UNIVERSALITY LIMITS AT THE SOFT EDGE OF THE SPECTRUM VIA CLASSICAL COMPLEX ANALYSIS

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ABSTRACT. We show that universality at the soft edge of the spectrum is equivalent to universality "along the diagonal", that is ratio asymptotics for Christoffel functions. The context is that of varying measures and limits involving the Airy kernel. In particular, we consider measures of the form $W_n^{2n}(x)\,dx$, where $\{W_n\}$ are a suitable sequence of weights. They do not need to be analytic, but instead should satisfy some hypotheses on the associated equilibrium measures.

1. INTRODUCTION AND RESULTS[1]

For $n \geq 1$, let μ_n be a finite positive Borel measure with support supp$[\mu_n]$ and infinitely many points in the support. If the support of μ_n is unbounded, we assume that all the power moments

$$\int x^j d\mu_n(x), \; j \geq 0,$$

are finite. Then we may define orthonormal polynomials

$$p_{n,m}(x) = \gamma_{n,m} x^m + ..., \gamma_{n,m} > 0,$$

$m = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_{n,j} p_{n,k} d\mu_n = \delta_{jk}.$$

Throughout we use μ_n' to denote the Radon-Nikodym derivative of μ_n.

The nth reproducing kernel for μ_n is denoted by

$$K_n(x, y) = \sum_{k=0}^{n-1} p_{n,k}(x)\, p_{n,k}(y). \tag{1.1}$$

The Christoffel-Darboux formula asserts that

$$K_n(x, y) = \frac{\gamma_{n,n-1}}{\gamma_{n,n}} \frac{p_{n,n}(x)\, p_{n,n-1}(y) - p_{n,n-1}(x)\, p_{n,n}(y)}{x - y}. \tag{1.2}$$

In random matrix theory, it is the normalized kernel

$$\widetilde{K}_n(x, y) = \mu_n'(x)^{1/2} \mu_n'(y)^{1/2} K_n(x, y) \tag{1.3}$$

Date: 2 August, 2010.

[1]Research supported by NSF grant DMS1001182 and US-Israel BSF grant 2008399

that is often more important than its non-normalized cousin. The simplest case of the universality law in the bulk is the limit

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n(\xi,\xi)} = \frac{\sin \pi (a-b)}{\pi (a-b)}.$$

Typically this holds uniformly for ξ in a compact subinterval of the interior of the support, and a, b in compact subsets of the real line. Of course, when $a = b$, we interpret $\frac{\sin \pi(a-b)}{\pi(a-b)}$ as 1. There is a large literature on this subject - some references may be found in [2], [4], [6], [7], [8] [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [31], [32], [38] and the proceedings devoted to the 60th birthday of Percy Deift [3].

In [20] a new approach was presented for proving universality for fixed measures on a compact set, using classical complex analysis, especially the theory of entire functions of exponential type. In [17], this method was used to prove universality in the bulk for varying weights, and hence also fixed exponential weights. The method has been adopted by Avila, Last and Simon [2], together with other ideas, to show universality in an ergodic setting.

The hypotheses in [20] involved the nth Christoffel function for μ_n, namely,

$$\lambda_n(x) = \lambda_n(\mu_n, x) = 1/K_n(x,x).$$

When μ_n is absolutely continuous, we shall use also the notation $\lambda_n(\mu_n', x)$. The Christoffel function admits the well known extremal property

$$\lambda_n(x) = \inf_{\deg(P)\leq n-1} \frac{\int P^2(t)\, d\mu_n(t)}{P^2(x)}.$$

To state one of the results from [17], we need some concepts from potential theory for external fields [30]. Let Σ be a closed set on the real line, and

$$W(x) = \exp(-Q(x))$$

be a continuous function on Σ. If Σ is unbounded, we assume that

$$\lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0. \tag{1.4}$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu}\left(\int\int \log\frac{1}{|x-t|} d\nu(x)\, d\nu(t) + 2\int Q\, d\nu\right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ν_W, characterized by the following conditions: let

$$V^{\nu_W}(z) = \int \log\frac{1}{|z-t|} d\nu_W(t)$$

denote the potential for ν_W. Then

$$\begin{aligned} V^{\nu_W} + Q &\geq c_W \text{ on } \Sigma; \\ V^{\nu_W} + Q &= c_W \text{ in supp}\,[\nu_W]\,. \end{aligned}$$

Here c_W is a characteristic constant. Usually ν_W is denoted μ_W, but we use a different symbol to avoid confusion with our measures of orthogonality $\{\mu_n\}$. One of the main results from [17] is:

Theorem 1.1
Let $W = e^{-Q}$ be a continuous non-negative function on the set Σ, which is assumed to consist of at most finitely many intervals. If Σ is unbounded, we assume also (1.4). Let h be a bounded positive continuous function on Σ, and for $n \geq 1$, let

$$d\mu_n(x) = \left(hW^{2n}\right)(x)\,dx.$$

Moreover, let $\tilde{K}_n$ denote the normalized nth reproducing kernel for μ_n.

Let J be a closed interval lying in the interior of supp$[\nu_W]$, where ν_W denotes the equilibrium measure for W. Assume that ν_W is absolutely continuous in a neighborhood of J, and that ν_W' and Q' are continuous in that neighborhood. Then uniformly for $\xi \in J$, and a, b in compact subsets of the real line, we have

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n(\xi,\xi)} = \frac{\sin \pi(a-b)}{\pi(a-b)}.$$

There was also a more general result in [17] for measures that are locally of the form $W_n^{2n}dx$.

In this paper, we consider asymptotics at the "soft" edge of the spectrum. For the classical Hermite weight $W(x) = \exp\left(-x^2\right)$, these take the form [39, p. 152]

$$\lim_{n\to\infty} \frac{1}{\sqrt{2}n^{1/6}} \tilde{K}_n\left(\sqrt{2n}\left(1 + \frac{a}{2n^{2/3}}\right), \sqrt{2n}\left(1 + \frac{b}{2n^{2/3}}\right)\right) = \mathbb{A}i(a,b), \tag{1.5}$$

and for the scaled (or contracted) Hermite weight $W^{2n}(x) = \exp\left(-2nx^2\right)$, these take the form

$$\lim_{n\to\infty} \frac{1}{2n^{2/3}} \tilde{K}_n\left(1 + \frac{a}{2n^{2/3}}, 1 + \frac{b}{2n^{2/3}}\right) = \mathbb{A}i(a,b), \tag{1.6}$$

where $\mathbb{A}i(\cdot,\cdot)$ is the Airy kernel, defined by

$$\mathbb{A}i(a,b) = \begin{cases} \frac{Ai(a)Ai'(b) - Ai'(a)Ai(b)}{a-b}, & a \neq b, \\ Ai'(a)^2 - aAi(a)^2, & a = b, \end{cases} \tag{1.7}$$

and Ai is the Airy function, defined on the real line by [28, p. 53]

$$Ai\left(x\right)=\frac{1}{\pi}\int_{0}^{\infty}\cos\left(\frac{1}{3}t^{3}+xt\right)dt. \tag{1.8}$$

The Airy function satisfies the differential equation

$$Ai''\left(z\right)-zAi\left(z\right)=0. \tag{1.9}$$

For $a=b=0$, (1.6) gives

$$\lim_{n\to\infty}\frac{1}{2n^{2/3}}\tilde{K}_{n}\left(1,1\right)=\mathbb{A}i\left(0,0\right), \tag{1.10}$$

so we may reformulate (1.6) as

$$\lim_{n\to\infty}\frac{\tilde{K}_{n}\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}a,1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}b\right)}{\tilde{K}_{n}\left(1,1\right)}=\frac{\mathbb{A}i\left(a,b\right)}{\mathbb{A}i\left(0,0\right)}. \tag{1.11}$$

It is this formulation of the universality limit that we seek to generalize in this paper. This limit has been established (with slightly different formulations) for varying exponential weights, using the Riemann-Hilbert method and $\overline{\partial}$ techniques by Miller and McLaughlin for a general class of non-analytic varying weights [25].

An important special case of our results is:

Theorem 1.2
Assume that for $n\geq 1$,

$$d\mu_{n}=W^{2n}\left(x\right)dx,\ x\in\mathbb{R}, \tag{1.12}$$

where $W=e^{-Q}$ and Q has the following properties:

(a) Q' satisfies a Lipschitz condition of some positive order on $\mathbb{R}$; and in some neighborhoods of ± 1, Q' satisfies a Lipschitz condition of order $>\frac{1}{2}$.
(b) The support of the equilibrium measure ν_W for Q, is $[-1,1]$.
(c) Q is convex on $\mathbb{R}$, or, Q is even and $xQ'(x)$ is strictly increasing on $(0,\infty)$.
Then the following are equivalent:
(I) For each real a,

$$\lim_{n\to\infty}\frac{\tilde{K}_{n}\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}a,1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}a\right)}{\tilde{K}_{n}\left(1,1\right)}=\frac{\mathbb{A}i\left(a,a\right)}{\mathbb{A}i\left(0,0\right)}. \tag{1.13}$$

(II) Uniformly for a,b in compact subsets of the real line,

$$\lim_{n\to\infty}\frac{\tilde{K}_{n}\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}a,1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_{n}(1,1)}b\right)}{\tilde{K}_{n}\left(1,1\right)}=\frac{\mathbb{A}i\left(a,b\right)}{\mathbb{A}i\left(0,0\right)}. \tag{1.14}$$

Remarks
(a) The result is useful because of the extremal property of the Christoffel function $\lambda_n(x)=1/K_n(x,x)$, which makes it much easier to establish the

asymptotic (1.13) along the diagonal, than the off-diagonal (1.14). To date, the most general class for which (1.13) and (1.14) have been established are the varying exponential weights of McLaughlin and Miller [25]. They used Riemann-Hilbert techniques, and the $\bar{\partial}$ method to treat non-anaytic weights, more specifically, the case where Q'' satisfies a Lipschitz condition of some positive order, together with some other conditions on the equilibrium measures. It seems likely that more elementary methods can be used to establish (1.13), as was done in the bulk by Totik [37]. However, this has not been achieved up till this time.
(b) Our proof shows that uniformly for u, v in compact subsets of the plane, (1.13) implies

$$\lim_{n\to\infty} \frac{K_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}v\right)}{K_n(1,1)} e^{-\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}nQ_n'(1)(u+v)}$$

$$= \frac{\mathbb{A}i(u,v)}{\mathbb{A}i(0,0)}. \tag{1.15}$$

Thus universality along the diagonal implies universality locally uniformly in the plane, not just on the real line.
(c) Since we are proving universality at 1, the hypothesis that Q' also satisfies a Lipschitz condition of order $> \frac{1}{2}$ at -1, can be dropped, at the expense of longer proofs.
(d) The convexity of Q or monotonicity of $xQ'(x)$ are needed primarily to ensure that the support of the equilibrium measure for W^λ is an interval for $\lambda > 1$ but sufficiently close to 1. There are more general conditions for this, due to Benko [5]. However, they do not seem to automatically guarantee properties that we require of the equilibrium density.
(e) Universality at the soft edge is also commonly studied in a different context, namely that of random Hermitian matrices with independently distributed entries, by numerous authors. See for example the work of Soshnikov, Tao, Erdős, and others [9], [33], [34], [35]. By contrast, in the context of this paper, the entries are not independently distributed.
(f) We note that

$$\mathbb{A}i(0,0) = \left(3^{1/3}\Gamma(1/3)\right)^{-2}.$$

This follows directly from (1.7) and a differentiation in (1.8):

$$\mathbb{A}i(0,0) = \left(Ai'(0)\right)^2 = \left(-\frac{1}{\pi}\int_0^\infty t\sin\left(\frac{t^3}{3}\right)dt\right)^2.$$

Now apply [12, p. 420, 3.761.4].

Recall that the modulus of continuity $\omega(\psi,\cdot)$ of a continuous function $\psi:[-1,1]\to\mathbb{R}$ is defined by

$$\omega(\psi,\delta) = \sup\{|\psi(s)-\psi(t)| : |s-t|\le\delta\}, \ \delta>0.$$

For sequences $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that for all n,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

We say a sequence of measures $\{\mu_n\}$ *admits a restricted range inequality to* $[-1,1]$ if there exists $A>0$ such that for $n \geq 1$ and all polynomials P of degree $\leq n-1$,

$$\int_{\mathbb{R}\setminus[-1,1]} P^2 d\mu_n \leq A \int_{-1}^{1} P^2 d\mu_n. \tag{1.16}$$

Theorem 1.2 is a special case of :

Theorem 1.3
Let $\{\mu_n\}$ be a sequence of positive Borel measures on the real line, each with all power moments finite, and that admits a restricted range inequality to $[-1,1]$. Assume that
(a) for some $\varepsilon > 0$, $\ell_0 \geq 1$, and all $n \geq 1$,

$$p_{n,n} \text{ has at most } \ell_0 \text{ zeros in } (1, 1+\varepsilon). \tag{1.17}$$

(b) in $\left[-1, 1+\frac{\log n}{n^{2/3}}\right]$,

$$d\mu_n = W_n^{2n}(x)\, dx, \tag{1.18}$$

where $W_n = e^{-Q_n}$ is continuous there;
(c) Let ν_n denote the equilibrium measure for the restriction of Q_n to $[-1,1]$. Assume that

$$\nu_n'(t) = \left(1-t^2\right)^{1/2} \psi_n(t),\ t \in [-1,1], \tag{1.19}$$

where for some $\Delta > 0$,

$$\lim_{n\to\infty} \psi_n(1) = \Delta, \tag{1.20}$$

and the moduli of continuity $\omega(\psi_n, \cdot)$ of ψ_n in $[-1,1]$ are such that the integrals

$$\int_0^4 \frac{\omega(\psi_n, t)}{t} dt \tag{1.21}$$

converge uniformly in n, and are uniformly bounded. Assume also that uniformly on $[-1,1]$ and in n,

$$\psi_n \sim 1. \tag{1.22}$$

(d) Assume moreover that there exists $\eta > 0$ such that the support of the equilibrium measure of W_n^λ is an interval for $n \geq 1$ and $\lambda \in [1, 1+\eta]$.
(e) Q_n' exists in $\left[1, 1 + \frac{\log n}{n^{2/3}}\right]$, satisfying there, uniformly in n,

$$Q_n'(x) - Q_n'(1) = o\left((1-x)^{1/2}\right). \tag{1.23}$$

Then (I) and (II) in Theorem 1.2 are equivalent.

Remarks
(a) The restricted range inequality (1.16) forces $[-1, 1]$ to be the "main support interval" for $\{\mu_n\}$. By primarily technical changes in the proofs, we can assume that the "main support" is a set J, consisting of finitely many disjoint intervals, one of which has 1 as a right endpoint. Instead of (1.16), we would assume there exists $A > 0$ such that for $n \geq 1$ and all polynomials P of degree $\leq n - 1$,

$$\int_{\mathbb{R}\setminus J} P^2 d\mu_n \leq A \int_J P^2 d\mu_n.$$

(b) We emphasize that ν_n is the equilibrium measure for Q_n restricted to $[-1, 1]$. We are not assuming that ν_n is the equilibrium measure of μ_n (if μ_n has one), nor of Q_n outside this interval.
(c) The hypothesis (1.17) is typically satisfied when $d\mu_n = W_n^{2n} dx$ in $(-\infty, \infty)$, and ν_n is the equilibrium measure for Q_n on $(-1, 1)$, for then one expects the zeros of $\{p_{n,n}\}_n$ to be concentrated in $(-1, 1)$. This notion can be made precise, if we assume more precise restricted range inequalities than (1.16).
(d) The hypothesis (d) on the support of W_n^λ is required only in establishing upper bounds for $\lambda_n\left(W_n^{2n}, 1\right)$. Thus it can be replaced by the more implicit assumption

$$\lambda_n\left(W_n^{2n}, 1\right)/W_n^{2n}(1) \leq Cn^{-2/3},\ n \geq 1. \tag{1.24}$$

(e) The sequence $\left\{\frac{\log n}{n^{2/3}}\right\}$ can be replaced by any sequence $\left\{\frac{\xi_n}{n^{2/3}}\right\}$ with $\lim_{n\to\infty} \xi_n = \infty$.
(f) As an application of the above results, one can prove universality at the soft edge for fixed weights. Let $K_n\left(W^2, x, y\right)$ denote the reproducing kernel for W^2 and $\tilde{K}_n\left(W^2, x, y\right) = W(x) W(y) K_n\left(W^2, x, y\right)$ denote its normalized cousin. The nth Mhaskar-Rakhmanov-Saff number a_n for an even weight $W^2 = e^{-2Q}$ is the positive root of the equation

$$n = \frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}} dt.$$

Recall that $\pm a_n$ constitute the "soft edge" of the spectrum for fixed weights. Let $\lambda \geq 6$ and

$$Q(x) = |x|^\lambda + g(x),\ x \in \mathbb{R},$$

where g is even and g' is continuous. Assume that $xQ'(x)$ is increasing on $(0,\infty)$, satisfying there

$$C_2 \geq \frac{xQ'(x)}{Q(x)} \geq C_1 > 1.$$

Assume moreover, that

$$g'(x) = o(x), \ x \to \infty;$$

and for some $\alpha > \frac{1}{2}$,

$$\left|g'(x) - g'(y)\right| \leq C|x-y|^{\alpha}, \text{ for all } x, y.$$

Then one can show that uniformly for a, b in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(W^2, a_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a\right), a_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}b\right)\right)}{\tilde{K}_n\left(W^2, a_n, a_n\right)} = \frac{\mathbb{A}i(a,b)}{\mathbb{A}i(0,0)}.$$

This can be achieved by applying Theorem 1.3 to the weights $W_n(x) = \exp\left(-\frac{1}{n}Q(a_n x)\right)$, $n \geq 1$.

This paper is organized as follows: in Section 2, we present the main ideas of proof. In Section 3, we present some notation, as well as some background on the Airy function and orthogonal polynomials. In Section 4, we establish some technical estimates on Q_n', which follow from our hypotheses on the equilibrium densities. In Sections 5 and 6, we establish lower and upper bounds on Christoffel functions. In Section 7, we prove a Markov-Bernstein inequality. In Section 8, we establish bounds on weighted polynomials and deduce normality of certain sequences of functions. In Section 9, we prove Theorems 1.2 and 1.3.

2. Ideas of Proof

In this section, we shall give the ideas of proof of Theorem 1.3, our most general result. Let us assume its hypotheses.

Step 1: A normal family

Let

$$\Psi(n) = -\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} n Q_n'(1).$$

We shall show that $|\Psi(n)| \geq Cn^{1/3}$. Define for all complex u, v,

$$f_n(u,v) = \frac{K_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}v\right)}{K_n(1,1)} e^{\Psi(n)(u+v)}.$$

A simple asymptotic estimate, which explains the presence of $\Psi(n)$, is the limit

$$\frac{W_n^{2n}\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a\right)}{W_n^{2n}(1)} = \exp\left(2\Psi(n)a + o(1)\right),$$

valid for real a. This allows us to write for real a, b,

$$f_n(a,b) = \frac{\tilde{K}_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}b\right)}{\tilde{K}_n(1,1)}(1+o(1)).$$

We show that f_n admits the bound

$$\begin{aligned} & |f_n(u,v)| \\ (2.1) \leq \quad & C\left|\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right)\right|^{1/2} \exp\left(-\frac{2}{3}\sigma_n^{3/2}\operatorname{Re}\left(u^{3/2}+v^{3/2}\right)\right). \end{aligned}$$

Here $\{\sigma_n\}$ is a bounded sequence, and like C, it is independent of u, v. However, for u, v in a given compact set $\mathcal{K}$, the bound holds for $n \geq n_0(\mathcal{K})$. The actual mechanism to establish this bound is non-trivial. We first need upper bounds for $K_n(a,b)$ for a, b close to 1, and lower bounds for $K_n(1,1)$. To obtain these, we use the extremal property of the Christoffel function

$$\lambda_n(\mu_n, x) = \frac{1}{K_n(x,x)},$$

for μ_n, namely

$$\lambda_n(\mu_n, x) = \inf_{\deg(P)\leq n-1} \frac{\int P^2 d\mu_n}{P^2(x)}.$$

To establish the lower bound for $\lambda_n(\mu_n, x)$, we use potential theoretic ideas in Section 5. The upper bound for $\lambda_n(\mu_n, 1)$ is established in section 6, by discretizing the integral equation for the equilibrium measure. It is there that we need our most stringent conditions on μ_n.

Once we have the lower bounds for $\lambda_n(\mu_n, x)$ for real x, we use weighted Bernstein-Walsh inequalities to move into the complex plane. This is done in Section 8. Finally, in Section 9, we can establish (2.1).

Step 2: The subsequential limit f

The bound (2.1) shows that $\{f_n\}$ is a normal family in each variable. Let f denote the limit of some subsequence $\{f_n\}_{n\in\mathcal{S}}$. For some appropriate $\sigma > 0$, and all u, v, it satisfies the bound

$$\begin{aligned} & |f(u,v)| \\ \leq \quad & C\left|\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right)\right|^{1/2} \exp\left(-\frac{2}{3}\sigma^{3/2}\operatorname{Re}\left(u^{3/2}+v^{3/2}\right)\right). \end{aligned}$$

It is entire in each variable, with order at most $\frac{3}{2}$ and type at most $\frac{2}{3}\sigma^{\frac{3}{2}}$ in each variable. From elementary properties of the reproducing kernel K_n, and scaling, and taking limits, we can show that for all $a \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(a,y)|^2 dy \leq \frac{1}{\mathbb{A}i(0,0)} f(a,\bar{a}).$$

Next, we establish that for all $b \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \left(\frac{f(b/\sigma, s/\sigma)}{f(b/\sigma, b/\sigma)} - \frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)} \right)^2 ds$$
$$\leq \frac{\sigma}{\mathbb{A}i(0,0) f(b/\sigma, b/\sigma)} - \frac{1}{\mathbb{A}i(b,b)}. \tag{2.2}$$

In order to do this, we use the fact that $\mathbb{A}i(b,s)$ is the reproducing kernel for a suitable space of entire functions of order $\leq 3/2$, that satisfy additional restrictions. That f itself belongs to this space, requires some series estimates on its zeros, and the Markov-Bernstein inequality established in Section 7. As a consequence of (2.2), we obtain

$$\sigma \geq \sup_{b \in \mathbb{R}} \frac{\mathbb{A}i(0,0) f(b/\sigma, b/\sigma)}{\mathbb{A}i(b,b)} \geq f(0,0) = 1. \tag{2.3}$$

Step 3 Use of the Markov-Stieltjes Inequalities
For the converse inequality to (2.3), we use Markov-Stieltjes inequalities, and classical formulae relating orders of entire functions and their zero distribution, to obtain

$$\sigma = 1.$$

This is achieved in Section 9. Then (2.2) becomes

$$\int_{-\infty}^{\infty} \left(\frac{f(b,s)}{f(b,b)} - \frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)} \right)^2 ds$$
$$\leq \frac{1}{\mathbb{A}i(0,0) f(b,b)} - \frac{1}{\mathbb{A}i(b,b)}. \tag{2.4}$$

Our hypothesis (1.13) implies that

$$f(b,b) = \lim_{n \to \infty, n \in \mathcal{S}} f_n(b,b) = \frac{\mathbb{A}i(b,b)}{\mathbb{A}i(0,0)},$$

so the right-hand side in (2.4) is 0, and thus

$$\frac{f(b,s)}{f(b,b)} = \frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)}.$$

As the limit is independent of the subsequence $\mathcal{S}$, the result follows.

3. Notation and Background

In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, y, s, t. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. As noted above, we use $\sim$ in the following sense: given real sequences $\{c_n\}$, $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 with

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

3.1. **Orthogonal Polynomials and Gauss Quadratures.** Throughout, $\{\mu_n\}$ denotes a sequence of finite positive Borel measures on the real line, each having all finite power moments. The Radon-Nikodym derivative of μ_n is denoted μ_n'. The corresponding orthonormal polynomials are denoted by $\{p_{n,k}\}_{k=0}^{\infty}$, so that

$$\int p_{n,k} p_{n,j} d\mu_n = \delta_{jk}.$$

We denote the zeros of $p_{n,n}$ by

$$x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n}. \tag{3.1}$$

We assume that

$$d\mu_n(x) = W_n^{2n}(x)\,dx \text{ in } \left[-1, 1 + \frac{\log n}{n^{2/3}}\right],$$

where

$$W_n(x) = e^{-Q_n(x)}$$

and Q_n is continuous on $\left[-1, 1 + \frac{\log n}{n^{2/3}}\right]$.

The nth reproducing kernel for μ_n is denoted by $K_n(x,t)$, and is defined by (1.1), while the normalized reproducing kernel is defined by (1.3). The nth Christoffel function for μ_n is

$$\lambda_n(x) = \lambda_n(\mu_n, x) = 1/K_n(x,x) = \inf_{\deg(P) \le n-1} \frac{\int P^2 d\mu_n}{P^2(x)}. \tag{3.2}$$

When μ_n is absolutely continuous, we shall often write $\lambda_n(\mu_n', x)$. In particular, $\lambda_n\left(W_n^{2n}, x\right)$ will denote the nth Christoffel function for the weight W_n^{2n}.

The Gauss quadrature formula asserts that whenever P is a polynomial of degree $\le 2n-1$,

$$\sum_{j=1}^{n} \lambda_n(x_{jn}) P(x_{jn}) = \int P \; d\mu_n. \tag{3.3}$$

In addition to this, we shall need another Gauss type of quadrature formula [10, p. 19 ff.]. Given a real number ξ, there are n or $n-1$ points $t_{jn} = t_{jn}(\xi)$, one of which is ξ, such that

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P \; d\mu_n, \tag{3.4}$$

whenever P is a polynomial of degree $\le 2n-3$. The $\{t_{jn}\}$ are zeros of

$$\psi_n(\xi, t) = p_{n,n}(\xi)\, p_{n,n-1}(t) - p_{n,n-1}(\xi)\, p_{n,n}(t), \tag{3.5}$$

regarded as a function of t.

In order to prove that universality holds at the edge of the spectrum, at the nth stage, we shall consider the quadrature that includes 1, so that

$$t_{jn} = t_{jn}(1) \text{ for all } j, \tag{3.6}$$

and $\{t_{jn}\}$ are the roots of

$$\psi_n(1,t) = p_{n,n}(1)\, p_{n,n-1}(t) - p_{n,n-1}(1)\, p_{n,n}(t). \tag{3.7}$$

Because we wish to focus on 1 as a right endpoint, we shall set $t_{0n} = 1$, and order the $\{t_{jn}\}$ around 1, as follows:

$$... < t_{2,n} < t_{1,n} < t_{0n} = 1 < t_{-1n} < ... .$$

The sequence of $\{t_{jn}\}$ consists of either $n-1$ or n points, so terminates. It is known [10, p. 19, proof of Theorem 3.1] that when $(p_{n,n}p_{n,n-1})(1) \neq 0$, then one zero of $\psi_n(1,t)$ lies in $(x_{jn}, x_{j-1,n})$ for each j, and the remaining zero lies outside (x_{nn}, x_{1n}).

We let

$$\Psi(n) = -\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} n Q_n'(1) \tag{3.8}$$

and

$$f_n(u,v) = \frac{K_n\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u, 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}v\right)}{K_n(1,1)} e^{\Psi(n)(u+v)}, \tag{3.9}$$

for u, $v \in \mathbb{C}$. We also let

$$\tau_n = 2n^{2/3}\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}. \tag{3.10}$$

(For the scaled Hermite weight $W_n^{2n}(x) = \exp\left(-2nx^2\right)$, $\tau_n \to 1$ as $n \to \infty$). The zeros of

$$f_n(0,t) = \frac{K_n\left(1, 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}t\right)}{K_n(1,1)} e^{\Psi(n)t}$$

will be denote by $\left\{\rho_{jn}\right\}_{j\neq 0}$. Thus, recalling (3.6) and (3.7), we have

$$\rho_{jn} = \frac{\tilde{K}_n(1,1)}{\mathbb{A}i(0,0)}(t_{jn} - 1). \tag{3.11}$$

We also set, corresponding to $t_{0n} = 1$,

$$\rho_{0n} = 0. \tag{3.12}$$

For an appropriate subsequence $\mathcal{S}$ of integers, we shall prove that there exists for all complex a, b,

$$f(a,b) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,b); \tag{3.13}$$

$$\tau = \lim_{n\to\infty, n\in\mathcal{S}} \tau_n. \tag{3.14}$$

We let

$$\sigma = \left(\sqrt{2}\pi\Delta\right)^{2/3} \frac{\tau}{2}. \tag{3.15}$$

The negative zeros of $f(0, \cdot)$ will be denoted by $\{\rho_j\}_{j\neq 0}$, and we set $\rho_0 = 0$. Our ordering of zeros is

$$... \leq \rho_3 \leq \rho_2 \leq \rho_1 < \rho_0 = 0.$$

3.2. **Potential Theory.** Throughout,

$$\varphi(z) = z + \sqrt{z^2 - 1} \tag{3.16}$$

denotes the usual conformal map of the exterior of $[-1, 1]$ onto the exterior of the unit ball, while $g(z, u)$ denotes the Green's function for $\bar{\mathbb{C}} \backslash [-1, 1]$ with pole at $u \notin [-1, 1]$. Thus $g(z, u)$ is harmonic in $\mathbb{C} \backslash [-1, 1]$, as a function of z, and has a finite limit at ∞. Moreover, $g(z, u) + \log|z - u|$ is bounded as $z \to u$, and $g(z, u)$ has limit 0 as z approaches any point in $(-1, 1)$ from the upper or lower half plane. It is well known that [29, p. 107, p. 109], [30, p. 122]

$$g(z, u) = \log \left| \frac{1 - \overline{\varphi(u)}\varphi(z)}{\varphi(z) - \varphi(u)} \right|. \tag{3.17}$$

For the external field Q_n, defined on $[-1, 1]$, its equilibrium measure is denoted by ν_n. Thus ν_n is the unique probability measure that minimizes

$$\int \int \log \frac{1}{|x - t|} d\nu(x) \, d\nu(t) + 2 \int Q_n(t) \, d\nu(t)$$

over all probability measures ν with support $[-1, 1]$. We let

$$V^{\nu_n}(z) = \int \log \frac{1}{|z - t|} d\nu_n(t)$$

denote the corresponding equilibrium potential. Our hypotheses will ensure that ν_n is absolutely continuous, and its support is all of $[-1, 1]$, so that

$$V^{\nu_n}(x) + Q_n(x) = c_n, \; x \in [-1, 1]. \tag{3.18}$$

Here c_n is a characteristic constant. In describing estimates for the Christoffel functions $\lambda_n(\mu_n, x)$, we shall use the function

$$\varphi_n(x) = \frac{1}{n} \left(1 - x^2 + n^{-2/3}\right)^{-1/2}. \tag{3.19}$$

This should not be confused with the conformal map φ above.

3.3. **Entire functions and Airy Functions.** Recall that an entire function g is of order $\leq \lambda$ and finite type $\leq \tau$, if for each $\varepsilon > 0$, there exists $C > 0$ such that for $r > 0$,

$$\sup_{|z|=r} |g(z)| \leq C \exp\left((\tau + \varepsilon) r^{\lambda} \right).$$

We let $n(g, r)$ denote the number of zeros of g in the ball center 0, radius r, and $n(g, [a, b])$ denote the number of zeros of g in $[a, b]$. For g entire of order λ, its indicator function is

$$h_g(\theta) = \limsup_{r \to \infty} \frac{\log \left| g\left(re^{i\theta}\right) \right|}{r^{\lambda}}, \ \theta \in [0, 2\pi].$$

We record some background on the Airy function and Airy kernel, defined by (1.8) and (1.9). The Airy function is of order $\frac{3}{2}$ and type $\frac{2}{3}$ and admits the asymptotic expansions [28, p. 103], [40, p. 15]

$$\begin{aligned} Ai(-x) &= \pi^{-1/2} x^{-1/4} \cos\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + o\left(x^{-1/4} \right), x \to \infty; \\ Ai'(-x) &= \pi^{-1/2} x^{1/4} \sin\left(\frac{2}{3} x^{3/2} - \frac{\pi}{4} \right) + o\left(x^{1/4} \right), x \to \infty \end{aligned}$$

(3.20)

so a simple calculation and (1.7) give as $x \to \infty$,

$$\mathbb{A}i(-x, -x) = \frac{x^{1/2}}{\pi} (1 + o(1)). \tag{3.21}$$

In the region $|\arg(z)| \leq \pi - \delta$, with $|z| \to \infty$, [28, p. 116],

$$Ai(z) = \frac{e^{-\frac{2}{3} z^{3/2}}}{2\pi^{1/2} z^{1/4}} \left\{ 1 + O\left(\frac{1}{z^{3/2}} \right) \right\}. \tag{3.22}$$

It is known that the Airy function has only real negative zeros. We let

$$0 > a_1 > a_2 > ...$$

denote these zeros. It is known [1, p. 450], [40, pp. 15-16] that

(3.23)

$$a_j = -\left[3\pi (4j - 1)/8 \right]^{2/3} \left(1 + O\left(\frac{1}{j^2} \right) \right) = -\left(\frac{3\pi j}{2} \right)^{2/3} (1 + o(1)),$$

so

$$a_j - a_{j-1} = -\frac{2}{3} \left(\frac{3\pi}{2} \right)^{2/3} j^{-1/3} (1 + o(1)) = -\pi |a_j|^{-1/2} (1 + o(1)). \tag{3.24}$$

4. Two Technicalities

We use our assumptions on the equilibrium densities $\{\nu_n'\}$ to prove:

Lemma 4.1
Assume the hypotheses of Theorem 1.3.
(a) Q_n' *exists in* $(-1,1]$ *and for* $x \in (-1,1]$,

$$PV\int_{-1}^{1}\frac{\nu_n'(t)}{x-t}dt = Q_n'(x). \tag{4.1}$$

For $x<1$*, the integral is taken in the Cauchy-Principal Value sense.*
(b) Uniformly in n*, as* $x \to 1-$,

$$Q_n'(1) - Q_n'(x) = o\left((1-x)^{1/2}\right). \tag{4.2}$$

We shall use the equilibrium relation

$$V^{\nu_n}(x) + Q_n(x) = c_n, x \in [-1,1].$$

Note that this holds throughout $[-1,1]$, since our hypothesis (1.19) shows that the support of ν_n is all of $[-1,1]$. While the differentiation in (a) is well known, we could not find a reference under our hypotheses, so provide a proof.

Proof of Lemma 4.1(a)
We assume $0 \le x < 1$. The cases $x = 1$ and $x \in (-1,0)$ are similar. Fix $\varepsilon > 0$ and let

$$|h| \le \frac{\varepsilon}{4} < \frac{1}{16}(1-x)^{1/2},$$

and define $\nu_n'(t) = 0$ outside $[-1,1]$. Then the equilibrium relation gives

$$\begin{aligned} &\frac{1}{h}\left[Q_n(x+h) - Q_n(x)\right] - PV\int_{-1}^{1}\frac{\nu_n'(t)}{x-t}dt \\ = &\int_{|x-t|\ge\varepsilon}\left\{\frac{1}{h}\log\left|1+\frac{h}{x-t}\right| - \frac{1}{x-t}\right\}\nu_n'(t)\,dt \\ &+\int_{|x-t|\le\varepsilon}\left\{\frac{1}{h}\log\left|1+\frac{h}{x-t}\right| - \frac{1}{x-t}\right\}\left\{\nu_n'(t)-\nu_n'(x)\right\}dt \\ &+\nu_n'(x)\int_{|x-t|\le\varepsilon}\frac{1}{h}\log\left|1+\frac{h}{x-t}\right|dt \\ = &: T_1 + T_2 + T_3. \end{aligned}$$

Firstly, as $h \to 0+$, the integrand in T_1 converges uniformly to 0 (recall that ν_n' is continuous), so

$$T_1 \to 0 \text{ as } h \to 0.$$

Next, our hypothesis (1.22) gives

$$\sup_{n\ge1}\|\psi_n\|_{L_\infty[-1,1]} < \infty. \tag{4.3}$$

Let

$$g(s)=s\log\left|1-\frac{1}{s}\right|+1,$$

which is bounded in any closed interval excluding 1. We see that

$$\begin{aligned}&\left|\frac{\nu_n'(t)-\nu_n'(x)}{t-x}\right|\\ =&\left|\frac{\left(1-t^2\right)^{1/2}\psi_n(t)-\left(1-x^2\right)^{1/2}\psi_n(x)}{t-x}\right|\\ \le&|\psi_n(t)|\left|\frac{\left(1-t^2\right)^{1/2}-\left(1-x^2\right)^{1/2}}{t-x}\right|+\left(1-x^2\right)^{1/2}\left|\frac{\psi_n(t)-\psi_n(x)}{t-x}\right|\\ \le&C\frac{1}{(1-t^2)^{1/2}+(1-x^2)^{1/2}}+C\frac{\omega(\psi_n;|x-t|)}{|x-t|}.\end{aligned}\tag{4.4}$$

We emphasize that C is independent of $h,n,t,x,$ and ε. Then if $|h|\le\frac{\varepsilon}{4}\le\frac{1}{16}(1-x)$,

$$\begin{aligned}|T_2|=&\left|\int_{|x-t|\le\varepsilon}g\left(\frac{t-x}{h}\right)\frac{\nu_n'(t)-\nu_n'(x)}{t-x}dt\right|\\ \le&C\int_{|x-t|\le\varepsilon}\left|g\left(\frac{t-x}{h}\right)\right|\frac{dt}{(1-t)^{1/2}}+C\int_{|x-t|\le\varepsilon}\left|g\left(\frac{t-x}{h}\right)\right|\frac{\omega(\psi_n;|x-t|)}{|x-t|}dt\\ =&C|h|\int_{|s|\le\varepsilon/|h|}|g(s)|\frac{ds}{(1-x+sh)^{1/2}}+C\int_{|s|\le\varepsilon/|h|}|g(s)|\frac{\omega(\psi_n;|h||s|)}{|s|}ds\\ \le&C|h|\int_{|s|\le\varepsilon/|h|;\ |s-1|\ge\frac12}\frac{ds}{(1-x+sh)^{1/2}}+C|h|\int_{|s|\le\varepsilon/|h|;\ |s-1|<\frac12}|g(s)|\frac{ds}{(1-x+sh)^{1/2}}\\ &+C\int_{|s|\le\varepsilon/|h|;\ |s-1|\ge\frac12}\frac{\omega(\psi_n;|h||s|)}{|s|}ds+C\int_{|s|\le\varepsilon/|h|;\ |s-1|\le\frac12}|g(s)|\frac{\omega(\psi_n;|h||s|)}{|s|}ds\\ \le&C\frac{\varepsilon}{(1-x)^{1/2}}+C\frac{|h|}{(1-x)^{1/2}}\int_{|s-1|\le\frac12}|g(s)|ds\\ &+C\int_0^\varepsilon\frac{\omega(\psi_n;u)}{u}du+C\omega(\psi_n;2|h|)\int_{|s-1|\le\frac12}|g(s)|ds\\ \le&C\frac{\varepsilon}{(1-x)^{1/2}}+C\int_0^\varepsilon\frac{\omega(\psi_n;u)}{u}du+C\omega(\psi_n;2|h|)\int_{2|h|}^{4|h|}\frac{dt}{t}\\ \le&C\frac{\varepsilon}{(1-x)^{1/2}}+C\int_0^\varepsilon\frac{\omega(\psi_n;u)}{u}du.\end{aligned}$$

Finally, a substitution shows that

$$\begin{aligned} |T_3| &= \nu_n'(x)\left|\int_{|s|\le\varepsilon/|h|}\log\left|1-\frac{1}{s}\right|ds\right| \\ &= \nu_n'(x)\left|\int_0^{\varepsilon/|h|}\log\left|1-\frac{1}{s^2}\right|ds\right| \\ &\to \nu_n'(x)\left|\int_0^{\infty}\log\left|1-\frac{1}{s^2}\right|ds\right| = 0, \end{aligned}$$

as $h\to 0$ (cf.[12, p. 560, no. 4.295.12]). Combining the above estimates gives

$$\begin{aligned} &\limsup_{h\to 0}\left|\frac{1}{h}\left[Q_n(x+h)-Q_n(x)\right]-PV\int_{-1}^{1}\frac{\nu_n'(t)}{x-t}dt\right| \\ \le\ & C\frac{\varepsilon}{(1-x)^{1/2}}+C\int_0^{\varepsilon}\frac{\omega(\psi_n;u)}{u}du. \end{aligned}$$

As C is independent of ε, and the left-hand side is independent of ε, we can let $\varepsilon\to 0+$ to obtain the result. ■

Proof of Lemma 4.1(b)

Recall first the well known identity [30, p. 225, (3.20)]

$$PV\int_{-1}^{1}\frac{1}{x-t}\frac{dt}{(1-t^2)^{1/2}}=0,\ x\in(-1,1).$$

This easily implies

$$PV\int_{-1}^{1}\frac{(1-t^2)^{1/2}}{x-t}dt = x\pi,$$

and hence

$$PV\int_{-1}^{1}\left(1-t^2\right)^{1/2}\left\{\frac{1}{x-t}-\frac{1}{1-t}\right\}dt=(x-1)\pi. \tag{4.5}$$

By (a) of the Lemma, and using (4.5),

$$\begin{aligned} &Q_n'(x)-Q_n'(1) \\ =\ & PV\int_{-1}^{1}\left(1-t^2\right)^{1/2}\psi_n(t)\left\{\frac{1}{x-t}-\frac{1}{1-t}\right\}dt \\ =\ & \psi_n(1)(x-1)\pi \\ & +PV\int_{-1}^{1}\left(1-t^2\right)^{1/2}\left\{\psi_n(t)-\psi_n(1)\right\}\left\{\frac{1}{x-t}-\frac{1}{1-t}\right\}dt \\ =\ & :\psi_n(1)(x-1)\pi+T. \qquad (4.6) \end{aligned}$$

Assume $x \in \left[\frac{3}{4}, 1\right]$. We split

$$\begin{aligned} & T \\ = \; & PV \left[\int_{-1}^{1-(1-x)^{1/5}} + \int_{1-(1-x)^{1/5}}^{1-2(1-x)} + \int_{1-2(1-x)}^{1} \right] \\ & \left(1-t^2\right)^{1/2} \left\{ \psi_n(t) - \psi_n(1) \right\} \left\{ \frac{1}{x-t} - \frac{1}{1-t} \right\} dt \\ = \; & : T_1 + T_2 + T_3. \end{aligned} \tag{4.7}$$

Here $t \leq 1 - (1-x)^{1/5} \Rightarrow 1 - t \geq (1-x)^{1/5} \geq 2(1-x)$. Also,

$$|x-t| = |(1-t) - (1-x)| \geq \frac{1}{2}(1-t).$$

Then

$$\begin{aligned} |T_1| & \leq C(1-x) \int_{-1}^{1-(1-x)^{1/5}} \left(1-t^2\right)^{1/2} \frac{|\psi_n(t) - \psi_n(1)|}{(1-t)^2} dt \\ & \leq C(1-x)^{\frac{3}{5}} \left\{ \int_{-1}^{1} (1-t)^{1/2} \psi_n(t)\, dt + \psi_n(1) \right\} \\ & \leq C(1-x)^{\frac{3}{5}}. \end{aligned} \tag{4.8}$$

Next,

$$\begin{aligned} |T_2| & \leq (1-x) \int_{1-(1-x)^{1/5}}^{1-2(1-x)} \left(1-t^2\right)^{1/2} \frac{|\psi_n(t) - \psi_n(1)|}{|x-t|\,|1-t|} dt \\ & \leq C(1-x)\, \omega\left(\psi_n; (1-x)^{1/5}\right) \int_{1-(1-x)^{1/5}}^{1-2(1-x)} \frac{dt}{(1-t)^{3/2}} \end{aligned}$$

$$\begin{aligned} & \leq C(1-x)^{1/2} \omega\left(\psi_n; (1-x)^{1/5}\right) \\ & \leq C(1-x)^{1/2} \int_{(1-x)^{1/5}}^{2(1-x)^{1/5}} \frac{\omega(\psi_n; s)}{s} ds. \end{aligned}$$

(4.9)

Next,

$$
\begin{aligned}
|T_3| &= \left| PV \int_{x-(1-x)}^{x+(1-x)} \left(1-t^2\right)^{1/2} \{\psi_n(t) - \psi_n(1)\} \left\{ \frac{1}{x-t} - \frac{1}{1-t} \right\} dt \right| \\
&= | \int_{x-(1-x)}^{x+(1-x)} \left(1-t^2\right)^{1/2} \frac{\psi_n(t) - \psi_n(x)}{x-t} dt \\
&\quad + \{\psi_n(x) - \psi_n(1)\} \, PV \int_{x-(1-x)}^{x+(1-x)} \frac{\left(1-t^2\right)^{1/2}}{x-t} dt \\
&\quad - \int_{x-(1-x)}^{x+(1-x)} \left(1-t^2\right)^{1/2} \left\{ \frac{\psi_n(t) - \psi_n(1)}{1-t} \right\} dt | \\
&\leq C(1-x)^{1/2} \int_{x-(1-x)}^{x+(1-x)} \left| \frac{\psi_n(t) - \psi_n(x)}{x-t} \right| dt \\
&\quad + |\psi_n(x) - \psi_n(1)| \left| \int_{x-(1-x)}^{x+(1-x)} \frac{\left(1-t^2\right)^{1/2} - \left(1-x^2\right)^{1/2}}{x-t} dt \right| \\
&\quad + C(1-x)^{1/2} \int_{x-(1-x)}^{x+(1-x)} \left| \frac{\psi_n(t) - \psi_n(1)}{t-1} \right| dt \\
&\leq C(1-x)^{1/2} \int_0^{1-x} \frac{\omega(\psi_n; u)}{u} du + C\omega(\psi_n; 1-x)(1-x)^{1/2} \\
&\quad + C(1-x)^{1/2} \int_0^{2(1-x)} \frac{\omega(\psi_n; u)}{u} du \\
&\leq C(1-x)^{1/2} \int_0^{2(1-x)} \frac{\omega(\psi_n; u)}{u} du,
\end{aligned}
$$

much as above. Adding all the estimates for T_1, T_2, T_3, and recalling (4.6) and (4.7), gives

$$
\left|Q_n'(x) - Q_n'(1)\right| \leq C(1-x)^{1/2} \left\{ (1-x)^{1/10} + \int_0^{2(1-x)^{1/5}} \frac{\omega(\psi_n; u)}{u} du \right\}.
$$

Our assumption that the integrals in (1.21) are uniformly convergent, now gives the result. ■

We shall also use:

Lemma 4.2
There exist C_1 and $\varepsilon_1 > 0$ such that

$$
\left|Q_n'(x)\right| \leq C, \; x \in [1-\varepsilon_1, 1]. \tag{4.10}
$$

Proof
Now

$$\begin{aligned} 0 &\leq Q_n'(1) = \int_{-1}^{1} \frac{\nu_n'(t)}{1-t} dt \\ &\leq C \int_{-1}^{1} \frac{\left(1-t^2\right)^{1/2}}{1-t} dt, \end{aligned}$$

by (1.19) and (1.22). The bound (4.10) then follows from the uniformity in Lemma 4.1(b). ■

5. Lower Bounds for Christoffel Functions

In this section, we establish lower bounds for the Christoffel function $\lambda_n(\mu_n, x)$. The method is identical to that in [15, Chapter 9], but the details are sufficently different to require some explanation. We shall prove:

Theorem 5.1
Assume the hypotheses of Theorem 1.3, except (1.21) and (1.22). Uniformly for $n \geq 1$ and $x \in [-1, 1]$,

$$\lambda_n(\mu_n, x) / W_n^{2n}(x) \geq C \frac{1}{n} \left(1 - x^2 + n^{-2/3}\right)^{-1/2}. \tag{5.1}$$

Recall that $g(z, u)$ denotes the Green's function for $\bar{\mathbb{C}} \backslash [-1, 1]$ with pole at $u \notin [-1, 1]$ and $\varphi(z) = z + \sqrt{z^2 - 1}$, and ν_n is the equilibrium measure for Q_n on $[-1, 1]$. We start with an analogue of Lemma 9.6 in [15, p. 260].

Lemma 5.2
Let $x \in (-1, 1)$ and $z = x + iy$, with $y \neq 0$. Then

$$\lambda_n(\mu_n, x) / W_n^{2n}(x) \geq \pi |y| e^{2n[V^{\nu_n}(z) - c_n + Q_n(x)] - \log|\varphi(z)|}. \tag{5.2}$$

Proof
Given a monic polynomial P of degree n, we can write

$$\log |P(z)| = \int \log |z - u| \, d\omega(u),$$

where ω has point masses at the zeros of P, of appropriate order. Let

$$h(z) = \frac{1}{n} \int \{\log |z - u| + g(z, u)\} \, d\omega(u) + V^{\nu_n}(z).$$

As the Green's function "cancels out" the zeros of P, h is harmonic in $\mathbb{C} \backslash [-1, 1]$. It also has a finite limit at ∞, since $\frac{1}{n}\omega$ and ν_n are both probability measures. It thus has a single valued harmonic conjugate $\tilde{h}$ in $\mathbb{C} \backslash [-1, 1]$. Let

$$H = \exp\left(h + i\tilde{h}\right),$$

so that H is analytic in $\mathbb{C}\backslash[-1,1]$, with a finite limit at ∞. By using Cauchy's integral formula on a contour enclosing $[-1,1]$, and shrinking the contour to $[-1,1]$, we obtain for $z\notin[-1,1]$,

$$\frac{H^{2n}(z)}{\varphi(z)}=\frac{1}{2\pi i}\int_{-1}^{1}\frac{\left(H^{2n}/\varphi\right)(x+i0)-\left(H^{2n}/\varphi\right)(x-i0)}{x-z}dx,$$

where $\left(H^{2n}/\varphi\right)(x\pm i0)$ denote boundary values from the upper and lower half planes respectively. Note that for $t\in(-1,1)$, the equilibrium relation (3.18) gives

$$\left|\left(H^{2n}/\varphi\right)(t\pm i0)\right|=\left|PW_n^n\right|^2(t)\,e^{2nc_n},$$

so for $z=x+iy$,

$$\left|\frac{H^{2n}(z)}{\varphi(z)}\right|\leq\frac{e^{2nc_n}}{\pi|y|}\int_{-1}^{1}\left|PW_n^n\right|^2(t)\,dt.$$

Here for $z\notin[-1,1]$,

$$\left|H^{2n}(z)\right|=|P(z)|^2\exp\left(2\int g(z,u)\,d\omega(u)+2nV^{\nu_n}(z)\right).$$

Thus

$$\begin{aligned}&\frac{\int_{-1}^{1}\left|PW_n^n\right|^2(t)\,dt}{|P(x)|^2}\\ \text{(5.3)}\qquad\geq\;&\pi|y|\,e^{2n[V^{\nu_n}(z)-c_n]-\log|\varphi(z)|+2\int g(z,u)d\omega(u)}\left(\frac{|P(z)|}{|P(x)|}\right)^2.\end{aligned}$$

Next, we claim that

$$\begin{aligned}&e^{2\int g(z,u)d\omega(u)}\left(\frac{|P(z)|}{|P(x)|}\right)^2\\ \text{(5.4)}\qquad=\;&\exp\left(2\int\left\{g(z,u)+\log\left|\frac{z-u}{x-u}\right|\right\}d\omega(u)\right)\geq 1.\end{aligned}$$

First observe that for fixed z, $g(z,u)+\log|z-u|$ is harmonic as a function of u, in $\mathbb{C}\backslash[-1,1]$. Next, $\log\frac{1}{|x-u|}$ is superharmonic as a function of $u\in\mathbb{C}\backslash[-1,1]$. So $g(z,u)+\log\left|\frac{z-u}{x-u}\right|$ is superharmonic as a function of u, and has a finite limit as $u\to\infty$. For $u\in(-1,1)$, we have

$$g(z,u)+\log\left|\frac{z-u}{x-u}\right|=0+\log\left|1+\frac{iy}{x-u}\right|\geq 0.$$

By the minimum principle for superharmonic functions, we have

$$g(z,u)+\log\left|\frac{z-u}{x-u}\right|\geq 0\text{ for all }u\in\mathbb{C}\backslash[-1,1].$$

Then (5.4) follows, and (5.3) gives

$$\begin{aligned}\lambda_n(\mu_n, x)/W_n^{2n}(x) &\geq \inf_{\deg(P)\leq n, P \text{ monic}} \frac{\int_{-1}^1 |PW_n^n|^2(t)\,dt}{|P(x)|^2 W_n^{2n}(x)} \\ &\geq \pi|y|\, e^{2n[V^{\nu_n}(z)-c_n+Q_n(x)]-\log|\varphi(z)|}.\end{aligned}$$

■

Proof of Theorem 5.1

Let us assume $x \in [0,1]$. Choose

$$y = \frac{1}{n}\left(1 - x + n^{-2/3}\right)^{-1/2}$$

and $z = x+iy$. Recall the bound (4.3) on ψ_n. Then our equilibrium relations give

$$\begin{aligned}& V^{\nu_n}(z) - c_n + Q_n(x) \\ = \;& V^{\nu_n}(z) - V^{\nu_n}(x) \\ = \;& -\frac{1}{2}\int_{-1}^1 \log\left(1+\left(\frac{y}{x-t}\right)^2\right)\psi_n(t)\left(1-t^2\right)^{1/2} dt \\ \geq \;& -Cy^2\int_{-1}^{-\frac{1}{2}} \psi_n(t)\left(1-t^2\right)^{1/2} dt - C\int_{-\frac{1}{2}}^1 \log\left(1+\left(\frac{y}{x-t}\right)^2\right)\left\{(1-x)^{1/2}+|x-t|^{1/2}\right\} dt \\ \geq \;& -Cy^2 - C(1-x)^{1/2}\, y\int_{-\infty}^{\infty}\log\left(1+\frac{1}{s^2}\right) ds - Cy^{3/2}\int_{-\infty}^{\infty}\log\left(1+\frac{1}{s^2}\right)|s|^{1/2}\, ds.\end{aligned}$$

In the last line, we made the substitution $x-t = ys$. Our choice of y ensures that

$$(1-x)^{1/2}\, y \leq \frac{1}{n} \text{ and } y^{3/2} \leq \frac{1}{n}.$$

Thus for some C independent of n and $x \in [0,1]$,

$$n\left[V^{\nu_n}(z) - c_n + Q_n(x)\right] \geq -C. \tag{5.5}$$

Finally, as $x+iy$ lies in a bounded set independent of n, $\log|\varphi(x+iy)|$ is bounded independent of n. Then Lemma 5.2 gives the result for the range $x \in [0,1]$. The range $x \in [-1,0]$ is similar. ■

6. Upper Bounds on Christoffel Functions

Throughout, we assume the hypotheses of Theorem 1.3. In particular, the equilibrium densities $\{\nu_n\}$ satisfy

$$\nu_n'(t) = \left(1-t^2\right)^{1/2}\psi_n(t),\; t \in (-1,1). \tag{6.1}$$

We shall prove:

Theorem 6.1
Assume the hypotheses of Theorem 1.3. Then

$$\lambda_n\left(\mu_n, 1\right)/W_n^{2n}\left(1\right) \leq Cn^{-2/3}. \tag{6.2}$$

Proof
By the restricted range inequality (1.16), for some $A > 1$, and for all $n \geq 1$, and polynomials P of degree $\leq n-1$,

$$\int_{\mathbb{R}\setminus[-1,1]} P^2 d\mu_n \leq A \int_{-1}^{1} P^2 d\mu_n = A \int_{-1}^{1} P^2 W_n^{2n}.$$

It follows that

$$\lambda_n\left(\mu_n, 1\right) \leq A\lambda_n\left(W_n^{2n}, 1\right). \tag{6.3}$$

Let

$$m = m\left(n\right) = n - \left[n^{1/3}\right].$$

Suppose that for large enough n, we can choose a polynomial R_n of degree $\leq m$ such that

$$\left\|R_n W_n^n\right\|_{L_\infty[-1,1]} \leq C_1 \tag{6.4}$$

and

$$\left(R_n W_n^n\right)\left(1\right) = 1. \tag{6.5}$$

Then

$$\begin{aligned}
\lambda_n\left(W_n^{2n}, 1\right)/W_n^{2n}\left(1\right) &\leq \inf_{\deg(P)\leq\left[n^{1/3}\right]-1} \int_{-1}^{1} P^2 R_n^2 W_n^{2n} / \left(P^2 R_n^2 W_n^{2n}\right)\left(1\right) \\
&\leq C_1^2 \inf_{\deg(P)\leq\left[n^{1/3}\right]-1} \left(\int_{-1}^{1} P^2\right)/P^2\left(1\right) \\
&\leq C_2 n^{-2/3},
\end{aligned}$$

by classical estimates for the Christoffel function for the Legendre weight on $[-1,1]$ [27, p. 108]. Combined with (6.3), this gives the result. ■

The rest of the section is devoted to constructing polynomials R_n with the properties (6.4) and (6.5). To do this we discretize the integral equation arising from the equilibrium potential, a procedure that has been studied by many. Its most refined form is due to Totik [36], and we follow many of his ideas. Recall that

$$\int_{-1}^{1} \log \frac{1}{|x-t|} \nu_n'\left(t\right) dt + Q_n\left(x\right) = c_n,\ x \in \left[-1, 1\right]. \tag{6.6}$$

Unfortunately, since we need a polynomial of degree $m = m\left(n\right) = n - \left[n^{1/3}\right]$, we cannot discretize (6.6), and instead have to consider a slightly different discretization problem. Let

$$\rho_n = \frac{n}{m} = 1 + \frac{\left[n^{1/3}\right]}{n} + O\left(n^{-4/3}\right); \tag{6.7}$$

$$Q_n^{\#}(x) = \rho_n Q_n(x), \; x \in [-1,1]; \tag{6.8}$$

$$W_n^{\#}(x) = e^{-Q_n^{\#}(x)} = W_n^{\rho_n}(x), \; x \in [-1,1]. \tag{6.9}$$

We let $\nu_n^{\#}$ denote the probability measure that is the equilibrium measure for $W_n^{\#}$. If $\mathcal{S}^{\#}$ is its support,

$$V^{\nu_n^{\#}}(x) + Q_n^{\#}(x) = c_n^{\#}, \text{ q.e. } x \in \mathcal{S}^{\#}. \tag{6.10}$$

Here q.e. means quasi-everywhere, that is outside a set of capacity 0. Since $\rho_n > 1$, it is known [30, p. 227, Thm. IV.4.1] that

$$\text{supp}\left[\nu_n^{\#}\right] \subset \text{supp}\left[\nu_n\right] = [-1,1].$$

It is also known [30, p. 236, Thm. IV.4.9] that

$$\nu_{n|\mathcal{S}^{\#}} \leq \frac{1}{\rho_n}\nu_n^{\#} + \left(1 - \frac{1}{\rho_n}\right)\omega_{\mathcal{S}^{\#}}, \tag{6.11}$$

where $\omega_{\mathcal{S}^{\#}}$ denotes the unweighted equilibrium measure for $\mathcal{S}^{\#}$. Moreover,

$$\nu_{n|\mathcal{S}^{\#}} \geq \frac{1}{\rho_n}\nu_n^{\#} + \left(1 - \frac{1}{\rho_n}\right)\omega_{[-1,1]|\mathcal{S}^{\#}}. \tag{6.12}$$

This last inequality forces $\nu_n^{\#}$ to be absolutely continuous.

Lemma 6.2
(a) If meas denotes linear Lebesgue measure, then for some $C > 0$ and all $n \geq 1$,

$$meas\left([-1,1] \setminus \mathcal{S}^{\#}\right) \leq Cn^{-2/3}. \tag{6.13}$$

(*b*) *For large enough n, $S^{\#}$ is an interval, say $[\alpha_n, \beta_n]$. We have*

$$1 + \alpha_n \leq Cn^{-2/3}; 1 - \beta_n \leq Cn^{-2/3}. \tag{6.14}$$

Proof
(a) By (6.12),

$$1 - \nu_n\left(\mathcal{S}^{\#}\right) \leq \left(1 - \frac{1}{\rho_n}\right)\left(1 - \omega_{[-1,1]}\left(\mathcal{S}^{\#}\right)\right).$$

Recall here that

$$\omega'_{[-1,1]}(x) = \frac{1}{\pi\sqrt{1-x^2}}, x \in (-1,1).$$

Using the above inequality, (1.19) and (6.7), we obtain for some C_1,

$$\int_{[-1,1]\setminus\mathcal{S}^{\#}} \left(1-t^2\right)^{1/2} dt \leq C_1 n^{-2/3} \int_{[-1,1]\setminus\mathcal{S}^{\#}} \left(1-t^2\right)^{-1/2} dt.$$

Observe that $|t| \leq 1 - 2C_1 n^{-2/3} \Rightarrow 1 - t^2 \geq 2C_1 n^{-2/3}$. Let $\mathcal{L} = \left\{t : |t| \leq 1 - 2C_1 n^{-2/3}\right\} \cap \left([-1,1] \backslash \mathcal{S}^{\#}\right)$. From the second last inequality,

$$\begin{aligned} & \int_{\mathcal{L}} \left(1-t^2\right)^{1/2} dt \\ \leq \; & C_1 n^{-2/3} \left\{ \int_{\mathcal{L}} \left(1-t^2\right)^{-1/2} dt + 2 \int_{1-2C_1 n^{-2/3}}^{1} \left(1-t^2\right)^{-1/2} dt \right\} \\ \leq \; & \frac{1}{2} \int_{\mathcal{L}} \left(1-t^2\right)^{1/2} dt + C_2 n^{-1}, \end{aligned}$$

by the definition of $\mathcal{L}$. Then

$$\frac{1}{2} \left(2C_1 n^{-2/3}\right)^{1/2} meas\left(\mathcal{L}\right) \leq \frac{1}{2} \int_{\mathcal{L}} \left(1-t^2\right)^{1/2} dt \leq C_2 n^{-1}$$

$$\Rightarrow meas\left(\mathcal{L}\right) \leq C_3 n^{-2/3}.$$

Then (6.13) follows from the definition of $\mathcal{L}$.
(b) Our hypothesis (d) in Theorem 1.3 asserts that the support $\mathcal{S}^{\#}$ of the equilibrium measure $W_n^{\rho_n}$ is an interval for n large enough. The estimates in (a) give (6.14). ■

We partition $\mathcal{S}^{\#} = [\alpha_n, \beta_n]$ for a given n and $m = m(n)$ as

$$\alpha_n = t_0 < t_1 < t_2 < \ldots < t_m = \beta_n,$$

so that

$$I_j := [t_j, t_{j+1}), 0 \leq j \leq m-1 \tag{6.15}$$

satisfies

$$\int_{I_j} d\nu_n^{\#} = \frac{1}{m}, 0 \leq j \leq m-1. \tag{6.16}$$

We follow Totik's idea of using the "weight point" $\xi_j \in I_j$ defined for $0 \leq j \leq m-1$ by

$$\int_{I_j} (t-\xi_j)\nu_n^{\#\prime}(t)dt = 0 \Leftrightarrow \xi_j = m \int_{I_j} t\nu_n^{\#\prime}(t)dt. \tag{6.17}$$

We define

$$Y_n(z) := \prod_{j=0}^{m-1} (z-\xi_j) \tag{6.18}$$

where

$$|I_j| = t_{j+1} - t_j.$$

Our task will be to estimate the quantity

$$\Gamma_n(u) := \log \mid Y_n(u) \mid + mV^{\nu_n^{\#}}(u). \tag{6.19}$$

Since $\int_{\alpha_n}^{\beta_n} d\nu_n^{\#} = \frac{1}{m}$, we may write

$$\Gamma_n(u) = \sum_{j=0}^{m-1} m \int_{I_j} \log \left| \frac{u-\xi_j}{u-t} \right| \nu_n^{\#\prime}(t)\, dt =: \sum_{j=0}^{m-1} \Gamma_{n,j}(u). \tag{6.20}$$

First we record a lower bound for $\Gamma_n(1)$:

Lemma 6.3

$$\Gamma_n(1) \geq 0. \tag{6.21}$$

Proof
We show that for $n \geq 1, 0 \leq j \leq n-1$,

$$\Gamma_{n,j}(1) \geq 0.$$

Let $t \in I_j$. We obtain via a Taylor expansion of $\log(1-t)$ about $t = \xi_j$, to second order,

$$\log \left| \frac{1-\xi_j}{1-t} \right| = \log(1-\xi_j) - [\log\left(1-\xi_j\right) - \frac{t-\xi_j}{1-\xi_j} - \frac{(t-\xi_j)^2}{2(1-s)^2}]$$

where s lies between $\xi_j, 1$. Then multiplying by $m\nu_n^{\#\prime}$ and integrating over I_j gives

$$\Gamma_{n,j}(1) \geq m \int_{I_j} \frac{t-\xi_j}{1-\xi_j} \nu_n^{\#\prime}(t)\, dt + 0 = 0$$

by definition of ξ_j. ■

Next, some inequalities for the discretisation points:

Lemma 6.4
(a) For $0 \leq j \leq m-1$,

$$|I_j| \geq \frac{C}{m}\left(1-t_j^2\right)^{-1/2}. \tag{6.22}$$

In particular, for $j \leq m-1$,

$$1 - t_j^2 \geq Cm^{-2/3}. \tag{6.23}$$

(b) For $0 \leq j \leq m-2$,

$$|I_j| \leq \frac{C}{m}\left(1-t_j^2\right)^{-1/2}. \tag{6.24}$$

(c) For $0 \leq j \geq m-2$,

$$1 - t_j \sim 1 - t_{j+1} \text{ and } |I_j| \sim |I_{j+1}|. \tag{6.25}$$

Proof

(a) Assume $t_j \geq 0$. We have by (6.12),

$$\begin{aligned} \frac{1}{m} &= \int_{t_j}^{t_{j+1}} \nu_n^{\#\prime} \leq \rho_n \int_{t_j}^{t_{j+1}} \nu_n' \\ &\leq C \int_{t_j}^{t_{j+1}} \left(1-t^2\right)^{1/2} dt \\ &\leq C\left(t_{j+1}-t_j\right)\left(1-t_j\right)^{1/2}. \end{aligned} \tag{6.26}$$

In the second last line, we used (1.19) and (4.3). Then also for $j \leq m-1$,

$$\frac{1}{m} \leq C\left(1-t_j\right)^{3/2}$$

so (6.23) follows.

(b) Assume $t_j \geq 0$. By (6.11), in $\left[t_j, t_{j+1}\right]$,

$$\begin{aligned} \nu_n^{\#\prime}(t) &\geq \rho_n \nu_n(t) - \left(\rho_n - 1\right) \omega_{\mathcal{S}^\#}'(t) \\ &\geq \rho_n \left(1-t^2\right)^{1/2} \left[\psi_n(t) - \frac{\rho_n - 1}{\rho \sqrt{1-t^2} \sqrt{\left(t-\alpha_n\right)\left(\beta_n - t\right)}}\right], \end{aligned}$$

recall that $\mathcal{S}^\# = \left[\alpha_n, \beta_n\right]$. Now if

$$I_j \subset \left[-1 + Kn^{-2/3}, 1 - Kn^{-2/3}\right] \tag{6.27}$$

for some large enough K, Lemma 6.2(b) allows us to continue this as

$$\nu_n^{\#\prime}(t) \geq \left(1-t^2\right)^{1/2} \left[\inf_{[-1,1]} \psi_n - \frac{Cn^{-2/3}}{Kn^{-2/3}}\right],$$

where C is independent of n, j, K, and arises from Lemma 6.2(b) and (6.7). It follows that if K is large enough (with some threshhold depending only on this C),

$$\nu_n^{\#\prime\prime}(t) \geq C_1 \left(1-t^2\right)^{1/2}, \; t \in I_j.$$

Here C_1 depends on K, but is independent of n, j, I_j. Then

$$\begin{aligned} \frac{1}{m} &= \int_{t_j}^{t_{j+1}} \nu_n^{\#\prime} \\ &\geq C_1 \int_{t_j}^{t_{j+1}} \left(1-t^2\right)^{1/2} dt \\ &\geq C_2 \left|I_j\right| \left(1-t_j^2\right)^{1/2}, \end{aligned}$$

and we obtain (6.24). If instead (6.27) fails, then again, we must have $I_j \subset \left[1 - Ln^{-2/3}, 1\right]$ for some $L > 0$, and the assertion becomes trivial.

(c) Now

$$\begin{aligned} 1 &\leq \frac{1-t_j}{1-t_{j+1}} \\ &= 1+\frac{t_{j+1}-t_j}{1-t_{j+1}} \\ &\leq 1+\frac{C}{m}\left(1-t_{j+1}\right)^{-3/2} \leq C, \end{aligned}$$

by first (6.24) and then (6.23). So the first assertion in (6.25) follows. The second assertion then follows from (6.22) and (6.24). ■

For a given $u \in (-1,1)$, we choose $j_0 = j_0(n,u)$ such that $u \in I_{j_0}$ and split

$$\Gamma_n(u) = \left[\sum_{j=0}^{j_0-3} + \sum_{j=j_0-2}^{j_0+2} + \sum_{j=j_0+3}^{m-1}\right] \Gamma_{n,j}(u) =: \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 . \tag{6.28}$$

We shall not explicitly display the dependence of j_0 on n and u. When $j_0 < 3$, we drop the first sum and replace the lower limit in the second sum by 0; similarly if $j_0 + 3 > m - 1$. Similarly if $u < \alpha_n$ or $u > \beta_n$, we drop two of the sums, as u is to the left or right of all the I_j. In any of these exceptional cases, we take I_{j_0} to be the closest interval to u, and $|I_{j_0}|$ to be the length of the interval closest to u.

Lemma 6.5
(a) For $n \geq n_0$,

$$\sum\nolimits_1 + \sum\nolimits_3 \leq C, u \in [-1,1] . \tag{6.29}$$

Here $C \neq C(n,u)$.
(b) For $|j - j_0| \leq 2$,

$$\Gamma_{n,j}(u) \leq C. \tag{6.30}$$

Proof
(a) We assume that $u \geq 0$; $u < 0$ is similar. First note that

$$|j - j_0| \geq 2 \Rightarrow \text{dist}\,(u, I_j) \geq C\,|I_j| . \tag{6.31}$$

Here $dist(u, I_j)$ denotes the distance from u to I_j, and this follows from Lemma 6.4(c). Let us assume that $j \leq j_0 - 2$, so that u is to the right of I_j. Then, via a Taylor expansion of $\log(u-t)$ about $t = \xi_j$, to second order,

$$\log\left|\frac{u-\xi_j}{u-t}\right| = \log(u-\xi_j) - [\log\left(u-\xi_j\right) - \frac{t-\xi_j}{u-\xi_j} - \frac{(t-\xi_j)^2}{2(u-s)^2}]$$

where s lies between ξ_j, u. Then multiplying by $m\nu_n^{\#\prime}$ and integrating over I_j gives

$$\begin{aligned}\Gamma_{n,j}(u) &= m\int_{I_j}\frac{t-\xi_j}{u-\xi_j}\nu_n^{\#\prime}(t)\,dt+\frac{m}{2}\int_{I_j}\frac{(t-\xi_j)^2}{(u-s)^2}\nu_n^{\#\prime}(t)\,dt\\ &\leq 0+\frac{|I_j|^2}{2dist\,(u,I_j)^2},\end{aligned}$$

by definition of ξ_j and I_j. It is an easy consequence of Lemma 6.4(c) that for $|\ j-j_0\ |\geq 2$, and $t\in I_j$,

$$dist(u,I_j)\sim|\ u-t\ |\ .$$

Moreover, from (6.23) and (6.24), for $t\in I_j\cap\left[-1+n^{-2/3},1-n^{-2/3}\right]$,

$$|I_j|\leq\frac{C}{n\left(1-|t|+n^{-2/3}\right)^{1/2}}.$$

This inequality also follows for $|t|\geq 1-n^{-2/3}$, since all $|I_j|=O\left(n^{-2/3}\right)$. We deduce that for $|j-j_0|\geq 2$,

$$\Gamma_{n,j}(u)\leq C\int_{I_j}\frac{dt}{n(u-t)^2\left(1-|t|+n^{-2/3}\right)^{1/2}}.$$

If $j=0$ or $m-1$, this inequality persists, with minor modifications to the above proofs. Adding over j gives

$$\begin{aligned}&\sum\nolimits_1+\sum\nolimits_3\\ \leq\ & C\int_{|u-t|\geq|I_{j_0}|}\frac{dt}{n(u-t)^2\left(1-|t|+n^{-2/3}\right)^{1/2}}.\end{aligned}$$

Using the substitution $1-t+n^{-2/3}=\left(1-u+n^{-2/3}\right)s$, we continue this as

$$\begin{aligned}&\leq \frac{C}{n}+\frac{C}{n}\int_{|u-t|\geq|I_{j_0}|,t\in[0,1]}\frac{dt}{n(u-t)^2\left(1-t+n^{-2/3}\right)^{1/2}}\\ &= \frac{C}{n}+\frac{C}{n\left(1-u+n^{-2/3}\right)^{3/2}}\int_{\left\{s:|s-1|\geq|I_{j_0}|/\left(1-u+n^{-2/3}\right)\right\}}\frac{ds}{(s-1)^2\,s^{1/2}}\\ &\leq \frac{C}{n}+\frac{C}{n\left(1-u+n^{-2/3}\right)^{3/2}}\left\{\frac{1-u+n^{-2/3}}{|I_{j_0}|}+1\right\}\leq C,\end{aligned}$$

as $n\left(1-u+n^{-2/3}\right)^{1/2}|I_{j_0}|\geq Cn\left(1-t_j\right)^{1/2}|I_{j_0}|\geq C$ and $n\left(1-u+n^{-2/3}\right)^{3/2}\geq 1$.

(b) Since $\Gamma_{n,j}(u)$ is subharmonic outside I_j and vanishes at ∞, it suffices, by the maximum principle, to prove the upper bound in (6.30) provided $u\in I_j$, that is $j=j_0$. Then $\left|u-\xi_j\right|\le|I_j|$. Therefore by (6.12), (1.19) and (1.22),

$$\begin{aligned}\Gamma_{n,j}(u) &\le m\int_{I_j}\log\left(\frac{|I_j|}{|u-t|}\right)\nu_n^{\#\prime}(t)\,dt\\ &\le m\rho_n\int_{I_j\cap\mathcal{S}^\#}\log\left(\frac{|I_j|}{|u-t|}\right)\nu_n'(t)\,dt\\ &\le Cn\int_{I_j\cap\mathcal{S}^\#}\log\left(\frac{|I_j|}{|u-t|}\right)\left(1-t^2\right)^{1/2}dt\\ &\le Cn\left(1-t_j^2\right)^{1/2}\int_{I_j}\log\left(\frac{|I_j|}{|u-t|}\right)dt\\ &\le Cn\left(1-t_j^2\right)^{1/2}|I_j|\int_{-1}^{1}\log\left(\frac{3}{|s|}\right)ds\le C,\end{aligned}$$

by Lemma 6.4(b) and the substitution $u-t=s\,|I_j|$.■

Proof of (6.4) and (6.5)
Recall (6.18) - (6.20) and Lemma 6.3. These give

$$\left|Y_ne^{mV^{\nu_n^\#}}\right|(1)\ge1, \tag{6.32}$$

while Lemma 6.5(a), (b) give

$$\left|Y_ne^{mV^{\nu_n^\#}}\right|(u)\le C,\ u\in[-1,1]. \tag{6.33}$$

We set

$$R_n(u)=e^{mc_n^\#}Y_n(u),$$

so that

$$\begin{aligned}&|R_n(u)\,W_n^n(u)|\\ =\;&\left|Y_ne^{mV^{\nu_n^\#}}\right|(u)\,e^{-m\left\{V^{\nu_n^\#}+Q_n^\#(u)-c_n^\#\right\}}.\end{aligned} \tag{6.34}$$

If we can show that

$$\left\{V^{\nu_n^\#}+Q_n^\#-c_n^\#\right\}(1)\le\frac{C_1}{m} \tag{6.35}$$

and

$$\left\{V^{\nu_n^\#}+Q_n^\#-c_n^\#\right\}(u)\ge0,\ u\in[-1,1], \tag{6.36}$$

then (6.4) and (6.5) follow from (6.32) - (6.33), apart from a multiplicative constant. Firstly (6.36) follows immediately from the equilibrium conditions for $\nu_n^{\#}$ [30, p. 27]. Next, using the equilibrium relations for Q_n and $Q_n^{\#}$,

$$\begin{aligned}
& \left\{V^{\nu_n^{\#}} + Q_n^{\#} - c_n^{\#}\right\}(1) \\
= & \left\{V^{\nu_n^{\#}} + Q_n^{\#} - c_n^{\#}\right\}(1) - \left\{V^{\nu_n^{\#}} + Q_n^{\#} - c_n^{\#}\right\}(\beta_n) \\
= & V^{\nu_n^{\#}}(1) - V^{\nu_n^{\#}}(\beta_n) + \rho_n \left\{Q_n(1) - Q_n(\beta_n)\right\} \\
= & V^{\nu_n^{\#}}(1) - V^{\nu_n^{\#}}(\beta_n) + \rho_n \left\{V^{\nu_n}(\beta_n) - V^{\nu_n}(1)\right\} \\
= & \rho_n \int \log\left|\frac{\beta_n - t}{1-t}\right| d\left\{\nu_n^{\#}/\rho_n - \nu_n\right\}(t) \\
= & \rho_n \left\{\int_{\alpha_n}^{\beta_n} + \int_{[-1,1]\backslash[\alpha_n,\beta_n]}\right\} \log\left|\frac{\beta_n - t}{1-t}\right| d\left\{\nu_n^{\#}/\rho_n - \nu_n\right\}(t) \\
(6.37) \quad = & : I_1 + I_2.
\end{aligned}$$

Here as $\left|\frac{\beta_n - t}{1-t}\right| \leq 1$ in the integrand in I_1, (6.11) gives

$$\begin{aligned}
|I_1| \leq & \rho_n \left(1 - \frac{1}{\rho_n}\right) \int_{\alpha_n}^{\beta_n} \left|\log\left|\frac{\beta_n - t}{1-t}\right|\right| d\omega_{[\alpha_n,\beta_n]}(t) \\
\leq & C n^{-2/3} \int_{\alpha_n}^{\beta_n} \left|\log\left|\frac{\beta_n - t}{1-t}\right|\right| \frac{dt}{\sqrt{(t-\alpha_n)(\beta_n - t)}} \\
\leq & C n^{-2/3} \left\{\int_{\alpha_n}^{1/2} \log(1 + 2(1-\beta_n)) \frac{dt}{\sqrt{(t-\alpha_n)(\beta_n - t)}} + \int_{1/2}^{\beta_n} \left|\log\left|\frac{\beta_n - t}{1-t}\right|\right| \frac{dt}{\sqrt{\beta_n - t}}\right\} \\
\leq & C n^{-2/3} \left\{(1-\beta_n) + (1-\beta_n)^{1/2} \int_1^{\infty} \left|\log\left|\frac{s-1}{s}\right|\right| \frac{ds}{(s-1)^{1/2}}\right\} \leq \frac{C}{n},
\end{aligned}$$

where we have used the substitution $1 - t = (1-\beta_n)s$ and Lemma 6.2(b). Next,

$$\begin{aligned}
I_2 = & -\rho_n \int_{[-1,1]\backslash[\alpha_n,\beta_n]} \log\left|\frac{\beta_n - t}{1-t}\right| d\nu_n(t) \\
\leq & C \int_{[-1,1]\backslash[\alpha_n,\beta_n]} \left|\log\left|\frac{\beta_n - t}{1-t}\right|\right| \left(1-t^2\right)^{1/2} dt \\
\leq & C(1-\beta_n) \int_{-1}^{\alpha_n} (1+t)^{1/2} dt + C \int_{\beta_n}^{1} \left|\log\left|\frac{\beta_n - t}{1-t}\right|\right| (1-t)^{1/2} dt \\
\leq & C(1-\beta_n)(1+\alpha_n)^{3/2} + C(1-\beta_n)^{3/2} \int_0^1 \left|\log\left|\frac{s-1}{s}\right|\right| s^{1/2} ds \leq \frac{C}{n},
\end{aligned}$$

where we have again used the substitution $1 - t = (1-\beta_n)s$ and Lemma 6.2. Substituting in the estimates for I_1 and I_2 into (6.37) gives (6.35) and the result. ■

7. A Local Markov-Bernstein Inequality

Recall that

$$\varphi_n(x) = \frac{1}{n}\left(1 - x^2 + n^{-2/3}\right)^{-1/2}, x \in [-1,1].$$

We shall use the method of [15, Chapter 10] to prove:

Theorem 7.1
There exists $\varepsilon_0 \in (0,1)$ such that for $n \geq 1$ and $\deg(P) \leq n-1$,

$$\int_{1-\varepsilon_0}^{1} \left\{\varphi_n(t)\left|\frac{d}{dt}\left(P(t)W_n^n\right)(t)\right|\right\}^2 dt \leq C\int_{-1}^{1} P^2(t)\,W_n^{2n}(t)\,dt. \tag{7.1}$$

Here $C \neq C(n,P)$.

We deduce a quadrature sum estimate:

Corollary 7.2
Let $n \geq 1$, $L \geq 1$, and $A > 0$. Assume that $\{s_j\}_{j=0}^{L}$ satisfy

$$1 - \varepsilon_0 \leq s_L < s_{L-1} < ... < s_1 < s_0 = 1 \tag{7.2}$$

and for $1 \leq j \leq L$,

$$s_{j-1} - s_j \leq A\varphi_n(s_j). \tag{7.3}$$

Then for polynomials P of degree $\leq n-1$,

$$\sum_{j=1}^{L} |PW_n^n|^2(s_j)(s_{j-1} - s_j) \leq C\int_{-1}^{1} |PW_n^n|^2. \tag{7.4}$$

Here C depends on A, but is independent of $L, n, P, \{s_j\}$.

We start with an analogue of Lemmas 10.5 and 10.6 in [15, pp. 299, 301]. For the purposes of this lemma, we set

$$Q_n(x) = Q_n(1), x > 1.$$

Lemma 7.3
Fix $x \in (-1,1)$ and let

$$F_x(z) = \exp\left(-nQ_n(x) - nQ_n'(x)(z-x)\right),\ z \in \mathbb{C}. \tag{7.5}$$

(a) Then for $\varepsilon > 0$,

$$\left|(PW_n^n)'(x)\,\varepsilon\right|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |PF_x|^2\left(x + \varepsilon e^{i\theta}\right)d\theta. \tag{7.6}$$

(b) There exists $\varepsilon_0 > 0$ such that for $n \geq n_0(\varepsilon_0)$, $x \in [1-\varepsilon_0, 1]$ and $|z - x| \leq \varphi_n(x)$, we have

$$|F_x(z)| \leq CW_n^n(|z|); \tag{7.7}$$

(7.8) $$n\left|Q_n(\operatorname{Re} z)-Q_n(|z|)\right|\leq C.$$

Here the constants do not depend on n or x.

Proof

(a) We have

$$F_x^{(j)}(x)=\left(W_n^n\right)^{(j)}(x),\ j=0,1.$$

By Cauchy's integral formula for derivatives,

$$\begin{aligned}\left|\left(PW_n^n\right)'(x)\right| &= \left|\left(PF_x\right)'(x)\right|\\ &= \left|\frac{1}{2\pi i}\int_{|t-x|=\varepsilon}\frac{\left(PF_x\right)(t)}{(t-x)^2}dt\right|\\ &\leq \frac{1}{\varepsilon}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|PF_x|^2\left(x+\varepsilon e^{i\theta}\right)d\theta\right\}^{1/2}.\end{aligned}$$

(b) Now

$$\begin{aligned}&\left|F_x(z)\right|/W_n^n(|z|)\\ =\ &\exp\left(n\left[Q_n(\operatorname{Re} z)-Q_n(x)-Q_n'(x)(\operatorname{Re} z-x)\right]+n\left[Q_n(|z|)-Q_n(\operatorname{Re} z)\right]\right).\end{aligned}$$

(7.9)

For some ζ between x and $\operatorname{Re} z$,

$$\begin{aligned}&Q_n(\operatorname{Re} z)-Q_n(x)-Q_n'(x)(\operatorname{Re} z-x)\\ =\ &\left\{Q_n'(\zeta)-Q_n'(x)\right\}(\operatorname{Re} z-x)\\ =\ &\left[o(1-\zeta)^{1/2}+o(1-x)^{1/2}\right](\operatorname{Re} z-x)\\ =\ &o\left((1-x)^{1/2}\varphi_n(x)\right)+o\left(\varphi_n(x)^{3/2}\right)=o\left(n^{-1}\right),\end{aligned}$$

by Lemma 4.1(b) and as $|z-x|\leq\varphi_n(x)$. Also, for some ξ between $|z|$ and $\operatorname{Re} z$,

$$Q_n(|z|)-Q_n(\operatorname{Re} z)=Q_n'(\xi)\left(|z|-\operatorname{Re} z\right).$$

Here (cf. [15, p. 302])

$$|z|-\operatorname{Re} z\leq\frac{|\operatorname{Im} z|^2}{2\operatorname{Re} z}\leq C\varphi_n(x)^2\leq\frac{C}{n^2}\left[n^{-2/3}+|1-x|\right]^{-1}=o\left(\frac{1}{n}\right),$$

while $|Q_n'(\xi)|\leq C$ by Lemma 4.2, so

$$Q_n(|z|)-Q_n(\operatorname{Re} z)=o\left(\frac{1}{n}\right).$$

Substituting all these estimates in (7.9) and choosing ε_0 small enough, we then obtain (7.7) and (7.8). ■

Next an analogue of Lemma 10.7 in [15, p. 303]:

Lemma 7.4

There is a function G_n analytic in $\mathbb{C}\setminus[-1,1]$ with a zero at ∞, such that
(a)

$$|G_n(z)| = \exp(V^{\nu_n}(z) - c_n), \; z \in \mathbb{C}\setminus[-1,1],$$

and

$$|G_n(x)| = W_n(x), \; x \in [-1,1]. \tag{7.10}$$

(b) For $x \in [1-\varepsilon_0, 1]$ and $|z-x| \le \varphi_n(x)$, we have for $n \ge n_0(\varepsilon_0)$

$$n\left[V^{\nu_n}(\operatorname{Re} z) - V^{\nu_n}(z)\right] \le C. \tag{7.11}$$

There exists $n_0 = n_0(\varepsilon_0)$ such that for $x \in [1-\varepsilon_0, 1]$ and $|z-x| \le \varphi_n(x)$, we have

$$W_n^n(|z|) \le C\,|G_n^n(z)| \tag{7.12}$$

Proof
(a) Let

$$G_n(z) = \exp\left(-\int_{-1}^{1} \log(z-t)\,d\nu_n(t) - c_n\right),$$

with the principal branch of the log. This is single valued in $\mathbb{C}\setminus[-1,1]$ as ν_n is a probability measure. We have

$$\begin{aligned}|G_n(z)| &= \exp\left(-\int_{-1}^{1} \log|z-t|\,d\nu_n(t) - c_n\right) \\ &= \exp(V^{\nu_n}(z) - c_n).\end{aligned}$$

For $x \in [-1,1]$, we can just use the equilibrium relation.
(b) The proof of (7.11) is virtually the same as the proof of (5.5) in the proof of Theorem 5.1 above, so is omitted. Next,

$$\begin{aligned}&W_n^n(|z|)\,/\,|G_n^n(z)| \\ =\;& \exp\left(n\left[-Q_n(|z|) - V^{\nu_n}(z) + c_n\right]\right) \\ =\;& \exp\left(n\left[Q_n(\operatorname{Re} z) - Q_n(|z|)\right] - n\left[V^{\nu_n}(\operatorname{Re} z) + Q_n(\operatorname{Re} z) - c_n\right] + n\left[V^{\nu_n}(\operatorname{Re} z) - V^{\nu_n}(z)\right]\right).\end{aligned}$$

The first term $n[Q_n(\operatorname{Re} z) - Q_n(|z|)]$ is bounded by (7.8). Next, if $\operatorname{Re} z \in [-1,1]$, the second term $V^{\nu_n}(\operatorname{Re} z) + Q_n(\operatorname{Re} z) - c_n = 0$, while if $\operatorname{Re} z \in [1, 1 + Cn^{-2/3}]$, a bound can be established by easy estimation (similar to the proof of Theorem 5.1). Finally, the third term is bounded by (7.11). ■

Lemma 7.5

$$\int_{1-\varepsilon_0}^{1} \left|(PW_n^n)'\right|^2 \varphi_n \le C \int_{-1}^{1} |PG_n^n|^2\, d\left[\nu_n^+ + \nu_n^-\right], \tag{7.13}$$

where $C \neq C(n,P)$, and for Borel measurable sets S with $S \subset \mathbb{C}$, with characteristic functions χ_S,

$$\begin{aligned} \nu_n^+(S) &= \int_{-1}^{1}\int_{0}^{\pi} \chi_S\left(x+\varphi_n(x)e^{i\theta}\right) d\theta\ dx; \\ \nu_n^-(S) &= \int_{-1}^{1}\int_{-\pi}^{0} \chi_S\left(x+\varphi_n(x)e^{i\theta}\right) d\theta\ dx. \end{aligned}$$

Proof
From Lemmas 7.3(b) and 7.4(b), for $x \in [1-\varepsilon_0, 1]$ and $|z-x| \leq \varphi_n(x)$,

$$|F_x(z)| \leq CW_n^n(|z|) \leq C|G_n^n(z)|,$$

with C independent of n, x. Then from Lemma 7.3(a), with $\varepsilon = \varphi_n(x)$,

$$\left|(PW_n^n)'(x)\varphi_n(x)\right|^2 \leq \frac{C}{2\pi}\int_{-\pi}^{\pi} |PG_n^n|^2\left(x+\varphi_n(x)e^{i\theta}\right) d\theta.$$

Integrating gives the result. ■

Proof of Theorem 7.1
This follows that in [15, pp. 304-307]. Recall that a non-negative measure ν with support in the upper-half plane is called a Carleson measure if there exists $A > 0$ such that

$$\nu(K) \leq Ah$$

for all squares K of side h, with base on the real axis. The smallest such A is denoted $N[\nu]$, the Carleson norm of ν. For functions F lying in the Hardy space H^2 of the upper half plane, there is the inequality [11, p. 63],

$$\int |F(z)|^2 d\nu(z) \leq CN[\nu]\int_{-\infty}^{\infty} |F(x)|^2 dx, \tag{7.14}$$

with C independent of ν and F. We claim that

$$\sup_{n\geq 1} N\left[\nu_n^+\right] < \infty. \tag{7.15}$$

Indeed, this was proved in [15, Lemma 10.10, pp. 305-306] for a much more general situation (our situation is subsumed in the proof for the case of the Hermite weight there, with $\varphi_n(x)$ given by (3.19)). Similarly,

$$\sup_{n\geq 1} N\left[\nu_n^-\right] < \infty.$$

Next, as P has degree $\leq n-1$, $(PG_n^n)(z) = O\left(z^{-1}\right)$ at ∞, so lies in H^2. Combining (7.13), (7.14), and (7.15),

$$\begin{aligned} &\int_{1-\varepsilon_0}^{1} \left\{\varphi_n \left|(PW_n^n)'\right|\right\}^2 \\ \leq\ & C\int_{-\infty}^{\infty} \left|PG_n^n\right|^2(x)\,dx \\ =\ & C\left\{\int_{-1}^{1} \left|PW_n^n\right|^2(x)\,dx + \int_{\mathbb{R}\setminus[-1,1]} \left|PG_n^n\right|^2(x)\,dx\right\}. \end{aligned} \tag{7.16}$$

By Lemma 4.3 in [15, p. 98],

$$\int_{\mathbb{R}\setminus[-1,1]} \left|PG_n^n\right|^2 \leq \int_{-1}^{1} \left|PG_n^n\right|^2 = \int_{-1}^{1} \left|PW_n^n\right|^2$$

and the result follows. ■

Proof of Corollary 7.2

We use a method of Paul Nevai. For $j \geq 1$, the fundamental theorem of calculus gives

$$\left|PW_n^n\right|^2(s_j) \leq \inf_{[s_j,s_{j-1}]} \left|PW_n^n\right|^2 + \int_{s_j}^{s_{j-1}} \left|\frac{d}{dt}\left\{(PW_n^n)^2(t)\right\}\right| dt$$

so

$$\left|PW_n^n\right|^2(s_j)(s_{j-1}-s_j) \leq \int_{s_j}^{s_{j-1}} \left|PW_n^n\right|^2 + 2A\int_{s_j}^{s_{j-1}} \left|PW_n^n\right| \left|(PW_n^n)'\right| \varphi_n.$$

Here we have used the hypothesis (7.3), and the monotonicity of φ_n, so that

$$s_{j-1} - s_j \leq A\varphi_n(s_j) \leq A\varphi_n(s),\ s \in [s_j, s_{j-1}].$$

Adding over j, and then using Cauchy-Schwarz, gives

$$\begin{aligned} &\sum_{j=1}^{L} \left|PW_n^n\right|^2(s_j)(s_{j-1}-s_j) \\ \leq\ & \int_{1-\varepsilon_0}^{1} \left|PW_n^n\right|^2 + 2A\left(\int_{1-\varepsilon_0}^{1} \left|PW_n^n\right|^2\right)^{1/2}\left(\int_{1-\varepsilon_0}^{1} \left\{\left|(PW_n^n)'\right|\varphi_n\right\}^2\right)^{1/2} \\ \leq\ & C\int_{-1}^{1} \left|PW_n^n\right|^2, \end{aligned}$$

by Theorem 7.1. ■

8. Bounds on Weighted Polynomials

We shall prove:

Theorem 8.1

Assume the hypotheses of Theorem 1.3. Let $\mathcal{K}$ be a compact subset of the *plane. Then there exist* n_0 *and* C *depending only on* $\mathcal{K}$ *with the following properties: for* $u \in \mathcal{K}$ *and* $n \geq n_0$*, and* P *a polynomial of degree* $\leq n$ *satisfying*

$$\left\| (PW_n^n)(x) \left(1 - x^2 + n^{-2/3}\right)^{-\frac{1}{4}} \right\|_{L_\infty[-1,1]} \leq 1, \tag{8.1}$$

we have

$$\begin{aligned} & \left|P\left(1 + n^{-2/3}u\right)\right| \exp\left(n\left\{-Q_n(1) - Q_n'(1)\, n^{-2/3} \operatorname{Re} u\right\}\right) \\ \leq & \left|\left(1 + \sqrt{2|u|}\right) n^{-1/3}\right|^{1/2} \exp\left(-\frac{2\sqrt{2}}{3}\pi\psi_n(1) \operatorname{Re}\left(u^{3/2}\right) + Cn^{-1/3}\right). \end{aligned}$$

(8.2)

Proof

We shall prove this, assuming some technical estimates that will be proved later. We may assume that P has actual degree n. (If not, consider $P(x) + \varepsilon x^n$ and then let $\varepsilon \to 0$). Form the Green's function $g\left(z, 1 + n^{-2/3}\right)$ with pole at $1 + n^{-2/3}$, and let

$$\begin{aligned} & G(z) \\ = & \log|P(z)| + n\left[V^{\nu_n}(z) - c_n\right] \\ & -\frac{1}{4}\left\{\log\left|1 + n^{-2/3} - z^2\right| + g\left(z^2, 1 + n^{-2/3}\right)\right\} + \frac{1}{2}\log|\varphi(z)|. \end{aligned}$$

Here $\log\left|1 + n^{-2/3} - z^2\right| + g\left(z^2, 1 + n^{-2/3}\right)$ is harmonic outside $[-1, 1]$, and behaves like $2\log|z| + O(1)$ at ∞. We see then that G is subharmonic outside $[-1, 1]$ and has a finite limit at ∞. Moreover, our hypothesis (8.1) shows that $G \leq 0$ on $(-1, 1)$. By the maximum principle for subharmonic functions,

$$G \leq 0 \text{ in } \mathbb{C}.$$

Then for all complex z,

$$\begin{aligned} & \log|P(z)| \\ \leq & -n\left[V^{\nu_n}(z) - c_n\right] + \frac{1}{4}\left\{\log\left|1 + n^{-2/3} - z^2\right| + g\left(z^2, 1 + n^{-2/3}\right)\right\} - \frac{1}{2}\log|\varphi(z)|. \end{aligned}$$

(8.3)

Now suppose

$$z = 1 + n^{-2/3}u = 1 + n^{-2/3}\rho e^{i\theta}.$$

By Lemma 8.2 below, with the principal branch of $\sqrt{}$, and with $r = n^{-2/3}$ and $a = 2u$,

$$\begin{aligned} & \log\left|1 + n^{-2/3} - z^2\right| + g\left(z^2, 1 + n^{-2/3}\right) \\ \leq & 2\log\left|\left(1 + \sqrt{2u}\right) n^{-1/3}\right| + O\left(n^{-2/3}\right). \end{aligned}$$

A simple calculation shows that

$$\log|\varphi(z)| = O\left(n^{-1/3}\right).$$

Then by (8.3) and Lemma 8.3 below for $u = \rho e^{i\theta} \in \mathcal{K}$ and $n \geq n_0(\mathcal{K})$, (and taking there $r = n^{-2/3}\rho$)

$$\begin{aligned}
& \log|P(z)| + n\left\{-Q_n(1) - Q_n'(1)\rho n^{-2/3}\cos\theta\right\} \\
\leq\ & \frac{1}{2}\log\left|\left(1+\sqrt{2\rho e^{i\theta}}\right)n^{-1/3}\right| \\
& -n\left[V^{\nu_n}(z) + Q_n(1) + Q_n'(1)\rho n^{-2/3}\cos\theta - c_n\right] + O\left(n^{-1/3}\right) \\
\leq\ & \frac{1}{2}\log\left|\left(1+\sqrt{2\rho e^{i\theta}}\right)n^{-1/3}\right| \\
& -\frac{2\sqrt{2}}{3}\pi\psi_n(1)\rho^{3/2}\cos\frac{3\theta}{2} + O\left(n^{-1/3}\right).
\end{aligned}$$

This is easily recast as (8.2). ■

Lemma 8.2
Uniformly for a in compact subsets of $\mathbb{C}\backslash\{z : |\sqrt{z} \pm 1| < 1\}$, as $r \to 0+$,

$$g\left(\left(1+\frac{a}{2}r\right)^2, 1+r\right) = \log\left|\frac{1+\sqrt{a}}{1-\sqrt{a}}\right| + O(r). \tag{8.4}$$

Moreover,

$$\begin{aligned}
& \log\left|(1+r) - \left(1+\frac{a}{2}r\right)^2\right| + g\left(\left(1+\frac{a}{2}r\right)^2, 1+r\right) \\
=\ & 2\log\left|\left(1+\sqrt{a}\right)\sqrt{r}\right| + O(r).
\end{aligned} \tag{8.5}$$

The branch of the square root is the principal one. Moreover, if $a < 0$, we interpret $\sqrt{a}$ as $i\sqrt{|a|}$.

Proof.
Recall from (3.17) that

$$g(z,w) = \log\left|\frac{1-\varphi(z)\overline{\varphi(w)}}{\varphi(z)-\varphi(w)}\right|, \tag{8.6}$$

where $\varphi(z) = z + \sqrt{z^2-1}$. We analyze this with

$$z = \left(1+\frac{a}{2}r\right)^2 = 1 + ar + \frac{a^2r^2}{4} \text{ and } w = 1+r,$$

where a lies in a compact subset of $\mathbb{C}\backslash\{z : |\sqrt{z} \pm 1| < 1\}$. First,

$$\varphi\left(\left(1+\frac{a}{2}r\right)^2\right) = 1 + \sqrt{2ar} + ar + O\left(r^{3/2}\right); \tag{8.7}$$

$$\varphi(1+r) = 1 + \sqrt{2r} + r + O\left(r^{3/2}\right), \tag{8.8}$$

so

$$\begin{aligned} & \left|\varphi\left(\left(1+\frac{a}{2}r\right)^2\right)-\varphi\left(1+r\right)\right| \\ = & \sqrt{2r}\left|1-\sqrt{a}\right|\left|1+\left(1+\sqrt{a}\right)\sqrt{\frac{r}{2}}+O\left(r\right)\right|. \end{aligned}$$

Then

$$\begin{aligned} & \log\left|\varphi\left(\left(1+\frac{a}{2}r\right)^2\right)-\varphi\left(1+r\right)\right| \\ = & \log\left(\sqrt{2r}\left|1-\sqrt{a}\right|\right)+\mathrm{Re}\left(\left(1+\sqrt{a}\right)\sqrt{\frac{r}{2}}\right)+O\left(r\right). \end{aligned}$$

Next, from (8.7) and (8.8),

$$\begin{aligned} & 1-\varphi\left(\left(1+\frac{a}{2}r\right)^2\right)\varphi\left(1+r\right) \\ = & -\sqrt{2r}\left(1+\sqrt{a}\right)-r\left(1+\sqrt{a}\right)^2+O\left(r^{3/2}\right). \end{aligned}$$

Then recalling our hypothesis on a,

$$\begin{aligned} & \log\left|1-\varphi\left(\left(1+\frac{a}{2}r\right)^2\right)\varphi\left(1+r\right)\right| \\ = & \log\left(2\sqrt{r}\left|1+\sqrt{a}\right|\right)+\mathrm{Re}\left(\left(1+\sqrt{a}\right)\sqrt{\frac{r}{2}}\right)+O\left(r\right), \end{aligned}$$

and so from (8.6),

$$g\left(\left(1+\frac{a}{2}r\right)^2,1+r\right)=\log\left|\frac{1+\sqrt{a}}{1-\sqrt{a}}\right|+O\left(r\right).$$

Then

$$\begin{aligned} & \log\left|\left(1+r\right)-\left(1+\frac{a}{2}r\right)^2\right|+g\left(\left(1+\frac{a}{2}r\right)^2,1+r\right) \\ = & \log\left|\left(1-a\right)r\right|+\log\left|\frac{1+\sqrt{a}}{1-\sqrt{a}}\right|+O\left(r\right) \\ = & 2\log\left|\left(1+\sqrt{a}\right)\sqrt{r}\right|+O\left(r\right). \end{aligned}$$

■

Lemma 8.3
As $r\to 0+$, uniformly for $\theta\in[0,\pi]$,

$$\begin{aligned} & -\left[V^{\nu_n}\left(1+re^{i\theta}\right)+Q_n\left(1\right)+Q_n'\left(1\right)r\cos\theta-c_n\right] \\ = & -\frac{2\sqrt{2}}{3}\pi\psi_n\left(1\right)r^{3/2}\cos\frac{3\theta}{2}+o\left(r^{3/2}\right). \end{aligned} \tag{8.9}$$

Proof
Write

$$\begin{aligned} \Gamma(r) &:= -\left[V^{\nu_n}\left(1+re^{i\theta}\right)+Q_n(1)+Q_n'(1)\,r\cos\theta-c_n\right] \\ &= -V^{\nu_n}\left(1+re^{i\theta}\right)+V^{\nu_n}(1)-Q_n'(1)\,r\cos\theta \\ &= \int_{-1}^{1}\left\{\log\left|1+\frac{re^{i\theta}}{1-t}\right|-\frac{r\cos\theta}{1-t}\right\}\left(1-t^2\right)^{1/2}\psi_n(t)\,dt, \end{aligned}$$

by (4.1) and (1.19). Fix θ and let

$$g(s)=\log\left|1+se^{i\theta}\right|-s\cos\theta,\ s\in[0,\infty).$$

By some simple manipulations, we see that

$$|g(s)|\le Cs^2,\ s\in\left[0,\frac{1}{2}\right], \tag{8.10}$$

and

$$|g(s)|\le Cs,\ s\in[2,\infty), \tag{8.11}$$

where C is independent of both s and θ. Then

$$\int_0^\infty \frac{|g(s)|}{s^{5/2}}ds<\infty, \tag{8.12}$$

the integral converging uniformly in θ. Write

$$\begin{aligned} \Gamma(r) &= \psi_n(1)\int_{-1}^{1} g\left(\frac{r}{1-t}\right)\left(1-t^2\right)^{1/2}dt \\ &\quad+\int_{-1}^{1} g\left(\frac{r}{1-t}\right)\frac{\psi_n(t)-\psi_n(1)}{1-t}(1-t)^{3/2}(1+t)^{1/2}\,dt \\ &=: T_1+T_2. \end{aligned} \tag{8.13}$$

The substitution $s=\frac{r}{1-t}$ shows that

$$T_1=\psi_n(1)\,r^{3/2}\int_{r/2}^{\infty}\frac{g(s)}{s^{5/2}}\left(2-\frac{r}{s}\right)^{1/2}ds.$$

Since $0\le 2-\frac{r}{s}\le 2$, Lebesgue's Dominated Convergence Theorem shows that as $r\to 0+$,

$$\begin{aligned} T_1 &= \psi_n(1)\,r^{3/2}\left(\sqrt{2}\int_0^\infty\frac{g(s)}{s^{5/2}}ds+o(1)\right) \\ &= \psi_n(1)\,r^{3/2}\sqrt{2}\,\mathrm{Re}\left(H\left(e^{i\theta}\right)\right)+o\left(r^{3/2}\right), \end{aligned}$$

where

$$H(z)=\int_0^\infty\{\log(1+sz)-sz\}\frac{ds}{s^{5/2}}.$$

The function H is analytic and single valued in $\mathbb{C}\backslash(-\infty,0]$, and $H(0)=0$. We see that

$$\begin{aligned} H'(z) &= \int_0^\infty \left\{ \frac{s}{1+sz} - s \right\} \frac{ds}{s^{5/2}} \\ &= -z\int_0^\infty \frac{1}{1+sz}\frac{ds}{s^{1/2}} \\ &= -z\int_{-\infty}^\infty \frac{1}{1+t^2 z} dt, \end{aligned}$$

by the substitution $s=t^2$. Using the residue calculus, we obtain at least for $\operatorname{Re} z>0$,

$$H'(z) = -\pi\sqrt{z},$$

and hence

$$H(z) = -\frac{2}{3}\pi z^{3/2}.$$

By analytic continuation, this holds for all $z\in\mathbb{C}\backslash(-\infty,0]$. Thus, uniformly in $\theta\in[0,\pi)$, as $r\to 0+$,

$$T_1 = -\frac{2\sqrt{2}}{3}\pi\psi_n(1)\, r^{3/2}\cos\frac{3\theta}{2} + o\left(r^{3/2}\right). \tag{8.14}$$

This also holds for $\theta=\pi$, by a continuity argument. Next, using first a substitution, and then our bounds (8.10), (8.11),

$$\begin{aligned} |T_2| &\le \int_{-1}^1 \left|g\left(\frac{r}{1-t}\right)\right| \frac{\omega(\psi_n;1-t)}{1-t}(1-t)^{3/2}(1+t)^{1/2}\,dt \\ &= \int_0^2 \left|g\left(\frac{r}{u}\right)\right| \frac{\omega(\psi_n;u)}{u} u^{3/2}(2-u)^{1/2}du \\ &\le C\int_0^{r/2} \frac{r}{u}\frac{\omega(\psi_n;u)}{u}u^{3/2}du + C\omega(\psi_n; r|\log r|)\int_r^{r|\log r|}\left|g\left(\frac{r}{u}\right)\right| u^{1/2}du \\ &\quad + C\int_{r|\log r|}^2 \left(\frac{r}{u}\right)^2 \frac{\omega(\psi_n;u)}{u}u^{3/2}du \\ &\le Cr^{3/2}\int_0^{r/2}\frac{\omega(\psi_n;u)}{u}du + C\left(\int_{r|\log r|}^{2r|\log r|}\frac{\omega(\psi_n;t)}{t}dt\right)\left(r^{3/2}\int_0^\infty\frac{|g(s)|}{s^{5/2}}ds\right) \\ &\quad + C\|\psi_n\|_{L_\infty[-1,1]}\, r^2\int_{r|\log r|}^2 \frac{du}{u^{3/2}} \\ &\le Cr^{3/2}\left\{\int_0^{2r|\log r|}\frac{\omega(\psi_n;t)}{t}dt + \frac{\|\psi_n\|_{L_\infty[-1,1]}}{|\log r|^{1/2}}\right\}. \end{aligned}$$

Since the integrals in (1.21) converge uniformly and because of (1.22), we deduce that uniformly in n and θ, as $r\to 0+$,

$$|T_2| = o\left(r^{3/2}\right).$$

This and (8.14) give the result. ■

9. Proof of Theorems 1.2 and 1.3

Throughout, we assume the hypotheses of Theorem 1.3. Recall that in Section 3, we defined

$$f_n(a,b) = \frac{K_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}b\right)}{K_n(1,1)} e^{\Psi(n)(a+b)}, \tag{9.1}$$

for a, $b \in \mathbb{C}$. Here $\Psi(n)$ is defined by (3.8). It will follow from Lemma 9.2 below, that uniformly for a, b in compact subsets of the real line,

$$f_n(a,b) = \frac{\tilde{K}_n\left(1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}b\right)}{\tilde{K}_n(1,1)}(1+o(1)). \tag{9.2}$$

Recall our notation from Section 3:

$$\tau = \lim_{n\to\infty, n\in\mathcal{S}} \tau_n = \lim_{n\to\infty, n\in\mathcal{S}} 2n^{2/3}\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}; \tag{9.3}$$

$$\sigma = \left(\sqrt{2}\pi\Delta\right)^{2/3}\frac{\tau}{2}. \tag{9.4}$$

Recall too our assumption (a) in Theorem 1.3 that $p_{n,n}$ has at most ℓ_0 zeros in $(1, 1+\varepsilon)$.

Theorem 9.1
Assume the hypotheses of Theorem 1.3.
(a) $\{f_n(u,v)\}_{n=1}^{\infty}$ *is uniformly bounded for* u, v *in compact subsets of the plane.*
(b) If $f(u,v)$ *is the locally uniform limit of some subsequence* $\{f_n(u,v)\}_{n\in\mathcal{S}}$ *of* $\{f_n(u,v)\}_{n=1}^{\infty}$*, with also (9.3) holding, then* $f(\cdot,\cdot)$ *is entire in* u, v*. Moreover, for some* C *independent of* $u, v \in \mathbb{C}$,

$$\begin{aligned} &|f(u,v)| \\ \le\; & C\left(\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right)\right)^{1/2}\exp\left(-\frac{2}{3}\sigma^{3/2}\operatorname{Re}\left(u^{3/2}+v^{3/2}\right)\right). \end{aligned} \tag{9.5}$$

(c) For each fixed real number u, $f(u,\cdot)$ *has only real zeros. At most* ℓ_0+1 *of these are positive.*

Remarks
(a) From (9.5) and (3.22), we see that for $|\arg(u)|, |\arg(v)| \le \pi - \delta$

$$|f(u/\sigma, v/\sigma)| \le C_\delta\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right)|Ai(u)||Ai(v)|. \tag{9.6}$$

(b) Note that $\operatorname{Re}\left(u^{3/2}+v^{3/2}\right)$ is well defined even for $u, v \in (-\infty, 0)$.
Proof

(a) By our lower bound in Theorem 5.1 for $\lambda_n = 1/K_n$, and by Cauchy-Schwarz, we have

$$\begin{aligned} & \frac{1}{n}\left|K_n(\xi,t)\right| W_n^n(\xi) W_n^n(t)\left(1-\xi^2+n^{-2/3}\right)^{-1/4}\left(1-t^2+n^{-2/3}\right)^{-1/4} \\ \leq\ & C \end{aligned}$$

for $\xi, t \in [-1,1]$ and $n \geq 1$. By Theorem 8.1, applied separately in each variable, we then have for $u, v \in \mathbb{C}$ with $|u|, |v| \geq 4$, and $n \geq n_0(u,v)$,

$$\begin{aligned} & \frac{1}{n}\left|K_n\left(1+\frac{u}{2n^{3/2}}, 1+\frac{v}{2n^{3/2}}\right)\right| \exp\left(n\left\{\begin{array}{c} -2Q_n(1) \\ -Q_n'(1)\, n^{-2/3}\operatorname{Re}\left(\frac{u}{2}+\frac{v}{2}\right) \end{array}\right\}\right) \\ \leq\ & C\left|\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right) n^{-2/3}\right|^{1/2} \times \\ & \exp\left(-\frac{2\sqrt{2}}{3}\pi\psi_n(1)\operatorname{Re}\left(\left(\frac{u}{2}\right)^{3/2}+\left(\frac{v}{2}\right)^{3/2}\right)+Cn^{-1/3}\right). \end{aligned}$$

In view of Theorem 6.1, we can recast this as

$$\begin{aligned} & \left|\frac{K_n\left(1+\frac{u}{2n^{3/2}}, 1+\frac{v}{2n^{3/2}}\right)}{K_n(1,1)}\right| \exp\left(n\left\{-Q_n'(1)\, n^{-2/3}\operatorname{Re}\left(\frac{u}{2}+\frac{v}{2}\right)\right\}\right) \\ \leq\ & \left|\left(1+\sqrt{|u|}\right)\left(1+\sqrt{|v|}\right)\right|^{1/2} \exp\left(-\frac{2\sqrt{2}}{3}\pi\psi_n(1)\operatorname{Re}\left(\left(\frac{u}{2}\right)^{3/2}+\left(\frac{v}{2}\right)^{3/2}\right)+Cn^{-1/3}\right). \end{aligned}$$

Recall the number τ_n defined by (3.10), and set

$$u = \tau_n a \text{ and } v = \tau_n b.$$

Since $\tau_n \sim 1$, as follows from Theorems 5.1 and 6.1, we obtain

$$\begin{aligned} & \left|f_n(a,b)\right| \\ \leq\ & C\left|\left(1+\sqrt{|a|}\right)\left(1+\sqrt{|b|}\right)\right|^{1/2} \exp\left(-\frac{2\sqrt{2}}{3}\pi\psi_n(1)\left(\frac{\tau_n}{2}\right)^{3/2}\operatorname{Re}\left(a^{3/2}+b^{3/2}\right)\right). \end{aligned}$$

Here C is independent of the compact set, and depends only on $\{\tau_n\}$. Recall too our hypothesis (1.20). The uniform boundedness of $\{f_n(a,b)\}$ for a, b in compact subsets of the plane follows.

(b) Now $\{f_n(u,v)\}_{n=1}^{\infty}$ is a normal family of the two variables $u, v \in \mathbb{C}$. If $f(u,v)$ is the locally uniform limit through the subsequence $\mathcal{S}$ of integers, we see that $f(u,v)$ is entire in u, v satisfying

$$\begin{aligned} & \left|f(a,b)\right| \\ \leq\ & C\left|\left(1+\sqrt{|a|}\right)\left(1+\sqrt{|b|}\right)\right|^{1/2} \exp\left(-\frac{2}{3}\sigma^{3/2}\operatorname{Re}\left(a^{3/2}+b^{3/2}\right)\right). \end{aligned}$$

Here C is independent of a, b.

(c) It is shown in [10, p. 18], that for each real ξ_n, $K_n(\xi_n, t)$ has only real simple zeros. Hence for real u, $f_n(u,v)$ has only real zeros as a function of v. Hurwitz's theorem shows that the same is true of $f(u,v)$.

We now discuss the positive zeros of f_n and f. Let

$$\xi_n = 1 + \frac{\mathbb{A}i\,(0,0)}{\tilde{K}_n\,(1,1)} a.$$

Since $p_{n,n}$ has at most ℓ_0 zeros in $(1, 1+\varepsilon)$ and the zeros of $K_n\,(\xi_n, t)$ interlace those of $p_{n,n}$, so $K_n\,(\xi_n, t)$ has at most $\ell_0 + 1$ zeros in $(1, 1+\varepsilon)$. It follows that there are at most ℓ_0+1 simple zeros of $f_n\,(u, \cdot)$ in $\left(0, \frac{\tilde{K}_n(1,1)}{\mathbb{A}i(0,0)}\varepsilon\right)$. Since $\frac{\tilde{K}_n(1,1)}{\mathbb{A}i(0,0)}\varepsilon$ has limit ∞, Hurwitz's Theorem shows that $f\,(u, \cdot)$ has at most $\ell_0 + 1$ positive zeros. ■

Lemma 9.2
(a) Uniformly for a in compact subsets of $\mathbb{R}$,

$$\frac{W_n^{2n}\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} a\right)}{W_n^{2n}\,(1)} = \exp\left(2\Psi\,(n)\,a + o\,(1)\right), \tag{9.7}$$

where, as in (3.8),

$$\Psi\,(n) = -\frac{\mathbb{A}i\,(0,0)}{\tilde{K}_n\,(1,1)} n Q_n'\,(1)\,. \tag{9.8}$$

Proof
(a) We have for some ζ between 1 and $1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} a$,

$$\begin{aligned} & \frac{W_n^{2n}\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} a\right)}{W_n^{2n}\,(1)} \\ = \;& \exp\left(2n\left[Q_n\,(1) - Q_n\left(1 + \frac{\mathbb{A}i\,(0,0)}{\tilde{K}_n\,(1,1)} a\right)\right]\right) \\ = \;& \exp\left(-2nQ_n'\,(\zeta)\,\frac{\mathbb{A}i\,(0,0)}{\tilde{K}_n\,(1,1)} a\right). \end{aligned}$$

Recall that $\tilde{K}_n\,(1,1) \sim n^{2/3}$. Then $|\zeta - 1| \leq \frac{C}{n^{2/3}}$, so by (4.2), if $a \leq 0$, and by (1.23) if $a > 0$,

$$\begin{aligned} & n\frac{\mathbb{A}i\,(0,0)}{\tilde{K}_n\,(1,1)} \left|Q_n'\,(\zeta) - Q_n'\,(1)\right| \\ = \;& O\left(n^{1/3}\right) o\left(|1-\zeta|^{1/2}\right) = o\,(1)\,. \end{aligned}$$

Now (9.7) follows. ■

Recall that $\{a_j\}$ denote the zeros of Ai.

Lemma 9.3

Assume that (1.13) holds.
(a) Uniformly for $u \in (-\infty, 0)$,

$$f(u,u) \sim (|u|+1)^{1/2}. \tag{9.9}$$

(b) For all $a \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(a,y)|^2 \, dy \leq \frac{1}{\mathbb{A}i(0,0)} f(a,\bar{a}). \tag{9.10}$$

(c) For each $a \in \mathbb{C}$,

$$\sum_{j=1}^{\infty} \frac{\left|f\left(a, \frac{a_j}{\sigma}\right)\right|^2}{|a_j|^{1/2}} < \infty. \tag{9.11}$$

Proof
(a) From (9.1), (9.7) and (1.13), for fixed each $u \in (-\infty, 0)$, as $n \to \infty$,

$$f_n(u,u) = \frac{\tilde{K}_n\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u, 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}u\right)}{\tilde{K}_n(1,1)}(1+o(1)) = \frac{\mathbb{A}i(u,u)}{\mathbb{A}i(0,0)}(1+o(1)).$$

Thus

$$f(u,u) = \frac{\mathbb{A}i(u,u)}{\mathbb{A}i(0,0)}$$

so (3.21) gives the result.
(b) We use the identity

$$K_n(s,\bar{s}) = \int |K_n(s,t)|^2 \, d\mu_n(t),$$

valid for all complex s. Let $a \in \mathbb{C}$, and

$$s = 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a.$$

Let $r > 0$. We drop most of the integral and make the substitution $t = 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}y$:

$$\begin{aligned}
1 &\geq \int_{1-\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}r}^{1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}r} \frac{|K_n(s,t)|^2}{K_n(s,\bar{s})} W_n^{2n}(t) \, dt \\
&= \mathbb{A}i(0,0) \int_{-r}^{r} \left| \frac{K_n\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}a, 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}y\right)}{K_n(1,1)} \right|^2 \frac{K_n(1,1)}{K_n(s,\bar{s})} \frac{W_n^{2n}\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}y\right)}{W_n^{2n}(1)} dy \\
&= \mathbb{A}i(0,0) \int_{-r}^{r} \frac{\left|f_n(a,y) e^{-\Psi(n)(a+y)}\right|^2}{f_n(a,\bar{a}) e^{-\Psi(n)(a+\bar{a})}} e^{\Psi(n)2y} (1+o(1)) \, dy \\
&= \mathbb{A}i(0,0) \int_{-r}^{r} \frac{|f_n(a,y)|^2}{f_n(a,\bar{a})} (1+o(1)) dy.
\end{aligned}$$

Here we have used Lemma 9.2. As $n \to \infty$ through the subsequence $\mathcal{S}$, the last right-hand side has lim inf at least

$$\mathbb{A}i\left(0,0\right) \int_{-r}^{r} \frac{\left|f\left(a,y\right)\right|^{2}}{f\left(a,\bar{a}\right)} dy.$$

Finally, let $r \to \infty$. Of course, we implicitly assumed that $f\left(a,\bar{a}\right) \neq 0$, but a continuity argument and (9.10) shows that this can never be 0.

(c) We apply the quadrature sum estimate in Corollary 7.2: let n be given and

$$s = 1 + \frac{\mathbb{A}i\left(0,0\right)}{\tilde{K}_{n}\left(1,1\right)} a,\ s_{j} = 1 + \frac{\mathbb{A}i\left(0,0\right)}{\tilde{K}_{n}\left(1,1\right)} \frac{a_{j}}{\sigma};\ 1 \leq j \leq n.$$

Then from (3.23) and (3.19), uniformly in j and n

$$1 - s_{j} \sim \left(\frac{j}{n}\right)^{2/3} \Rightarrow \varphi_{n}\left(s_{j}\right) \sim j^{-1/3} n^{-2/3}.$$

Then using also the spacing (3.24) of $\{a_{j}\}$,

$$s_{j-1} - s_{j} \sim j^{-1/3} n^{-2/3} \sim \varphi_{n}\left(s_{j}\right). \tag{9.12}$$

Thus (7.3) is satisfied. We can then apply Corollary 7.2 to

$$P\left(t\right) = K_{n}\left(s,t\right)$$

so that for fixed $L \geq 1$,

(9.13)

$$\sum_{j=1}^{L} \left|K_{n}\left(s,s_{j}\right)\right|^{2} W_{n}^{2n}\left(s_{j}\right)\left(s_{j-1}-s_{j}\right) \leq C \int_{-1}^{1} \left|K_{n}\left(s,t\right)\right|^{2} W_{n}^{2n}\left(t\right) dt \leq C K_{n}\left(s,\bar{s}\right).$$

We emphasize that C is independent of L, n. We multiply (9.13) by $\frac{e^{\Psi(n)(a+\bar{a})}}{K_{n}(1,1)}$, apply (9.12) and Lemma 9.2, and the definition of f_{n} to obtain

$$\sum_{j=1}^{L} \left|f_{n}\left(a, \frac{a_{j}}{\sigma}\right)\right|^{2} j^{-1/3} \leq C \left|f_{n}\left(a,\bar{a}\right)\right|. \tag{9.14}$$

Letting $n \to \infty$ through $\mathcal{S}$ gives

$$\sum_{j=1}^{L} \left|f\left(a, \frac{a_{j}}{\sigma}\right)\right|^{2} j^{-1/3} \leq C \left|f\left(a,\bar{a}\right)\right|.$$

Since C is independent of L, we can let $L \to \infty$ to obtain the result. ■

Remark

We note that the hypothesis that $d\mu_{n} = W_{n}^{2n} dx$ in $\left[-1, 1 + \frac{\log n}{n^{2/3}}\right]$ is needed only for (9.10) - so that we obtain an integral over $[-r, r]$, rather than over $[-r, 0]$. Everywhere else in this paper, we could assume that $d\mu_{n} = W_{n}^{2n} dx$ in $[-1, 1]$.

Now for our main inequality:

Lemma 9.4
(a) We have for all $b \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \left(\frac{f(b/\sigma, s/\sigma)}{f(b/\sigma, b/\sigma)} - \frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)} \right)^2 ds$$

$$\leq \frac{\sigma}{\mathbb{A}i(0,0) f(b/\sigma, b/\sigma)} - \frac{1}{\mathbb{A}i(b,b)}. \tag{9.15}$$

(b)

$$\sigma \geq \sup_{b\in\mathbb{R}} \frac{\mathbb{A}i(0,0) f(b/\sigma, b/\sigma)}{\mathbb{A}i(b,b)} \geq 1. \tag{9.16}$$

Proof
(a) The left-hand side in (9.15) equals

$$\frac{1}{f\left(\frac{b}{\sigma}, \frac{b}{\sigma}\right)^2} \int_{-\infty}^{\infty} f\left(\frac{b}{\sigma}, \frac{s}{\sigma}\right)^2 ds - \frac{2}{f\left(\frac{b}{\sigma}, \frac{b}{\sigma}\right)} \int_{-\infty}^{\infty} f\left(\frac{b}{\sigma}, \frac{s}{\sigma}\right) \frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)} ds$$
$$+ \frac{1}{\mathbb{A}i(b,b)^2} \int_{-\infty}^{\infty} \mathbb{A}i(b,s)^2 ds. \tag{9.17}$$

Firstly, from Lemma 9.3(b),

$$\frac{1}{f\left(\frac{b}{\sigma}, \frac{b}{\sigma}\right)^2} \int_{-\infty}^{\infty} \left| f\left(\frac{b}{\sigma}, \frac{s}{\sigma}\right) \right|^2 d\frac{s}{\sigma} \leq \frac{1}{\mathbb{A}i(0,0) f\left(\frac{b}{\sigma}, \frac{b}{\sigma}\right)}.$$

We claim next that

$$\int_{-\infty}^{\infty} f\left(\frac{b}{\sigma}, \frac{s}{\sigma}\right) \mathbb{A}i(b,s)\, ds = f\left(\frac{b}{\sigma}, \frac{b}{\sigma}\right). \tag{9.18}$$

Indeed the reproducing kernel identity

$$\int_{-\infty}^{\infty} g(s)\, \mathbb{A}i(b,s)\, ds = g(b)$$

was established in [18] for entire functions g of order at most $3/2$, whose restriction to the real line is in $L_2(\mathbb{R})$, and which in addition satisfy the following: whenever $0 < \delta < \pi$, there exists C_δ and $L > 0$ such that for $|\arg(z)| \leq \pi - \delta$;

$$|g(z)| \leq C_\delta (1 + |z|)^L \left| e^{-\frac{2}{3} z^{3/2}} \right|;$$

$$\sum_{j=1}^{\infty} \frac{|g(a_j)|^2}{|a_j|^{1/2}} < \infty.$$

The second condition is Lemma 9.3(c) for $g(z) = f\left(\frac{b}{\sigma}, \frac{z}{\sigma}\right)$, and the bound on growth follows from Theorem 9.1. We thus obtain (9.18). Choosing

$g(s) = \mathbb{A}i(b,s)$, we also obtain

$$\int_{-\infty}^{\infty} \mathbb{A}i^2(b,s)\,ds = \mathbb{A}i(b,b).$$

On substituting this and (9.18) into (9.17), we obtain the upper bound

$$\frac{\sigma}{\mathbb{A}i(0,0)\,f(b/\sigma,b/\sigma)} - \frac{1}{\mathbb{A}i(b,b)}$$

for the left-hand side of (9.15).
(b) The left inequality in (9.16) follows as the left-hand side of (9.15) is non-negative. Since $f(0,0)=1$, the remaining inequality also follows. ■

Recall from Section 3, the Gauss type quadrature formula, with nodes $\{t_{jn}\}$ including the point 1:

$$\sum_j \lambda_n(t_{jn})\,P(t_{jn}) = \int P(t)\,d\mu_n(t),$$

for all polynomials P of degree $\leq 2n-2$. Recall that we order the nodes as

$$... < t_{2n} < t_{1n} < t_{0n} = 1 < t_{-1,n} < ...,$$

and for $j \geq 0$, write for some $\rho_{jn} < 0$,

$$t_{jn} = 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}\rho_{jn}. \tag{9.19}$$

Lemma 9.5
(a) For each fixed $j \geq 0$, as $n \to \infty$ through $\mathcal{S}$,

$$\rho_{jn} \to \rho_j, \tag{9.20}$$

where $\rho_0 = 0$ and

$$0 > \rho_1 \geq \rho_2 \geq ... \,.$$

(b) The function $f(0,z)$ has (possibly multiple) zeros at $\rho_j, j \neq 0$, and no other zeros, except possibly for at most $\ell_0 + 1$ positive zeros.
Proof
(a), (b) We know that $f_n(0,z) = \left(K_n\left(1, 1+\frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)}z\right)/K(1,1)\right)e^{\Psi(n)z}$ has only real zeros, with simple zeros at ρ_{jn}, and perhaps positive zeros. Moreover as $n \to \infty$ through our subsequence, this sequence converges to $f(0,z)$, uniformly for z in compact sets, and $f(0,z)$ is not identically 0. Finally, $f(0,z)$ is entire of order $\leq 3/2$ and has at most ℓ_0+1 positive zeros by Theorem 9.1, while $f(0,\cdot) \in L_2(\mathbb{R})$, by Lemma 9.3. Then Hadamard's factorization theorem shows that $f(0,\cdot)$ must have infinitely many zeros. The result then follows by Hurwitz' theorem. ■

Lemma 9.6
For each fixed $k > \ell > 0$,

$$\sum_{j=\ell+1}^{k-1} \frac{1}{f\left(\rho_j, \rho_j\right)} \leq \mathbb{A}i\left(0,0\right)\left(\rho_\ell - \rho_k\right) \leq \sum_{j=\ell}^{k} \frac{1}{f\left(\rho_j, \rho_j\right)}. \tag{9.21}$$

Proof
We need the Markov-Stieltjes inequalities [17, Lemma 5.3], [10, p. 33]: for each $1 \leq \ell < k \leq n$, and $B \in \mathbb{R}$,

$$\sum_{t_{kn} < t_{jn} < t_{\ell n}} \lambda_n\left(t_{jn}\right) e^{Bt_{jn}} \leq \int_{t_{kn}}^{t_{\ell n}} e^{Bt} d\mu_n\left(t\right) \leq \sum_{t_{kn} \leq t_{jn} \leq t_{\ell n}} \lambda_n\left(t_{jn}\right) e^{Bt_{jn}}.$$

Now choose

$$B = -2\Psi\left(n\right) \frac{\tilde{K}_n\left(1,1\right)}{\mathbb{A}i\left(0,0\right)}.$$

Then after multiplying by $\frac{K_n(1,1)}{\mathbb{A}i(0,0)}$ we see that

$$\frac{K_n\left(1,1\right)}{\mathbb{A}i\left(0,0\right)} \sum_{j=\ell+1}^{k-1} \frac{e^{Bt_{jn}}}{K_n\left(t_{jn}, t_{jn}\right)} \leq \int_{t_{kn}}^{t_{\ell n}} \frac{K_n\left(1,1\right)}{\mathbb{A}i\left(0,0\right)} e^{Bt} d\mu_n\left(t\right) \leq \frac{K\left(1,1\right)}{\mathbb{A}i\left(0,0\right)} \sum_{j=\ell}^{k} \frac{e^{Bt_{jn}}}{K_n\left(t_{jn}, t_{jn}\right)}.$$

Using the substitution $t = 1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} s$ in the integral in the last line gives

$$\begin{aligned}
&\int_{t_{kn}}^{t_{\ell n}} \frac{K_n\left(1,1\right)}{\mathbb{A}i\left(0,0\right)} e^{Bt} W_n^{2n}\left(t\right) dt \\
= \quad & e^B \int_{\rho_{kn}}^{\rho_{\ell n}} e^{-2\Psi(n)s} \frac{W_n^{2n}\left(1 + \frac{\mathbb{A}i(0,0)}{\tilde{K}_n(1,1)} s\right)}{W_n^{2n}\left(1\right)} ds \\
= \quad & e^B \left(\rho_\ell - \rho_k + o\left(1\right)\right),
\end{aligned}$$

by Lemma 9.2 and the convergence in Lemma 9.5. Next, for each fixed j, as $n \to \infty$ through $\mathcal{S}$,

$$\begin{aligned}
\frac{K_n\left(1,1\right)}{K_n\left(t_{jn}, t_{jn}\right)} e^{Bt_{jn}} &= e^B \frac{1}{f_n\left(\rho_{jn}, \rho_{jn}\right)} \\
&= e^B \frac{1 + o\left(1\right)}{f\left(\rho_j, \rho_j\right)}.
\end{aligned}$$

Thus for each fixed $k > \ell$,

$$\frac{1}{\mathbb{A}i\left(0,0\right)} \sum_{j=\ell+1}^{k-1} \frac{1}{f\left(\rho_j, \rho_j\right)} \leq \rho_\ell - \rho_k \leq \frac{1}{\mathbb{A}i\left(0,0\right)} \sum_{j=\ell}^{k} \frac{1}{f\left(\rho_j, \rho_j\right)}.$$

■

Lemma 9.7
Assume that for all large enough positive x,

$$f(-x,-x) = \frac{\mathbb{A}i(x,x)}{\mathbb{A}i(0,0)}. \tag{9.22}$$

Then

$$\lim_{r\to\infty} \frac{n(f(0,\cdot),[-r,0])}{r^{3/2}} = \lim_{r\to\infty} \frac{n(f(0,\cdot),r)}{r^{3/2}} = \frac{2}{3\pi}. \tag{9.23}$$

Proof
By (3.21), we have as $x \to \infty$

$$f(-x,-x) = \frac{x^{1/2}}{\pi \mathbb{A}i(0,0)}(1+o(1)). \tag{9.24}$$

Let $\delta \in (0,1)$. Choose C_0 such that for $x \geq C_0$,

$$\frac{1}{1+\delta} \leq f(-x,-x) / \left[\frac{x^{1/2}}{\pi \mathbb{A}i(0,0)}\right] \leq 1+\delta \tag{9.25}$$

and choose J such that

$$\left|\rho_j\right| \geq C_0 \text{ for } j \geq J. \tag{9.26}$$

By the previous lemma, for each fixed $k > \ell \geq J$,

$$\begin{aligned} &\pi(1+\delta)^{-1} \sum_{j=\ell+1}^{k-1} \frac{1}{\left|\rho_j\right|^{1/2}} \\ \leq \quad & \rho_\ell - \rho_k \leq \pi(1+\delta) \sum_{j=\ell}^{k} \frac{1}{\left|\rho_j\right|^{1/2}}, \end{aligned}$$

and hence, for each fixed $k > \ell \geq J$,

$$\begin{aligned} &\pi(1+\delta)^{-1} |\rho_k|^{-1/2} (k-\ell-1) \\ \leq \quad & \rho_\ell - \rho_k \leq \pi(1+\delta) |\rho_\ell|^{-1/2} (k-\ell+1). \end{aligned} \tag{9.27}$$

Fortunately, the rate of change of the function $x^{-1/2}$ is slow enough to allow us to effectively remove the difference between $|\rho_\ell|^{-1/2}$ and $|\rho_k|^{-1/2}$ in the last inequality: we can do this by choosing a subsequence of k for which $|\rho_k|$ grows geometrically, taking account of the fact that the spacing between successive ρ_j approaches 0.

Let us now make this rigorous. Recall that we fixed $\delta \in (0,1)$. Fix some large r, and choose m and an increasing sequence $\{k_j\}_{j=0}^m$ such that for $1 \leq j < m$,

$$\left|\rho_{k_j}\right| \geq (1+\delta)\left|\rho_{k_{j-1}}\right| \tag{9.28}$$

but

$$\left|\rho_{k_j-1}\right| < (1+\delta)\left|\rho_{k_{j-1}}\right| \tag{9.29}$$

while $k_0 = J$ and

$$\left|\rho_{k_m-1}\right| < r \leq \left|\rho_{k_m}\right|. \tag{9.30}$$

Note that it is possible to choose such a sequence, at least if J is large enough. Indeed, (9.21) and (9.24) show that $|\rho_k| - \left|\rho_{k-2}\right| = O\left(|\rho_k|^{-1/2}\right) \to 0$ as $k \to \infty$. Moreover,

$$\begin{aligned} \left|\rho_{k_j}\right| &= \left|\rho_{k_j-1}\right| + \left(\left|\rho_{k_j}\right| - \left|\rho_{k_j-1}\right|\right) \\ &\leq (1+\delta)\left|\rho_{k_{j-1}}\right| + O\left(\left|\rho_{k_j}\right|^{-1/2}\right). \end{aligned}$$

It follows (perhaps by increasing J) that for $j \geq 1$,

$$\left|\rho_{k_j}\right| \leq \left(1 + \frac{3}{2}\delta\right)\left|\rho_{k_{j-1}}\right|. \tag{9.31}$$

Similarly,

$$\left|\rho_{k_m}\right| < r + O\left(\left|\rho_{k_m}^{-1/2}\right|\right). \tag{9.32}$$

Next, (9.27) with $k = k_j$ and $\ell = k_{j-1}$ gives

$$\begin{aligned} &\pi (1+\delta)^{-1}\left|\rho_{k_j}\right|^{-1/2}(k_j - k_{j-1} - 1) \\ \leq\ & \left|\rho_{k_j}\right| - \left|\rho_{k_{j-1}}\right| \leq \pi(1+\delta)\left|\rho_{k_{j-1}}\right|^{-1/2}(k_j - k_{j-1} + 1) \end{aligned}$$

and hence also

$$\begin{aligned} &\pi\left(1+\frac{3}{2}\delta\right)^{-3/2}(k_j - k_{j-1} - 1) \\ \leq\ & \int_{\left|\rho_{k_{j-1}}\right|}^{\left|\rho_{k_j}\right|} t^{1/2}dt \leq \pi\left(1+\frac{3}{2}\delta\right)^{3/2}(k_j - k_{j-1} + 1). \end{aligned}$$

Next, (9.21) shows that $\rho_{j+2} - \rho_j > 0$, so each ρ_j is at most a double zero of $f(0,z)$. Moreover, each double zero will be repeated in the sequence $\{\rho_j\}$. It follows that $n\left(f(0,\cdot), [\rho_{k_j}, \rho_{k_{j-1}})\right) = k_j - k_{j-1}$ or $k_j - k_{j-1} + 1$. Thus, adding this last inequality over $1 \leq j \leq m$, gives

$$\begin{aligned} &\pi\left(1+\frac{3}{2}\delta\right)^{-3/2} n(f(0,\cdot), [-r,0]) + O(m) \\ \leq\ & \int_0^r t^{1/2}dt + O\left(r^{1/2}\right) \\ \leq\ & \pi\left(1+\frac{3}{2}\delta\right)^{3/2} n(f(0,\cdot), [-r,0]) + O(m). \end{aligned} \tag{9.33}$$

Here, in bounding the order terms, we have used the fact that the number of zeros in $\left[\rho_{k_0}, 0\right] = [\rho_J, 0]$ is independent of r, and (9.32). Next, from (9.28),

$$\left|\rho_{k_m}\right| \geq (1+\delta)^m \left|\rho_{k_0}\right|$$

so

$$r + o(1) \geq (1+\delta)^m \left|\rho_J\right|.$$

We deduce that

$$m = O(\log r).$$

Now combine this with (9.33), divide by $r^{3/2}$ and let $r \to \infty$. We obtain

$$\begin{aligned} & \pi\left(1+\frac{3}{2}\delta\right)^{-3/2} \limsup_{r\to\infty} \frac{n\left(f(0,\cdot), [-r,0]\right)}{r^{3/2}} \\ \leq \; & \frac{2}{3} \leq \pi\left(1+\frac{3}{2}\delta\right)^{3/2} \liminf_{r\to\infty} \frac{n\left(f(0,\cdot), [-r,0]\right)}{r^{3/2}}. \end{aligned}$$

Finally, let $\delta \to 0+$ to deduce the result. ■

Lemma 9.8

$$\sigma = 1.$$

Proof
Let $g(z) = f(0,z)$ and

$$R(z) = L(z) \prod_{j=1}^{\infty} \left(1 - \frac{z}{\rho_j}\right) e^{z/\rho_j}$$

denote the canonical product whose zeros are those of $f(0,\cdot)$. Here if $f(0,\cdot)$ has no positive zero, L is just the constant 1, while if $f(0,\cdot)$ has positive zeros, L is a polynomial of degree at most $\ell_0 + 1$ with those zeros. Since $f(0,z)$ is of order $\leq 3/2$, the Hadamard Factorization Theorem shows that for some constant c,

$$g(z) = f(0,z) = e^{cz} R(z).$$

We have $n(R,r) = n(f(0,\cdot), r)$, so that

$$\lim_{r\to\infty} \frac{n(R,r)}{r^{3/2}} = \frac{2}{3\pi}. \tag{9.34}$$

Moreover, if we define the indicator functions of R and g,

$$\begin{aligned} h_R(\theta) &= \limsup_{r\to\infty} \frac{\log\left|R\left(re^{i\theta}\right)\right|}{r^{3/2}}; \\ h_g(\theta) &= \limsup_{r\to\infty} \frac{\log\left|g\left(re^{i\theta}\right)\right|}{r^{3/2}}, \end{aligned}$$

we see that

$$h_g(\theta) = h_R(\theta).$$

Next, by (9.23) and by Theorem 8.1 of [14, p. 81], applied to $R(-z)$,

$$\sup_{\theta\in(0,2\pi)}\left|\log R\left(-re^{i\theta}\right)+\frac{2}{3}e^{i\frac{3}{2}(\theta-\pi)}r^{3/2}\right|\sin\frac{\theta}{2}=o\left(r^{3/2}\right),$$

as $r\to\infty$. Hence, as $r\to\infty$,

$$\sup_{t\in(-\pi,\pi)}\left|\frac{\log\left|R\left(re^{it}\right)\right|}{r^{3/2}}+\frac{2}{3}\operatorname{Re}\left(e^{i\frac{3}{2}t}\right)\right|\cos\frac{t}{2}=o(1).$$

Hence for $t\in(-\pi,\pi)$,

$$h_R(t)=-\frac{2}{3}\cos\frac{3}{2}t. \tag{9.35}$$

Also from Theorem 9.1, and the decomposition above, as $r\to\infty$.

$$\begin{aligned}\frac{\log\left|R\left(re^{it}\right)\right|}{r^{3/2}} &= \frac{\log\left|g\left(re^{it}\right)\right|}{r^{3/2}}+o(1)\\ &\leq -\frac{2}{3}\sigma^{3/2}\operatorname{Re}\left(e^{i\frac{3}{2}t}\right)+o(1)\\ &= -\frac{2}{3}\sigma^{3/2}\cos\frac{3}{2}t+o(1).\end{aligned}$$

Thus for $t\in(-\pi,\pi)$,

$$-\frac{2}{3}\cos\frac{3}{2}t\leq-\frac{2}{3}\sigma^{3/2}\cos\frac{3}{2}t.$$

As $\cos\frac{3}{2}t$ assumes both positive and negative values, we deduce that $\sigma^{3/2}=1$. ■

Proof of Theorem 1.3

We must show the equivalence of (1.13) and (1.14), that is (I)$\Longleftrightarrow$(II).

(II)$\Rightarrow$(I) is immediate.

(I)$\Rightarrow$(II).

Let f be as in Theorem 9.1. It follows directly from (1.13) that for all real a,

$$f(a,a)=\frac{\mathbb{A}i(a,a)}{\mathbb{A}i(0,0)}.$$

Next, we apply Lemma 9.4 with $\sigma=1$:

$$\begin{aligned}&\int_{-\infty}^{\infty}\left(\frac{f(b,s)}{f(b,b)}-\frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)}\right)^2 ds\\ \leq\ &\frac{1}{\mathbb{A}i(0,0)\,f(b,b)}-\frac{1}{\mathbb{A}i(b,b)}=0,\end{aligned}$$

by our hypothesis (1.13). Thus for all s, and all b,

$$\frac{f(b,s)}{f(b,b)}=\frac{\mathbb{A}i(b,s)}{\mathbb{A}i(b,b)},$$

and hence

$$f(b,s)=\frac{\mathbb{A}i(b,s)}{\mathbb{A}i(0,0)}.$$

Since the limit is independent of the subsequence $\mathcal{S}$ of Theorem 9.1, we obtain for all real b,s,

$$\lim_{n\to\infty}f_n(b,s)=\frac{\mathbb{A}i(b,s)}{\mathbb{A}i(0,0)}.$$

Moreover, because the left-hand side is uniformly bounded for b,s in compact subsets of the plane, this limit holds uniformly for a,b in compact subsets of the plane. Then (1.14) follows from Lemma 9.2, and (1.15) follows as well. ■

Proof of Theorem 1.2

We show that all the hypotheses of Theorem 1.3 are fulfilled with $Q_n=Q$, $n\geq 1$. If Q is convex, then it follows from Theorem 4.1 in [15, p. 95] that for not identically vanishing polynomials P of degree $\leq 2n-2$, we have

$$\int_{\mathbb{R}\setminus[-1,1]}|P(x)|\,W^{2n}(x)\,dx<\int_{-1}^{1}|P(x)|\,W^{2n}(x)\,dx. \tag{9.36}$$

Indeed, we choose $p=1$ and $t=2n$ there, and note that since $[-1,1]$ is the support of the equilibrium measure for Q, so 1 is the Mhaskar-Rakhmanov-Saff number a_{2n} for W^{2n}. The exact same proof works for the case where Q is even and $xQ'(x)$ is increasing on $[0,1]$. Of course, (9.36) yields (1.16) with $A=1$, so $\left\{W^{2n}dx\right\}$ admits a restricted range inequality to $[-1,1]$.

Next, it is a consequence of (9.36) that at most one zero of $p_{n,n}$ lies to the right of 1, so (1.17) holds in a stronger form. To see this, let us assume on the contrary that $x_{2n}>1$, and let us apply (9.36), in the form

$$\begin{aligned}\int_{\mathbb{R}\setminus[-1,1]}\left|\frac{p_{n,n}^2(x)}{(x-x_{1n})(x-x_{2n})}\right|W^{2n}(x)\,dx &< \int_{-1}^{1}\left|\frac{p_{n,n}^2(x)}{(x-x_{1n})(x-x_{2n})}\right|W^{2n}(x)\,dx\\ &= \int_{-1}^{1}\frac{p_{n,n}^2(x)}{(x-x_{1n})(x-x_{2n})}W^{2n}(x)\,dx.\end{aligned}$$

But by orthogonality,

$$\int_{-\infty}^{\infty}\frac{p_{n,n}^2(x)}{(x-x_{1n})(x-x_{2n})}W^{2n}(x)\,dx=0,$$

and we obtain a contradiction.

Next, if Q is even and $xQ'(x)$ is increasing, ν_W is absolutely continuous on $[-1,1]$ and

$$\nu_W'(x)=\frac{2\sqrt{1-x^2}}{\pi}\int_0^1\frac{xQ'(x)-tQ'(t)}{x^2-t^2}\frac{dt}{\sqrt{1-t^2}}. \tag{9.37}$$

See [30, p. 226, proof of Thm. IV.3.2] and note that since we assumed $[-1,1]$ is the support of ν_W,

$$\frac{1}{\pi}\int_{-1}^{1}\frac{tQ'(t)}{\sqrt{1-t^2}}dt=1.$$

A little manipulation shows that

$$\begin{aligned}(9.38)\qquad \nu_W'(x) &= \frac{\sqrt{1-x^2}}{\pi}\int_{-1}^{1}\frac{Q'(x)-Q'(t)}{x-t}\frac{dt}{\sqrt{1-t^2}}\\ &= \sqrt{1-x^2}\psi(x)\text{, say.}\end{aligned}$$

When Q is instead convex, this formula still holds, see for example, Theorem 2.5 in [15, p. 42]. Here

$$\begin{aligned}\Delta &= \psi(1)=\frac{1}{\pi}\int_{-1}^{1}\frac{Q'(1)-Q'(t)}{1-t}\frac{dt}{\sqrt{1-t^2}}\\ &= \frac{2}{\pi}\int_{0}^{1}\frac{Q'(1)-tQ'(t)}{1-t^2}\frac{dt}{\sqrt{1-t^2}}\end{aligned}$$

is finite as we assumed that Q' satisfies a Lipschitz condition of order larger than $1/2$ near 1. It is also positive because $xQ'(x)$ increasing.Moreover, as Q' satisfies a Lipschitz condition of positive order in $[-1,1]$, while

$$\psi(x)=\frac{PV}{\pi}\int_{0}^{\pi}\frac{Q'(\cos\theta)}{\cos\theta-x}d\theta,$$

Privalov's Theorem shows that ψ satisfies a Lipschitz condition of positive order in $(-1+\varepsilon,1-\varepsilon)$ for each $\varepsilon>0$, while the condition on Q' near 1 ensures that ψ satisfies a Lipschitz condition of some positive order throughout $[-1,1]$. Then both (1.21) and (1.22) follow for $\psi_n=\psi$, $n\geq 1$. The positivity of ψ in $[-1,1]$ follows from (9.37) or (9.38), and its continuity ensures that it has a positive lower bound. Finally, since either Q is even, or $xQ'(x)$ is increasing in $[0,1]$ and Q is even, the support of the equilibrium measure for $W^\lambda=e^{-\lambda Q}$ is an interval [30, p. 198, Thm. IV.1.10]. Finally, (1.23) follows as Q' satisfies a Lipschitz condition of order $>\frac{1}{2}$ in a neighborhood of 1. ■

Acknowledgement

The authors gratefully acknowledge the insights of Alexandre Eremenko regarding the zero distribution of entire functions of non-integer order.

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