int_{2^j \beta \leq |x-t| \leq 2^{j+1}\beta} A_n(t)\, d\mu(t) \leq \mathcal{M}[A_n d\mu](x),$$

so we can continue this as

$$\Psi_n(x,r) \leq \left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 \frac{A_n(x)}{K_n(x,x)} \beta^{-1} \mathcal{M}[A_n d\mu](x)\, 8$$

$$= \frac{8}{r}\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^2 A_n(x)\, \mu'(x)\, \mathcal{M}[A_n d\mu](x).$$

Now a.e. $x \in \mathrm{supp}\,[\mu]$ is a Lebesgue point of the measure $A_n d\mu$, so for a.e. x, we have

$$\mathcal{M}[A_n d\mu](x) \geq \lim_{h \to 0+} \frac{1}{2h} \int_{x-h}^{x+h} A_n(t)\, d\mu(t) = A_n(x)\, \mu'(x).$$

In the case when x is not in the support of μ, this holds trivially as $\mu'(x) = 0$. $\qquad\square$

We want to translate this into estimates for Φ_n, but first need a number of estimates involving K_n:

Lemma 7.2. *Let $\varepsilon > 0$. There exists n_0 and $\delta > 0$, with the following properties: for $n \geq n_0$, there is a set $\mathcal{H}_n = \mathcal{H}_n(\varepsilon)$ of measure $\leq \varepsilon$, such that for $\xi \in \{\mu' > 0\} \backslash \mathcal{H}_n$, and for all $n \geq n_0$,*

(a)

$$(7.2) \qquad\qquad \delta^{-1} \geq \tilde{K}_n(\xi,\xi)/n \geq \delta;$$

(b) *for all complex u, v,*

$$(7.3) \qquad\qquad \left|K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)\right| \leq C_1 n e^{C_2(|u|+|v|)/\varepsilon};$$

  Both C_1 and C_2 are independent of n, ξ, u, v, while C_2 is also independent of ε.

(c)

$$(7.4) \qquad \frac{\tilde{K}_n(\xi,\xi)}{2} \int_{|t-\xi| \leq \frac{1}{\tilde{K}_n(\xi,\xi)}} \left(1 - \chi_{\{\mu'>0\}}(t)\right) dt \leq \varepsilon;$$

(d) *The bound (5.1) holds for $f_n(u,v,\xi)$ and all complex u, v.*

Proof. (a), (b) It is easily seen from Corollary 3.2 and Lemma 4.3 that there is a set $\mathcal{H}_n$ of measure $< \varepsilon/2$, for which (7.2) and (7.3) hold. Of course one also uses that for some $C > 1$,

$$C^{-1} \leq \mu' \leq C$$

in $\{\mu' > 0\}$, except on a set of small measure.

(c) By Lebesgue's density theorem, for a.e. $\xi \in \{\mu' > 0\}$,

$$\lim_{h \to 0+} \frac{1}{2h} meas\left(\{\mu' > 0\} \cap [\xi - h, \xi + h]\right) = 1.$$

As also

$$\lim_{n \to \infty} \tilde{K}_n(\xi, \xi) = \infty$$

in $\{\mu' > 0\}$, except at jump discontinuities of μ [15, p. 63, Thm. 2.1], so for a.e. $\xi \in \{\mu' > 0\}$,

$$\lim_{n \to \infty} \frac{\tilde{K}_n(\xi, \xi)}{2} \int_{|t - \xi| \leq \frac{1}{\tilde{K}_n(\xi, \xi)}} \left(1 - \chi_{\{\mu' > 0\}}\right) = 0.$$

We can now simply apply Egorov's theorem to get uniform convergence except on a set of small measure, and append this to $\mathcal{H}_n$ above.

(d) One can simply append the set $\mathcal{G}_n$ of Theorem 5.1 to $\mathcal{H}_n$. $\square$

We can now deduce estimates for Γ_n and I_n, defined respectively by (1.19) and (1.20).

Lemma 7.3. *Let $\varepsilon > 0$ and $n_0(\varepsilon), \delta, \mathcal{H}_n$ be as in Lemma 7.2. Let $r > 0$ and $n \geq n_0(\varepsilon)$.*

(a) Assume that $\xi \notin \mathcal{H}_n$. For $|u|, |v| \leq \frac{r\delta}{2}$,

$$(7.5) \qquad \Gamma_n(u, v, \xi, r) \leq \left[f_n(u, u, \xi) \Phi_n\left(\xi + \frac{u}{\tilde{K}_n(\xi, \xi)}, \frac{r}{2}\right)\right]^{1/2}$$

$$\times \left[f_n(v, v, \xi) \Phi_n\left(\xi + \frac{v}{\tilde{K}_n(\xi, \xi)}, \frac{r}{2}\right)\right]^{1/2}.$$

(b) Assume that $\xi \notin \mathcal{H}_n$ and $r\delta/2 \geq 1$. Then

$$(7.6) \qquad I_n(\xi, r) \leq \left(\mathcal{M}\left[\Phi_n\left(\cdot, \frac{r}{2}\right)^{1/4} \chi_{\{\mu' > 0\}}\right](\xi) + \varepsilon\right)^2.$$

(c) For all $\lambda > 0$ and for some C_1 independent of $\varepsilon, \delta, r, \lambda$,

(7.7)
$$meas\left\{\xi \in \{\mu' > 0\} : I_n(\xi, r) > (\lambda + \varepsilon)^2\right\} \leq \frac{3}{\lambda}\left(C_1(r\delta)^{-1/4} + \varepsilon\right) + \varepsilon.$$

Proof. (a) Let

$$U = \xi + \frac{u}{\tilde{K}_n(\xi, \xi)}; \quad V = \xi + \frac{v}{\tilde{K}_n(\xi, \xi)},$$

and let $s \geq r$. From the reproducing kernel relation,

$$\frac{K_n(U,V)}{K_n(\xi,\xi)} - \int_{|y-\xi| \leq \frac{s}{n}} \frac{K_n(U,y)}{K_n(\xi,\xi)} \frac{K_n(V,y)}{K_n(\xi,\xi)} \tilde{K}_n(\xi,\xi) \frac{d\mu(y)}{\mu'(\xi)}$$

$$= \int_{|y-\xi| > \frac{s}{n}} \frac{K_n(U,y)}{\sqrt{K_n(\xi,\xi)}} \frac{K_n(V,y)}{\sqrt{K_n(\xi,\xi)}} d\mu(y).$$

We now make the substitution $y = \xi + \frac{t}{\tilde{K}_n(\xi,\xi)}$, in the first integral only, recasting the last equation as

$$f_n(u,v,\xi) - \int_{-s\frac{\tilde{K}_n(\xi,\xi)}{n}}^{s\frac{\tilde{K}_n(\xi,\xi)}{n}} f_n(u,t,\xi) f_n(v,t,\xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)}$$

$$= f_n(u,u,\xi)^{1/2} f_n(v,v,\xi)^{1/2} \int_{|y-\xi| > \frac{s}{n}} \frac{K_n(U,y)}{\sqrt{K_n(U,U)}} \frac{K_n(V,y)}{\sqrt{K_n(V,V)}} d\mu(y).$$

(7.8)

Next, observe that for $\xi \notin \mathcal{H}_n$, (7.2) gives

$$|y - \xi| \geq \frac{s}{n} \Rightarrow |y - U| \geq \frac{s}{n} - \frac{|u|}{n} \frac{n}{\tilde{K}_n(\xi,\xi)} \geq \frac{s}{n} - \frac{|u|}{\delta n} \geq \frac{s}{2n},$$

as $|u| \leq \frac{\delta r}{2} \leq \frac{\delta s}{2}$. Now use Cauchy-Schwarz on the right-hand side of (7.8), and the fact that $s \geq r$:

$$\Gamma_n(u,v,\xi,r)$$

$$= \sup_{s \geq r} \left| f_n(u,v,\xi) - \int_{-s\frac{\tilde{K}_n(\xi,\xi)}{n}}^{s\frac{\tilde{K}_n(\xi,\xi)}{n}} f_n(u,t,\xi) f_n(v,t,\xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right|$$

$$\leq \left[f_n(u,u,\xi) f_n(v,v,\xi) \int_{|y-U| > \frac{r}{2n}} \frac{K_n^2(U,y)}{K_n(U,U)} d\mu(y) \right.$$

$$\left. \times \int_{|y-V| > \frac{r}{2n}} \frac{K_n^2(V,y)}{K_n(V,V)} d\mu(y) \right]^{1/2}.$$

We obtain (7.5), on taking account of the definition (1.14) of Φ_n.

(b) Using (a) and integrating, gives

$$I_n(\xi,r) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} \Gamma_n(u,v,\xi,r)^{1/2} (f_n(u,u,\xi) f_n(v,v,\xi))^{-1/4} du\, dv$$

$$\leq \left(\frac{1}{2} \int_{-1}^{1} \Phi_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \frac{r}{2}\right)^{1/4} du \right)^2$$

$$= \left(\frac{\tilde{K}_n(\xi,\xi)}{2} \int_{|t-\xi| \leq \frac{1}{\tilde{K}_n(\xi,\xi)}} \Phi_n\left(t, \frac{r}{2}\right)^{1/4} dt \right)^2$$

$$\leq \left(\frac{\tilde{K}_n\left(\xi,\xi\right)}{2} \int_{|t-\xi|\leq\frac{1}{\tilde{K}_n(\xi,\xi)}} \Phi_n\left(t,\frac{r}{2}\right)^{1/4} \chi_{\{\mu'>0\}}(t)\,dt + \varepsilon \right)^2,$$

by Lemma 7.2(c), and as $\Phi_n \leq 1$. Taking account of the definition of the maximal function, we obtain (7.6).

(c) From (b), for $\xi \notin \mathcal{H}_n$,

$$I_n\left(\xi,r\right) > \left(\lambda+\varepsilon\right)^2$$

(7.9)
$$\Rightarrow \mathcal{M}\left[\Phi_n\left(\cdot,\frac{r}{2}\right)^{1/4}\chi_{\{\mu'>0\}}\right](\xi) > \lambda.$$

Moreover, by the classical weak type (1,1) inequality for maximal functions [39, p. 138],

$$meas\left\{\xi : \mathcal{M}\left[\Phi_n\left(\cdot,\frac{r}{2}\right)^{1/4}\chi_{\{\mu'>0\}}\right](\xi) > \lambda\right\}$$

$$\leq \frac{3}{\lambda}\int_{-\infty}^{\infty}\Phi_n\left(t,\frac{r}{2}\right)^{1/4}\chi_{\{\mu'>0\}}(t)\,dt$$

$$\leq \frac{3}{\lambda}\left(\int_{\{\mu'>0\}\backslash\mathcal{H}_n}\Phi_n\left(t,\frac{r}{2}\right)^{1/4}dt + \int_{\{\mu'>0\}\cap\mathcal{H}_n}\Phi_n\left(t,\frac{r}{2}\right)^{1/4}dt\right)$$

(7.10)
$$\leq \frac{3}{\lambda}\left(\int_{\{\mu'>0\}\backslash\mathcal{H}_n}\Phi_n\left(t,\frac{r}{2}\right)^{1/4}dt + \varepsilon\right),$$

as $\Phi_n \leq 1$ and $\mathcal{H}_n$ has measure at most ε. Next, let

$$C_0 = \sup_n \frac{\gamma_{n-1}}{\gamma_n}.$$

This is finite as μ has compact support. By (7.2) of Lemma 7.2, for $t \in \{\mu' > 0\}\backslash\mathcal{H}_n$, followed by Lemma 7.1,

$$\Phi_n\left(t,\frac{r}{2}\right) \leq \Psi_n\left(t,\frac{r\delta}{2}\right) \leq \frac{16C_0^2}{r\delta}\mathcal{M}\left[A_n d\mu\right](t)^2.$$

Then, by the distributional formula for integrals [39, p. 172], and as $\Phi_n \leq 1$,

$$\int_{\{\mu'>0\}\backslash\mathcal{H}_n}\Phi_n\left(t,\frac{r}{2}\right)^{1/4}dt$$

$$= \frac{1}{4}\int_0^1 s^{-3/4}meas\left\{t \in \{\mu'>0\}\backslash\mathcal{H}_n : \Phi_n\left(t,\frac{r}{2}\right) > s\right\}ds$$

$$\leq \frac{1}{4}\int_0^1 s^{-3/4}meas\left\{t \in \{\mu'>0\} : \mathcal{M}\left[A_n d\mu\right](t) > \frac{(sr\delta)^{1/2}}{4C_0}\right\}ds.$$

Here, again, by the classical weak type (1,1) inequality for maximal functions,

$$meas \left\{ t : \mathcal{M} \left[A_n d\mu \right] (t) > \frac{(sr\delta)^{1/2}}{4C_0} \right\} \leq \frac{12C_0}{(sr\delta)^{1/2}} \int A_n \, d\mu = \frac{24C_0}{(sr\delta)^{1/2}}.$$

Thus, if we let $L = meas \{\mu' > 0\}$,

$$\int_{\{\mu'>0\}\backslash\mathcal{H}_n} \Phi_n \left(t, \frac{r}{2} \right)^{1/4} dt \leq \frac{1}{4} \int_0^1 s^{-3/4} \min \left\{ L, \frac{24C_0}{(sr\delta)^{1/2}} \right\} ds$$

$$\leq C_1 \left(r\delta \right)^{-1/4}.$$

Here C_1 is independent of r, ε, δ. Substituting into (7.10), gives

$$meas \left\{ \xi : \mathcal{M} \left[\Phi_n \left(\cdot, \frac{r}{2} \right)^{1/4} \chi_{\{\mu'>0\}} \right] (\xi) > \lambda \right\} \leq \frac{3}{\lambda} \left(C_1 \left(r\delta \right)^{-1/4} + \varepsilon \right).$$

Taking account of (7.9), and again using the fact that $\mathcal{H}_n$ has measure at most ε, gives the result. $\qquad\square$

8. Proof of Theorem 1.1 and the Corollaries

Lemma 8.1. *Let $\varepsilon \in (0, 1)$. There exists $n_0 (\varepsilon)$ and r_ε depending on ε, but independent of n, such that for $n \geq n_0 (\varepsilon)$,*

$$(8.1) \qquad \mathcal{J}_n (\varepsilon) = \left\{ \xi \in \{\mu' > 0\} : I_n (\xi, r_\varepsilon) > \varepsilon^2 \right\}$$

has

$$(8.2) \qquad meas \left(\mathcal{J}_n (\varepsilon) \right) \leq 4\varepsilon.$$

Proof. From Lemma 7.3 (c), with ε replaced by ε^2, and $\lambda = \varepsilon$,

$$meas \left\{ \xi \in \{\mu' > 0\} : I_n (\xi, r) > \left(\varepsilon + \varepsilon^2 \right)^2 \right\}$$

$$\leq \frac{3}{\varepsilon} \left(C_1 \left(r\delta \right)^{-1/4} + \varepsilon^2 \right) + \varepsilon^2.$$

Here δ depends on ε, but the crucial feature is that C_1 is independent of r. Choose $r = r_\varepsilon$ by the equation

$$C_1 \left(r\delta \right)^{-1/4} = \varepsilon^2.$$

Then

$$meas \left\{ \xi \in \{\mu' > 0\} : I_n (\xi, r_\varepsilon) > 4\varepsilon^2 \right\} \leq 7\varepsilon.$$

On replacing ε by $\varepsilon/2$, and relabelling r_ε, we obtain the result. $\qquad\square$

We use the above lemma to prove:

Lemma 8.2. *Let μ be a compactly supported measure with infinitely many points in its support. Let $\varepsilon > 0$. There exists for $n \geq n_0 (\varepsilon)$, a set $\mathcal{I}_n$ of measure $\leq \varepsilon$, with the following properties: let ξ be given, and $\mathcal{S}$ be a*

sequence of positive integers such that for $n \in \mathcal{S}$, we have $\xi \in \{\mu' > 0\} \setminus \mathcal{I}_n$. Then there is a subsequence $\mathcal{T}$ of $\mathcal{S}$, for which

$$(8.3) \qquad \lim_{n \to \infty, n \in \mathcal{T}} f_n(u, v, \xi) = \frac{\sin \pi (u - v)}{\pi (u - v)}$$

uniformly for u, v in compact subsets of the plane.

Proof. Let $\mathcal{J}_n(\varepsilon)$ be defined by (8.1). It exists for $n \geq n_0(\varepsilon)$. Let

$$\mathcal{I}_n = \bigcup_{j \geq 0 : n_0(2^{-j}\varepsilon) \leq n} \mathcal{J}_n\left(2^{-j}\varepsilon\right).$$

Here $n_0\left(2^{-j}\varepsilon\right)$ is as in the previous lemma. Of course, given any positive integer k, we have for large enough n,

$$\mathcal{I}_n \supseteq \bigcup_{j=0}^{k} \mathcal{J}_n\left(2^{-j}\varepsilon\right).$$

From the previous lemma,

$$meas\left(\mathcal{I}_n\right) \leq 8\varepsilon.$$

Let $\mathcal{S}$ be a sequence of positive integers such that for $n \in \mathcal{S}$, we have $\xi \in \{\mu' > 0\} \setminus \mathcal{I}_n$. Fix $j \geq 1$. For large enough $n \in \mathcal{S}$, we have

$$I_n\left(\xi, r_{2^{-j}\varepsilon}\right) \leq \left(2^{-j}\varepsilon\right)^2,$$

so for each $s \geq r_{2^{-j}\varepsilon} \frac{\tilde{K}_n(\xi,\xi)}{n}$, and in particular if $s \geq \delta^{-1} r_{2^{-j}\varepsilon}$, (recall (7.2) holds outside $\mathcal{H}_n$, and $\mathcal{H}_n$ is contained in the exceptional set $\mathcal{I}_n$)

$$\int_{-1}^{1} \int_{-1}^{1} \left| f_n(u, v, \xi) - \int_{-s}^{s} f_n(u, t, \xi) f_n(v, t, \xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} \right|^{1/2} \times$$

$$\times \left(f_n(u, u, \xi) f_n(v, v, \xi)\right)^{-1/4} du \, dv$$

$$\leq 4\left(2^{-j}\varepsilon\right)^2.$$

Because $\{f_n(\cdot, \cdot, \xi)\}$ is a normal family, recall Lemma 7.2(d) and Theorem 5.1, we can also assume that $f_n(u, v, \xi) \to f(u, v, \xi)$, locally uniformly as $n \to \infty$ through a subsequence $\mathcal{T}$ of $\mathcal{S}$. The subsequence and limit function are independent of j. Moreover, as we may assume ξ is a Lebesgue point (as a.e. ξ is), we have as $n \to \infty$ through the subsequence,

$$\int_{-s}^{s} f_n(u, t, \xi) f_n(v, t, \xi) \frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)}$$

$$= \int_{-s}^{s} f_n(u, t, \xi) f_n(v, t, \xi) \, dt$$

$$+ \int_{-s}^{s} f_n(u,t,\xi) f_n(v,t,\xi) \left[\frac{d\mu\left(\xi + \frac{t}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} - dt \right]$$

$$\rightarrow \int_{-s}^{s} f(u,t,\xi) f(v,t,\xi)\, dt.$$

We deduce that for each fixed $s \geq \delta^{-1} r_{2^{-j}\varepsilon}$,

$$\int_{-1}^{1} \int_{-1}^{1} \left| f(u,v,\xi) - \int_{-s}^{s} f(u,t,\xi) f(v,t,\xi)\, dt \right|^{1/2}$$
$$\times \left(f(u,u,\xi) f(v,v,\xi) \right)^{-1/4} du\, dv \leq 4 \left(2^{-j}\varepsilon \right)^2.$$

Consequently, also

$$\int_{-1}^{1} \int_{-1}^{1} \left| f(u,v,\xi) - \int_{-\infty}^{\infty} f(u,t,\xi) f(v,t,\xi)\, dt \right|^{1/2}$$
$$\times \left(f(u,u,\xi) f(v,v,\xi) \right)^{-1/4} du\, dv \leq 4 \left(2^{-j}\varepsilon \right)^2.$$

Here, we also used the convergence of the integrals in (5.3). As the left-hand side is independent of j, we obtain

$$\int_{-1}^{1} \int_{-1}^{1} \left| f(u,v,\xi) - \int_{-\infty}^{\infty} f(u,t,\xi) f(v,t,\xi)\, dt \right|^{1/2}$$
$$\times \left(f(u,u,\xi) f(v,v,\xi) \right)^{-1/4} du\, dv = 0.$$

So for a.e. $(u,v) \in [-1,1] \times [-1,1]$,

$$f(u,v,\xi) = \int_{-\infty}^{\infty} f(u,t,\xi) f(v,t,\xi)\, dt.$$

Because both sides are entire in (u,v), we obtain that this holds for all complex u,v. This fulfils the major hypothesis (iv) of Theorem 6.1, with $F(u,v) = f(u,v,\xi)$. Since each $f_n(0,0,\xi) = 1$, so also $f(0,0,\xi) = 1$. The remaining hypotheses (i), (ii), (iii) were already established in Theorem 5.1. From Theorem 6.1,

$$f(u,v,\xi) = \frac{\sin \pi(u-v)}{\pi(u-v)}.$$

$\square$

Proof of Theorem 1.1. Fix r and $\varepsilon > 0$. For $\xi \in \{\mu' > 0\}$, let

$$g_n(\xi) = \sup_{|u|,|v| \leq r} \left| f_n(u,v,\xi) - \frac{\sin \pi(u-v)}{\pi(u-v)} \right|.$$

Let

(8.4) $$\mathcal{L}_n(\varepsilon, r) = \mathcal{L}_n = \left\{ \xi \in \{\mu' > 0\} : g_n(\xi) \geq \varepsilon \right\}.$$

We have to show that as $n \rightarrow \infty$,

$$meas(\mathcal{L}_n) \rightarrow 0.$$

Let us suppose that this is false. Then for some infinite sequence $\mathcal{N}$ of integers, and some $\eta > 0$, we have

$$meas\left(\mathcal{L}_n\right) \geq \eta, \; n \in \mathcal{N}.$$

Next, by Lemma 8.2, with ε replaced by $\eta/2$, there exists for $n \geq n_0\left(\eta/2\right)$ a set $\mathcal{I}_n$ of measure $\leq \eta/2$ with the following property: let $\mathcal{S}$ be an infinite sequence of integers. If $\xi \in \{\mu' > 0\} \setminus \mathcal{I}_n$, $n \in \mathcal{S}$, then there is a subsequence $\mathcal{T}$ of $\mathcal{S}$ such that $\{f_n\}_{n \in \mathcal{T}}$ converges to the sinc kernel, uniformly in compact sets. Now consider

$$\mathcal{C} = \limsup_{n \to \infty, n \in \mathcal{N}} \left(\mathcal{L}_n \setminus \mathcal{I}_n\right) = \bigcap_{k=n_0(\eta/2)}^{\infty} \bigcup_{n \geq k, n \in \mathcal{N}} \left(\mathcal{L}_n \setminus \mathcal{I}_n\right).$$

Since $\infty > \text{meas}(\mathcal{L}_n \setminus \mathcal{I}_n) \geq \eta/2$, so $\text{meas}(\mathcal{C}) \geq \eta/2$. Indeed, the sets $\bigcup_{n \geq k, n \in \mathcal{N}} \left(\mathcal{L}_n \setminus \mathcal{I}_n\right)$ decrease as k increases, and each has measure $\geq \eta/2$, and each is contained in the bounded set $\{\mu' > 0\}$. Let $\xi \in \mathcal{C}$. Then for infinitely many $n \in \mathcal{N}$, we have $\xi \in \mathcal{L}_n \setminus \mathcal{I}_n$ - say this occurs for $n \in \mathcal{N}_1$. By the above mentioned property from Lemma 8.2, there is a subsequence $\mathcal{M}$ of $\mathcal{N}_1$ such that $\{f_n\}_{n \in \mathcal{M}}$ converges uniformly in compact sets to the sinc kernel. But since $\mathcal{M}$ is a subsequence of $\mathcal{N}$, also for $n \in \mathcal{M}$, $\xi \in \mathcal{L}_n$, so

$$g_n\left(\xi\right) = \sup_{|u|,|v| \leq r} \left| f_n\left(u, v, \xi\right) - \frac{\sin \pi\left(u - v\right)}{\pi\left(u - v\right)} \right| \geq \varepsilon.$$

Thus we have a contradiction. So we have the desired convergence in measure. $\qquad \square$

Proof of Corollary 1.2. This follows easily from Theorem 1.1: for $\varepsilon, r > 0$, let $\mathcal{L}_n\left(\varepsilon, r\right)$ denote the set defined by (8.4). From Theorem 1.1, we can choose a subsequence $\mathcal{T} = \{n_j\}$ of the given sequence $\mathcal{S}$, such that

$$meas\left(\mathcal{L}_{n_j}\left(\frac{1}{j}, j\right)\right) \leq j^{-2}, \quad j \geq 1.$$

Let

$$\mathcal{E} = \limsup_{j \to \infty} \mathcal{L}_{n_j}\left(\frac{1}{j}, j\right).$$

Since $\sum_j j^{-2} < \infty$, $meas\left(\mathcal{E}\right) = 0$. Let $\xi \in \{\mu' > 0\} \setminus \mathcal{E}$. Let $r > 0$. We have for large enough j, $\xi \notin \mathcal{L}_{n_j}\left(\frac{1}{j}, j\right)$, so

$$\sup_{|u|,|v| \leq r} \left| f_{n_j}\left(u, v, \xi\right) - \frac{\sin \pi\left(u - v\right)}{\pi\left(u - v\right)} \right| < \frac{1}{j}.$$

It follows that for each $r > 0$, and uniformly for $|u|, |v| \leq r$,

$$\lim_{j \to \infty} f_{n_j}\left(u, v, \xi\right) = \frac{\sin \pi\left(u - v\right)}{\pi\left(u - v\right)}.$$

$\qquad \square$

Proof of Corollary 1.3. Let $r > 0$, and J be a compact subset of O. Since O is open, we may assume that J is a single closed interval inside O. Now by hypothesis,

$$C^{-1} \le \mu' \le C$$

a.e. in O. Standard methods then show that in some closed interval J_1 containing J in its interior, we have

$$C_1 \le \frac{K_n(\xi, \xi)}{n} \le C_2, \quad n \ge 1, \quad \xi \in J_1.$$

See for example [36, p. 116, Thm. 20]. Bernstein's growth lemma (cf. [27, pp. 383–384]) enables one to show that given $r > 0$, we have for $n \ge n_0(r)$, all $\xi \in J_1$, and $|u|, |v| \le r$,

$$\left| K_n\left(\xi + \frac{u}{n}, \xi + \frac{v}{n}\right)\right| \le C_3 e^{C_4(|\operatorname{Im} u| + |\operatorname{Im} v|)}.$$

Here C_3 and C_4 are independent of ξ, r, u, v. It follows that for $n \ge n_0(r)$,

$$\sup_{|u|, |v| \le r, \xi \in J} |f_n(u, v, \xi)| \le C_5,$$

where C_5 is independent of n. This boundedness, and the convergence in measure in Theorem 1.1, immediately yield (1.8). $\qquad\square$

Proof of Corollary 1.4. A double Taylor series expansion gives

$$\frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{u}{\tilde{K}_n(\xi, \xi)}\right)}{K_n(\xi, \xi)} - \frac{\sin \pi(u - v)}{\pi(u - v)}$$

$$= \sum_{j,k=0}^{\infty} \left[\frac{K_n^{(j,k)}(\xi, \xi)}{K_n(\xi, \xi) \tilde{K}_n(\xi, \xi)^{j+k}} - \pi^{j+k} \tau_{j,k} \right] \frac{u^j v^k}{j! k!}.$$

Moreover, by Cauchy's estimates,

$$\left| \frac{K_n^{(j,k)}(\xi, \xi)}{K_n(\xi, \xi) \tilde{K}_n(\xi, \xi)^{j+k}} - \pi^{j+k} \tau_{j,k} \right|$$

$$\le \frac{j! k!}{r^{j+k}} \sup_{|u|, |v| \le r} \left| \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{u}{\tilde{K}_n(\xi, \xi)}\right)}{K_n(\xi, \xi)} - \frac{\sin \pi(u - v)}{\pi(u - v)} \right|.$$

Integrating and using Corollary 1.3, gives the result. $\qquad\square$

REFERENCES

[1] N. I. Achieser, *Theory of Approximation* (transl. by C.J. Hyman), Dover, New York, 1992.

[2] A. Avila, Y. Last, and B. Simon, *Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with a.c. spectrum,* to appear in Analysis and PDE.

[3] J. Baik, T. Kriecherbauer, K. T-R. McLaughlin, P.D. Miller, *Uniform Asymptotics for Polynomials Orthogonal with respect to a General Class of Discrete Weights and Universality Results for Associated Ensembles*, Princeton Annals of Mathematics Studies, 2006.

[4] C. Bennett and R. Sharpley, *Interpolation of Functions*, Academic Press, Orlando, 1988.

[5] P. Bleher and A. Its, *Random matrix models and their applications,* Cambridge University Press, Cambridge, 2001.

[6] J. Breuer, Y. Last, B. Simon, *The Nevai Condition*, to appear in Constr. Approx.

[7] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[8] P. Deift, D. Gioev, *Random Matrix Theory: Invariant Ensembles and Universality,* Courant Institute Lecture Notes, Vol. 18, New York University Pres, New York, 2009.

[9] P. Deift, T. Kriecherbauer, K. T-R. McLaughlin, S. Venakides and X. Zhou, *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory*, Communications in Pure and Applied Maths., 52(1999), 1335–1425.

[10] L. Erdős, B. Schlein, H.T. Yau, J. Yin, *The local relaxation flow approach to universality of the local statistics for random matrices*, to appear in Annales Inst. H. Poincaré (B), Probability and Statistics

[11] L. Erdős, S. Peche, J. Ramirez, B. Schlein, H.T.Yau, *Bulk Universality for Wigner Matrices,* Communications in Pure and Applied Mathermatics, 63(2010), 895–925.

[12] E. Findley, *Universality for Regular Measures satisfying Szegő's Condition,* J. Approx. Theory, 155 (2008), 136–154.

[13] A. Foulquie Moreno, A. Martinez-Finkelshtein, V. Sousa, *Asymptotics of Orthogonal Polynomials for a Weight with a Jump on* $[-1, 1]$, manuscript.

[14] P. J. Forrester, *Log-gases and Random matrices*, online book,
http://www.ms.unimelb.edu.au/~matpjf/matpjf.html

[15] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[16] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, Orlando, 1981.

[17] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, San Diego, 1979.

[18] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Mathematics Research Notices, 30(2002), 1575–1600.

[19] A.B. Kuijlaars and M. Vanlessen, *Universality for eigenvalue correlations at the origin of the spectrum*, Comm. Math. Phys. 243 (2003), 163–191.

[20] N.S. Landkof, *Foundations of Modern Potential Theory*, Springer, Berlin, 1972.

[21] B. Ya. Levin, in collaboration with Yu. Lyubarskii, M. Sodin, V. Tkachenko, *Lectures on Entire Functions*, Translations of Mathematical Monographs, Vol. 150, American Mathematical Society, Providence, 1996.

[22] Eli Levin and D.S. Lubinsky, *Universality Limits in the Bulk for Varying Measures,* Advances in Mathematics, 219(2008), 743–779.

[23] Eli Levin and D.S. Lubinsky, *Universality Limits for Exponential Weights*, Constructive Approximation, 29(2009), 247–275.

[24] Eli Levin and D.S. Lubinsky, *Universality Limits at the Soft Edge of the Spectrum via Classical Complex Analysis,* to appear in International Maths Research Notices.

[25] D.S. Lubinsky, *A New Approach to Universality Limits at the Edge of the Spectrum*, Contemporary Mathematics (60th Birthday of Percy Deift), 458(2008), 281–290.

[26] D.S. Lubinsky, *Universality Limits at the Hard Edge of the Spectrum for Measures with Compact Support,* International Mathematics Research Notices, 2008(2008), Article ID rnn099, 39 pages.

[27] D.S. Lubinsky, *Universality limits in the bulk for arbitrary measures on compact sets*, J. d'Analyse Mathematique, 106 (2008), 373–394.

[28] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, Annals of Mathematics, 170(2009), 915–939.

[29] D.S. Lubinsky, *Universality Limits for Random Matrices and de Branges Spaces of Entire Functions*, Journal of Functional Analysis, 256(2009), 3688–3729.

[30] D.S. Lubinsky, *Universality in the Bulk holds close to Given Points*, to appear in Journal of Approximation Theory.

[31] D.S. Lubinsky, *A Maximal Function Approach to Christoffel Functions and Nevai's Operators*, to appear in Constructive Approximation.

[32] A. Mate, P. Nevai, V. Totik, *Szegő's Extremum Problem on the Unit Circle*, Annals of Mathematics, 134(1991), 433–453.

[33] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights*, Intern. Math. Res. Papers, 2006, Article ID 48673, pages 1–77.

[34] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights*, Int. Math. Res. Not. IMRN 2008, Art. ID rnn 075, 66 pp.

[35] M.L. Mehta, *Random Matrices*, 2nd edn., Academic Press, Boston, 1991.

[36] P. Nevai, *Orthogonal Polynomials*, Memoirs of the AMS no. 213 (1979).

[37] L. Pastur and M. Shcherbina, *Universality of the local eigenvalue statistics for a class of Unitary Invariant Random Matrix Ensembles*, J. Statistical Physics, 86(1997), 109–147.

[38] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995.

[39] W. Rudin, Real and Complex Analysis, Wiley, McGraw Hill, Singapore, 1987.

[40] E.B. Saff and V. Totik, *Logarithmic Potentials with External Fields,* Springer, New York, 1997.

[41] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.

[42] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, J. d'Analyse Mathematique, 105 (2008), 345–362.

[43] B. Simon, *The Christoffel-Darboux Kernel*, (in) Perspectives in PDE, Harmonic Analysis and Applications, a volume in honor of V.G. Maz'ya's 70th birthday, Proceedings of Symposia in Pure Mathematics, 79 (2008), 295–335.

[44] A. Soshnikov, *Universality at the Edge of the Spectrum in Wigner Random Matrices,* Comm. Math. Phys., 207(1999), 697–733.

[45] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[46] F. Stenger, *Numerical Methods Based on Sinc and Analytic Function*s, Springer, New York, 1993.

[47] T. Tao, V. Vu, *From the Littlewood-Offord Problem to the Circular Law: Universality of the Spectral Distribution of Random Matrices*, Bull. Amer. Math. Soc., 46 (2009), 377–396.

[48] T. Tao, V. Vu, Random Covariance Matrices: Universality of Local Statisics of Eigenvalues, manuscript.

[49] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d' Analyse de Mathematique, 81(2000), 283–303.

[50] V. Totik, *Universality and fine zero spacing on general sets*, Arkiv för Matematik, 47(2009), 361–391.

[51] V. Totik, *Christoffel Functions on Curves and Domains,* Trans. Amer. Math. Soc. 362 (2010), 2053–2087.

[52] C.A. Tracy and H. Widom, *Level-Spacing Distributions of the Airy Kernel,* Comm. Math. Phys., 159(1994), 151–174.

DORON S. LUBINSKY

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160 USA, lubinsky@math.gatech.edu