UNIVERSALITY LIMITS IN THE BULK FOR VARYING MEASURES

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Abstract. Universality limits are a central topic in the theory of random matrices. We establish universality limits in the bulk of the spectrum for varying measures, using the theory of entire functions of exponential type. In particular, we consider measures that are of the form $W_n^{2n}(x)\,dx$ in the region where universality is desired. W_n does not need to be analytic, nor possess more than one derivative - and then only in the region where universality is desired. We deduce universality in the bulk for a large class of weights of the form $W^{2n}(x)\,dx$, for example, when $W = e^{-Q}$ where Q is convex and Q' satisfies a Lipschitz condition of some positive order. We also deduce universality for a class of fixed exponential weights on a real interval.

1. Introduction and Results[1]

Let $\mathcal{M}(n)$ denote the space of n by n Hermitian matrices $M = (m_{ij})_{1\le i,j\le n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$\begin{aligned} P^{(n)}(M) &= ce^{-F_n(M)}dM \\ &= ce^{-F_n(M)}\left(\prod\nolimits_{j=1}^{n} dm_{jj}\right)\left(\prod\nolimits_{j<k} d(\operatorname{Re} m_{jk})\, d(\operatorname{Im} m_{jk})\right). \end{aligned}$$

Here $F_n(M)$ is a function defined on $\mathcal{M}(n)$, and c is a normalizing constant. The most important case is

$$F_n(M) = 2n\ \mathrm{tr}Q_n(M),$$

for appropriate functions $\{Q_n\}$ defined on $\mathcal{M}(n)$. In particular, the choice

$$F_n(M) = 2n\ \mathrm{tr}\left(M^2\right),$$

leads to the Gaussian unitary ensemble (apart from scaling) that was considered by Wigner. One may identify $P^{(n)}$ above with a probability density on the eigenvalues $x_1 \le x_2 \le ... \le x_n$ of M,

$$P^{(n)}(x_1, x_2, ..., x_n) = ce^{-\sum_{j=1}^{n} 2nQ_n(x_j)}\prod\nolimits_{i<j}(x_i - x_j)^2.$$

See [15, p. 102 ff.]. Again, c is a normalizing constant.

It is at this stage that orthogonal polynomials with a respect to a sequence of measures arise [15], [41]. For $n \ge 1$, let μ_n be a finite positive Borel

Date: May 19, 2008.

[1]Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353

measure with support supp$[\mu_n]$ and infinitely many points in the support. If the support of μ_n is unbounded, we assume that at least the first $2n+1$ power moments

$$\int x^j d\mu_n(x), \ 0 \leq j \leq 2n,$$

are finite. Then we may define orthonormal polynomials

$$p_{n,m}(x) = \gamma_{n,m} x^m + ..., \gamma_{n,m} > 0,$$

$m = 0, 1, 2, ...n$, satisfying the orthonormality conditions

$$\int p_{n,j} p_{n,k} d\mu_n = \delta_{jk}.$$

Throughout we use μ_n' to denote the Radon-Nikodym derivative of μ_n. The nth reproducing kernel for μ_n is

$$K_n(x,y) = \sum_{k=0}^{n-1} p_{n,k}(x) p_{n,k}(y) \tag{1.1}$$

and the normalized kernel is

$$\widetilde{K}_n(x,y) = \mu_n'(x)^{1/2} \mu_n'(y)^{1/2} K_n(x,y). \tag{1.2}$$

When

$$d\mu_n(x) = e^{-2nQ_n(x)} dx,$$

there is the basic formula for the probability distribution $P^{(n)}$ [15, p.112]:

$$P^{(n)}(x_1, x_2, ..., x_n) = \frac{1}{n!} \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \leq i,j \leq n}.$$

One may use this to compute a host of statistical quantities - for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One particularly important quantity is the $m-$point correlation function for $M(n)$ [15, p. 112]:

$$\begin{aligned} R_m(x_1, x_{2,...}, x_m) &= \frac{n!}{(n-m)!} \int ... \int P^{(n)}(x_1, x_2..., x_n) \, dx_{m+1} \, dx_{m+2} \, ...dx_n \\ &= \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \leq i,j \leq m}. \end{aligned}$$

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, ξ in a suitable subset of the (common) supports of $\{\mu_n\}$, and real $a_1, a_2, ..., a_m$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\xi,\xi)^m} R_m\left(\xi + \frac{a_1}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi,\xi)}, ..., \xi + \frac{a_m}{\tilde{K}_n(\xi,\xi)}\right) \\ &= \det\left(\frac{\sin \pi(a_i - a_j)}{\pi(a_i - a_j)}\right)_{1 \leq i,j \leq m}. \end{aligned}$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin \pi(a_i - a_j)}{\pi(a_i - a_j)}$ as 1. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n \to \infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \frac{\sin \pi (a-b)}{\pi (a-b)}. \tag{1.3}$$

Typically, this is established uniformly for a, b in compact subsets of the real line. Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials.

As suggested above, in many of the most important applications, $Q_n = Q$, and we consider measures of the form

$$d\mu_n(x) = e^{-2nQ(x)} dx.$$

In analyzing this case, Riemann-Hilbert methods have yielded spectacular advances - asymptotics of orthogonal polynomials, with complete asymptotic expansions for remainder terms, that can be substituted directly into the Christoffel-Darboux formula

$$K_n(x,y) = \frac{\gamma_{n,n-1}}{\gamma_{n,n}} \frac{p_{n,n}(x)\, p_{n,n-1}(y) - p_{n,n-1}(x)\, p_{n,n}(y)}{x-y}. \tag{1.4}$$

For example, if Q is real analytic on the real axis, and $Q(x)/\log\left(1+x^2\right)$ has limit ∞ at $\pm\infty$, then Deift et al [17] established (1.3), and they can derive remainder terms in the limit as well. Subsequently, McLaughlin and Miller [39], [40] used the $\bar{\partial}$ technique to replace analyticity by conditions on the second derivative of Q. There is an extensive literature on random matrices and Riemann-Hilbert methods. A (very!) partial list is [2], [3], [4], [8], [9], [10], [11], [12], [13], [14], [16], [25], [27], [28], [29], [38], [57].

Another established approach that has yielded very useful results involves classical analysis and operator theory, especially Toeplitz and Hankel operators [5], [6], [54], [55], [56], [58]. Further approaches, often with a mathematical physics origin, appear in [1], [18], [19], [20], [22], [43], [44]. Again, this list is incomplete. The online book by Forrester [20] and the lecture notes by Deift [15] may be used as an introduction to the subject. The recent conference proceedings of the 60th birthday conference of Percy Deift will contain up to date references [4].

In [36] and [37] two new approaches were presented for proving universality for fixed measures on a compact set. The first new approach [36] involved a comparison inequality, and applied to regular measures (in the sense of Stahl and Totik [48]) on $[-1, 1]$. It required only absolute continuity of the measure μ in a neighborhood of the point where universality is desired, together with positivity and continuity of μ' at that point.

It was subsequently extended to regular measures on arbitrary compact subsets of the real line using a host of other ideas by Barry Simon [47] and Vili Totik [53]. Totik used polynomial pullbacks for the extension to general sets, and showed that continuity of μ' may be weakened to a Lebesgue point type condition. Moreover, when $\log \mu'$ is integrable in an interval, then universality holds a.e. in that interval. Simon used the theory of Jost functions for the extension to general sets. The approach of [36] has been applied at the edge of the spectrum [35], on the unit circle [34], and to spacing of zeros of orthogonal polynomials [33]. It has also been applied to fixed exponential weights [32], together with other ideas. There new ways were introduced to prove universality for exponential weights, showing that first order asymptotics of orthogonal polynomials suffice.

The second new approach [37] is more powerful, and direct, and uses the theory of entire functions of exponential type. It avoids the assumption of regularity of the measure, and shows that universality is equivalent to universality along the diagonal - namely that (1.3) holds with $a = b$. In this paper, we use that method to handle varying weights, and subsequently fixed exponential weights. The hypotheses involve the nth Christoffel function for μ_n, namely,

$$\lambda_n(x) = \lambda_n(\mu_n, x) = 1/K_n(x, x). \tag{1.5}$$

When μ_n is absolutely continuous, we shall use also the notation $\lambda_n(\mu_n', x)$. There is the well known extremal property

$$\lambda_n(x) = \inf_{\deg(P) \leq n-1} \frac{\int P^2(t)\, d\mu_n(t)}{P^2(x)}.$$

In addition, we need some concepts from potential theory for external fields [45]. Let Σ be a closed set on the real line, and

$$W(x) = \exp(-Q(x))$$

be a continuous function on Σ. If Σ is unbounded, we assume that

$$\lim_{|x| \to \infty, x \in \Sigma} W(x)|x| = 0.$$

Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu} \left(\int \int \log \frac{1}{|x-t|} d\nu(x)\, d\nu(t) + 2 \int Q\, d\nu \right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ν_W, characterized by the following conditions: let

$$V^{\nu_W}(z) = \int \log \frac{1}{|z-t|} d\nu_W(t)$$

denote the potential for ν_W. Then

$$\begin{aligned} V^{\nu_W} + Q &\geq F_W \text{ on } \Sigma; \\ V^{\nu_W} + Q &= F_W \text{ in } \operatorname{supp}[\nu_W]. \end{aligned}$$

Here the number F_W is a constant. Usually ν_W is denoted μ_W, but we use a different symbol to avoid confusion with our measures of orthogonality $\{\mu_n\}$.

Our first result imposes similar hypotheses to those of Vili Totik [51], who studied asymptotics for Christoffel functions for varying weights.

Theorem 1.1
Let $W = e^{-Q}$ be a continuous non-negative function on the set Σ, which is assumed to consist of at most finitely many intervals. If Σ is unbounded, we assume also

$$\lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0.$$

Let h be a bounded positive continuous function on Σ, and for $n \geq 1$, let

$$d\mu_n(x) = \left(hW^{2n}\right)(x)\,dx. \tag{1.6}$$

Moreover, let $\tilde{K}_n$ denote the normalized nth reproducing kernel for μ_n.

Let J be a closed interval lying in the interior of supp$[\nu_W]$, where ν_W denotes the equilibrium measure for W. Assume that ν_W is absolutely continuous in a neighborhood of J, and that ν_W' and Q' are continuous in that neighborhood, while $\nu_W' > 0$ there. Then uniformly for $\xi \in J$, and a, b in compact subsets of the real line, we have

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n(\xi,\xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{1.7}$$

In particular, when Q' satisfies a Lipschitz condition of some positive order in a neighborhood of J, then [45, p. 216] ν_W' is continuous there, and hence we obtain universality except near zeros of ν_W'. Note too that when Q is convex in Σ, or $xQ'(x)$ is increasing there, then the support of ν_W consists of at most finitely many intervals, with at most one interval per component of Σ [45, p. 199]. More generally, if $\exp(Q)$ is convex in Σ, it is still true that the support of ν_W consists of at most finitely many intervals, with at most one interval per component of Σ [7, Theorem 5].

The proof of Theorem 1 depends heavily on Totik's asymptotics for Christoffel functions [51]. We note that prior to this result, the most general universality result for varying weights places global conditions on Q'' [40]. That paper is based on the $\bar{\partial}$ Riemann Hilbert method. The original powerful Riemann-Hilbert methods required Q to be real analytic [17]. Theorem 1.1 follows easily from the following general result:

Theorem 1.2
For $n \geq 1$, let μ_n be a positive Borel measure on the real line, with at least the first $2n+1$ power moments finite. Let I be a compact interval in which each μ_n is absolutely continuous. Assume moreover that in I,

$$d\mu_n(x) = h(x)\,W_n^{2n}(x)\,dx, \tag{1.8}$$

where

$$W_n = e^{-Q_n} \tag{1.9}$$

is continuous on I, and h is a bounded positive continuous function on I. Let ν_{W_n} denote the equilibrium measure for the restriction of W_n to I. Let J be a compact subinterval of I^o. Assume that
(a) $\left\{\nu'_{W_n}\right\}_{n=1}^{\infty}$ are positive and uniformly bounded in some open interval containing J;
(b) $\{Q'_n\}_{n=1}^{\infty}$ are equicontinuous and uniformly bounded in some open interval containing J;
(c) For some $C_1, C_2 > 0$, and for $n \geq 1$ and $\xi \in I$, the Christoffel functions $\lambda_n(\cdot) = \lambda_n(\mu_n, \cdot)$ satisfy

$$C_1 \leq \lambda_n^{-1}(\xi) W_n^{2n}(\xi)/n \leq C_2. \tag{1.10}$$

(d) Uniformly for $\xi \in J$ and a in compact subsets of the real line,

$$\lim_{n\to\infty} \frac{\lambda_n\left(\xi + \frac{a}{n}\right)}{\lambda_n(\xi)} \frac{W_n^{2n}(\xi)}{W_n^{2n}\left(\xi + \frac{a}{n}\right)} = 1. \tag{1.11}$$

Then uniformly for $\xi \in J$, and a, b in compact subsets of the real line, we have (1.7).

Remarks

(i) We note that we think of W_n as defined only on the interval I, and ν'_{W_n} is the equilibrium density for W_n defined only on I. In contrast, μ_n is typically defined on a larger interval. In applications, W_n might also be defined on a larger interval, and in this case the equilibrium measures ν_{W_n} should be thought of as equilibrium measures for the restriction of W_n to I. This can also be seen from our hypothesis (1.10), that the bounds for the Christoffel functions for μ_n hold on all of I, which in applications forces I to be a proper subset of the support of μ_n.

(ii) We can weaken the equicontinuity assumption (b) on $\{Q'_n\}$. We actually need only that for some open interval J_2 containing J, and each fixed $a > 0$,

$$\sup_{t\in J_2, |h|\leq a} \left| Q'_n(t) - Q'_n\left(t + \frac{h}{n}\right)\right| \to 0 \text{ as } n \to \infty. \tag{1.12}$$

In fact, we shall need this weaker hypothesis in Section 7, where we consider fixed exponential weights.

(iii) Under mild additional conditions on $\{Q'_n\}$, such as them satisfying a uniform Lipschitz condition, of some positive order, on some open interval containing J, one can establish (a) and (c) in Theorem 1.2, using methods in [31] or [51]. Moreover, one can use the methods of [31], or perhaps in greater generality, those of [50], [51], to establish (d). However, we omit these here, as this would substantially lengthen the paper, and distract from the new techniques that are used here.

(iv) Our proof actually establishes the following limit, uniformly for $\xi \in J$ and a, b in compact subsets of the complex plane, not just the real line:

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} e^{-\frac{n}{\widetilde{K}_n(\xi,\xi)} Q'_n(\xi)(a+b)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{1.13}$$

This paper is organised as follows. In Section 2, we present some of the main ideas of proof. In Section 3, we present notation and background for Sections 4 through 6. In Section 4, we use normality to establish some elementary properties, and in Section 5, we prove Theorem 1.2. In Section 6, we deduce Theorem 1.1. In Section 7, we shall establish universality for fixed exponential weights.

2. The Ideas of Proof

We start with the hypothesis (c) from Theorem 1.2. It may be reformulated as

$$C_1 \leq \frac{1}{n} K_n(\xi, \xi) W_n^{2n}(\xi) \leq C_2 \tag{2.1}$$

for $n \geq 1$ and $\xi \in I$. Using Cauchy-Schwarz's inequality, we obtain

$$\frac{1}{n} \left| K_n(\xi, t) \right| W_n^n(\xi) W_n^n(t) \leq C$$

for $n \geq 1$ and $\xi, t \in I$. The elements of potential theory for external fields enable us to extend this bound into the complex plane. For this, we also use the uniform boundedness of the equilibrium densities $\left\{\nu'_{W_n}\right\}$. Applying these methods in each variable ξ, t above leads to the estimate

$$\frac{1}{n} \left| K_n\left(\xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right| W_n^n\left(\xi + \frac{\operatorname{Re} a}{n}\right) W_n^n\left(\xi + \frac{\operatorname{Re} b}{n}\right) \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}.$$

Here $a, b \in \mathbb{C}$ and C_1 and C_2 are independent of n, a, b, ξ. However, for a, b in a given compact subset $\mathcal{K}$ of the plane, the estimate holds for $n \geq n_0(\mathcal{K})$. Using (2.1) again, and recalling our notation (1.2), we obtain

$$\left| \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi, \xi)} \right| \frac{W_n^n\left(\xi + \frac{\operatorname{Re} a}{\tilde{K}_n(\xi,\xi)}\right) W_n^n\left(\xi + \frac{\operatorname{Re} b}{\tilde{K}_n(\xi,\xi)}\right)}{W_n^{2n}(\xi)} \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}.$$

Of course, the constants C_1 and C_2 might be different. Our assumptions on $\{Q_n\}$ ensure that

$$\frac{W_n^n\left(\xi + \frac{\operatorname{Re} a}{\tilde{K}_n(\xi,\xi)}\right) W_n^n\left(\xi + \frac{\operatorname{Re} b}{\tilde{K}_n(\xi,\xi)}\right)}{W_n^{2n}(\xi)} = e^{\Psi(\xi,n)(\operatorname{Re} a + \operatorname{Re} b)} (1 + o(1)),$$

where

$$\Psi(\xi, n) = -\frac{n}{\tilde{K}_n(\xi, \xi)} Q'_n(\xi).$$

Define

$$f_n(a, b) = \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi, \xi)} e^{\Psi(\xi,n)(a+b)},$$

an entire function of exponential type in each variable a, b. Given $A > 0$, we obtain for $n \geq n_0(A)$ and $|a|, |b| \leq A$, that

$$|f_n(a, b)| \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}. \tag{2.2}$$

Thus $\{f_n(a,b)\}_{n=1}^{\infty}$ is a normal family for a, b in the complex plane.

Let $f(a,b)$ be the limit of some subsequence $\{f_n(\cdot,\cdot)\}_{n\in\mathcal{S}}$ of $\{f_n(\cdot,\cdot)\}_{n=1}^{\infty}$. It is an entire function in a, b, but (2.2) shows more: for all complex a, b,

$$|f(a,b)| \leq C_1 e^{C_2(|\operatorname{Im} a|+|\operatorname{Im} b|)}. \tag{2.3}$$

So f is bounded for $a, b \in \mathbb{R}$, and is an entire function of exponential type in each variable. Our goal is to show

$$f(a,b) = \frac{\sin \pi (a-b)}{\pi (a-b)}.$$

Our main tool is to scale up properties of the reproducing kernel K_n, and after taking limits, to deduce that an analogous property is true for f. Let us fix a. Since for each real ξ, $K_n(\xi,t)$ has only real zeros, the same is true of $f(a,\cdot)$. Moreover, $f(a,\cdot)$ has countably many such zeros. Using elementary properties of the reproducing kernel K_n, we can show that for all $a \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(a,s)|^2 \, ds \leq f(a,\bar{a}).$$

If σ is the exponential type of $f(a,\cdot)$, we can show that σ is independent of a, using interlacing properties of zeros of K_n. Using the fact that $\frac{\sin \pi s}{\pi s}$ is a reproducing kernel for the entire functions of exponential type that are also in $L_2(\mathbb{R})$, we can establish the useful inequality

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \sigma (s-a)}{\sigma (s-a)} \right)^2 ds \\ &\leq \frac{1}{f(a,a)} - \frac{\pi}{\sigma}. \end{aligned} \tag{2.4}$$

From this we deduce

$$\sigma \geq \pi \sup_{x\in\mathbb{R}} f(x,x) \geq \pi.$$

For the converse inequality, we use Markov-Stieltjes inequalities, and a formula relating exponential type of entire functions and their zero distribution, to obtain

$$\sigma \leq \pi \sup_{x\in\mathbb{R}} f(x,x).$$

Thus,

$$\sigma = \pi \sup_{x\in\mathbb{R}} f(x,x),$$

and (2.4) becomes

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin \sigma (a-s)}{\sigma (a-s)} \right)^2 ds \\ \leq \; & \frac{1}{f(a,a)} - \frac{1}{\sup_{x\in\mathbb{R}} f(x,x)}. \end{aligned} \tag{2.5}$$

Assuming the hypothesis (1.11) of Theorem 1.2, we immediately obtain

$$f(x,x) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(x,x) = 1$$

for all x, and then $\sigma = \pi$. Substituting this back into (2.5), completes the proof of Theorem 1.2.

3. Notation and Background

In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, y, s, t. The same symbol does not necessarily denote the same constant in different occurences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. We use $\sim$ in the following sense: given real sequences $\{c_n\}$, $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 with

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions.

Throughout, μ_n denotes a finite positive Borel measure on the real line, having at least the first $2n+1$ power moments finite. The Radon-Nikodym derivative of μ_n is denoted μ_n'. The corresponding orthonormal polynomials are denoted by $\{p_{n,k}\}_{k=0}^{n}$, so that

$$\int p_{n,k} p_{n,j} d\mu_n = \delta_{jk}.$$

We denote the zeros of $p_{n,n}$ by

$$x_{nn} < x_{n-1,n} < ... < x_{2n} < x_{1n}. \tag{3.1}$$

The nth reproducing kernel for μ_n is denoted by $K_n(x,t)$, and is defined by (1.1), while the normalized reproducing kernel is defined by (1.2). The nth Christoffel function for μ_n is

$$\lambda_n(x) = \lambda_n(\mu_n, x) = 1/K_n(x,x) = \inf_{\deg(P)\leq n-1} \frac{\int P^2 d\mu_n}{P^2(x)}. \tag{3.2}$$

When μ_n is absolutely continuous, we shall often write $\lambda_n(\mu_n', x)$. In particular, $\lambda_n\left(hW_n^{2n}, x\right)$ will denote the nth Christoffel function for the weight hW_n^{2n}.

The Gauss quadrature formula asserts that whenever P is a polynomial of degree $\leq 2n-1$,

$$\sum_{j=1}^{n} \lambda_n(x_{jn}) P(x_{jn}) = \int P \, d\mu_n. \tag{3.3}$$

In addition to this, we shall need another Gauss type of quadrature formula [21, p. 19 ff.]. Given a real number ξ, there are n or $n-1$ points $t_{jn} = t_{jn}(\xi)$, one of which is ξ, such that

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P \, d\mu_n, \tag{3.4}$$

whenever P is a polynomial of degree $\leq 2n-3$. The $\{t_{jn}\}$ are zeros of

$$\psi_n(\xi, t) = p_{n,n}(\xi) p_{n,n-1}(t) - p_{n,n-1}(\xi) p_{n,n}(t), \tag{3.5}$$

regarded as a function of t. Note that only the finiteness of the first $2n+1$ moments is required for the existence of $\{t_{jn}\}$. This is well known, and obvious from the proofs in Freud [21].

In order to prove that universality holds uniformly for ξ in J, we shall fix a sequence $\{\xi_n\}$ of points in J, rather than a fixed ξ. At the nth stage, we shall consider the quadrature that includes ξ_n, so that

$$t_{jn} = t_{jn}(\xi_n) \text{ for all } j. \tag{3.6}$$

Because we wish to focus on ξ_n, we shall set $t_{0n} = \xi_n$, and order the $\{t_{jn}\}$ around ξ_n, treated as the origin:

$$\ldots < t_{-2,n} < t_{-1,n} < t_{0n} = \xi_n < t_{1n} < \ldots . \tag{3.7}$$

The sequence of $\{t_{jn}\}$ consists of either $n-1$ or n points, so terminates, and it is possible that all t_{jn} lie to the left or right of ξ_n. However in the limiting situations we treat, where ξ_n lies in the interior of the support, this will not occur. It is known [21, p. 19, proof of Theorem 3.1] that when $(p_{n,n}p_{n,n-1})(\xi_n) \neq 0$, then one zero of $\psi_n(\xi_n, t)$ lies in $(x_{jn}, x_{j-1,n})$ for each j, and the remaining zero lies outside $[x_{nn}, x_{1n}]$.

Throughout I and J will be the intervals in Theorem 1.2. Recall that

$$\mu_n' = hW_n^{2n} \text{ in } I.$$

We shall often abbreviate the equilibrium measure ν_{W_n} of W_n as ν_n. In addition to I and J, we shall need compact intervals J_1 and J_2 such that

$$I^o \supset J_2 \text{ and } J_2^o \supset J_1 \text{ and } J_1^o \supset J. \tag{3.8}$$

We assume that our hypotheses (a) and (b) in Theorem 1.2 hold in the following more detailed form:

$$0 < \nu_n'(x) \leq C_1 \text{ for } n \geq 1 \text{ and } x \in J_2; \tag{3.9}$$

$$\left\{Q_n'\right\} \text{ are uniformly bounded in } J_2; \tag{3.10}$$

For each fixed $a > 0$,

$$\sup_{t \in J_2, |h| \leq a} \left| Q_n'(t) - Q_n'\left(t + \frac{h}{n}\right) \right| \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Of course, this is the condition (1.12), which is weaker than the equicontinuity assumed in Theorem 1.2(b), but is all we shall use in our proofs.

For the given sequence $\{\xi_n\}$ in J, we shall define for $n \geq 1$,

$$f_n(a,b) = \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} e^{\Psi(\xi_n,n)(a+b)}, \tag{3.12}$$

where

$$\Psi(\xi_n,n) = -\frac{n}{\tilde{K}_n(\xi_n,\xi_n)} Q_n'(\xi_n). \tag{3.13}$$

The zeros of

$$f_n(0,t) = \frac{K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} e^{\Psi(\xi_n,n)t}$$

will be denoted by $\{\rho_{jn}\}_{j\neq 0}$. Thus, recalling (3.5) and (3.6), if $t_{jn} = t_{jn}(\xi_n)$, we have

$$\rho_{jn} = \tilde{K}_n(\xi_n,\xi_n)(t_{jn} - \xi_n).$$

We also set, corresponding to $t_{0n} = \xi_n$,

$$\rho_{0n} = 0.$$

For an appropriate subsequence $\mathcal{S}$ of integers, we shall let

$$f(a,b) = \lim_{n\to\infty, n\in\mathcal{S}} f_n(a,b). \tag{3.14}$$

The zeros of $f(0,\cdot)$ will be denoted by $\{\rho_j\}_{j\neq 0}$, and we set $\rho_0 = 0$. Our ordering of zeros is

$$... \leq \rho_{-2} \leq \rho_{-1} < \rho_0 = 0 < \rho_1 \leq \rho_2 \leq ... .$$

We shall denote the (exponential) type of $f(a,\cdot)$ by σ_a - it will be defined shortly. We shall show that σ_a is independent of a, and then just use σ to denote the type. Initially, this type will be associated with the specific subsequence $\mathcal{S}$.

We next review some theory of entire functions of exponential type. Most of this can be found in the elegant series of lectures of B. Ya. Levin [30]. Recall that if g is entire of order 1, then its *exponential type* σ is

$$\sigma = \limsup_{r\to\infty} \frac{\max_{|z|=r} \log|g(z)|}{r}. \tag{3.15}$$

We say that an entire function g belongs to the *Cartwright class* and write $g \in \mathcal{C}$ if it is of exponential type and

$$\int_{-\infty}^{\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty. \tag{3.16}$$

Here $\log^+ s = \max\{0, \log s\}$.

We let $n(g,r)$ denote the number of zeros of g in the ball center 0, radius r, counting multiplicity. It is known [26, p. 66], [30, Theorem 1, p. 127] that for $g \in \mathcal{C}$ that is real along the real axis,

$$\lim_{r\to\infty} \frac{n(g,r)}{2r} = \frac{\sigma}{\pi}. \tag{3.17}$$

When f is entire of exponential type $\leq \sigma$ and bounded along the real axis, we have [30, p. 38, Theorem 3]

$$|f(z)| \leq e^{\sigma|\operatorname{Im} z|} \|f\|_{L_\infty(\mathbb{R})}, \ z \in \mathbb{C}. \tag{3.18}$$

When g is entire of exponential type σ and $g \in L_2(\mathbb{R})$, we write, as did B. Ya. Levin, $g \in L^2_\sigma$. Here, we have instead of the last inequality, [30, p. 149]

$$|g(z)| \leq \left(\frac{2}{\pi}\right)^{1/2} e^{\sigma(|\operatorname{Im} z|+1)} \|g\|_{L_2(\mathbb{R})}, \ z \in \mathbb{C}. \tag{3.19}$$

An important identity is the reproducing kernel identity [49, p. 95], [24, (6.75), p. 58]

$$g(x) = \int_{-\infty}^{\infty} g(t) \frac{\sin \sigma(x-t)}{\pi(x-t)} dt, \ x \in \mathbb{R}, \tag{3.20}$$

when $g \in L^2_\sigma$. We shall also use [23, p. 414, no. 3.741.3]

$$\int_{-\infty}^{\infty} \left(\frac{\sin \pi s}{\pi s}\right)^2 ds = 1. \tag{3.21}$$

Of course this integral may be deduced from (3.20) by choosing $\sigma = \pi$, $g(t) = \frac{\sin \pi t}{\pi t}$ and $x = 0$.

4. Normality

We start by bounding the growth of weighted polynomials in the complex plane. Recall our assumption from Theorem 1.2 on the equilibrium measure ν_{W_n} of W_n restricted to I, which we abbreviate as ν_n. For some $C > 0$, and some J_2 satisfying (3.8),

$$0 < \nu_n'(x) \leq C, \ n \geq 1, \ x \in J_2. \tag{4.1}$$

Also, by definition,

$$\int_I d\nu_n = 1, \ n \geq 1. \tag{4.2}$$

Inasmuch as ν_n is the equilibrium measure for the continuous function W_n on I, we have then supp$[\nu_n] \subset I$ and [45, Lemma 2.2, p. 36]

$$V^{\nu_n}(x) + Q_n(x) = c_n \text{ on } \operatorname{supp}[\nu_n]. \tag{4.3}$$

Here c_n is a characteristic constant, called the equilibrium constant. Moreover we have equality in (4.3) for all $x \in J_2$, since $J_2 \subset$supp$[\nu_n]$, as (4.1) shows.

Lemma 4.1

There exists C_2 such that for $n \geq 1$, for polynomials P_n of degree $\leq n$, for $x \in J_1$ and a real, we have

$$\left|P_n\left(x+i\frac{a}{n}\right)\right| W_n^n(x) \leq e^{C_2|a|} \left\|PW_n^n\right\|_{L_\infty(I)}. \tag{4.4}$$

Proof
It is an easy consequence of the maximum principle for subharmonic functions [45, Theorem 2.1, p. 153] that for $z \in \mathbb{C}\backslash I$,

$$|P_n(z)|\, e^{n[V^{\nu_n}(z)-c_n]} \leq \left\|P_n e^{n[V^{\nu_n}-c_n]}\right\|_{L_\infty(I)} = \left\|PW_n^n\right\|_{L_\infty(I)}.$$

Then, using (4.3),

$$\left|P_n\left(x+i\frac{a}{n}\right)\right| W_n^n(x) \leq e^{n\left[V^{\nu_n}(x)-V^{\nu_n}\left(x+i\frac{a}{n}\right)\right]} \left\|PW_n^n\right\|_{L_\infty(I)}. \tag{4.5}$$

Here, for $x \in J_1$,

$$\begin{aligned}
& V^{\nu_n}(x) - V^{\nu_n}\left(x+i\frac{a}{n}\right) \\
= \; & \frac{1}{2}\int_I \log\left(1+\left(\frac{a}{n(x-t)}\right)^2\right) d\nu_n(t) \\
\leq \; & C_1 \int_{J_2} \log\left(1+\left(\frac{|a|}{n(x-t)}\right)^2\right) dt \\
& + \log\left(1+\left(\frac{|a|}{n dist(J_1, I\backslash J_2)}\right)^2\right) \int_{I\backslash J_2} d\nu_n(t) \\
\leq \; & C_1 \frac{|a|}{n} \int_{-\infty}^{\infty} \log\left(1+\frac{1}{s^2}\right) ds + C_3\left(\frac{|a|}{n\; dist(J_1, I\backslash J_2)}\right) \\
\leq \; & C_2 \frac{|a|}{n}.
\end{aligned}$$

Here we used (4.1) and (4.2), and made the substitution $x - t = \frac{s|a|}{n}$. We also used the inequality $\log\left(1+x^2\right) \leq C|x|$. Now the result follows from (4.5). ■

Next, we prove

Lemma 4.2
(a) Uniformly for a in compact subsets of the real line, and $\xi \in J_2$,
(4.6)
$$\frac{\mu_n'\left(\xi+\frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{\mu_n'(\xi)} = \frac{\left(hW_n^{2n}\right)\left(\xi+\frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{\left(hW_n^{2n}\right)(\xi)} = \exp\left(2\Psi(\xi,n)\, a + o(1)\right),$$

where, as in (3.13), $\Psi(\xi,n) = -\frac{n}{\tilde{K}_n(\xi,\xi)} Q_n'(\xi)$*.*

(b)

$$\sup_{\xi \in J_2, n \geq 1} |\Psi(\xi,n)| < \infty. \tag{4.7}$$

Proof

Since h is positive and continuous in compact I, we have, uniformly for a in compact subsets of the real line,

$$\frac{h\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{h(\xi)} = 1 + o(1).$$

We have for some ζ between ξ and $\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}$,

$$\begin{aligned} & \frac{W_n^{2n}\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{W_n^{2n}(\xi)} \\ = \ & \exp\left(-2nQ_n'(\zeta)\frac{a}{\tilde{K}_n(\xi,\xi)}\right) \\ = \ & \exp\left(\left[2\Psi(\xi,n) + \delta\right] a\right), \end{aligned}$$

where

$$\delta = \frac{2n}{\tilde{K}_n(\xi,\xi)}\left(Q_n'(\xi) - Q_n'(\zeta)\right).$$

Recalling that $\tilde{K}_n$ is defined by (1.2), and that $d\mu_n$ is defined by (1.8), while $h \sim 1$ in I, we may reformulate (1.10) as

$$\tilde{K}_n(\xi,\xi) \sim n \text{ uniformly in } n \text{ and } \xi \in I. \tag{4.8}$$

As $|\zeta - \xi| \leq \frac{C}{n}$, our hypothesis (3.11) gives, uniformly in ξ,

$$\begin{aligned} \delta &= \frac{2n}{\tilde{K}_n(\xi,\xi)}\left(Q_n'(\xi) - Q_n'(\zeta)\right) \\ &= o(1). \end{aligned}$$

Finally (4.8) and the boundedness of $\{Q_n'\}$ give (4.7). ■

Next, for the given sequence $\{\xi_n\}$ in J, we let

$$f_n(a,b) = \frac{K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} e^{\Psi(\xi_n,n)(a+b)}, \tag{4.9}$$

for all complex a and b. Note that $f_n(a,b)$ is actually an entire function of exponential type in each variable a and b. Moreover, by Lemma 4.2, and (4.8), uniformly for a,b in compact subsets of the real line,

$$f_n(a,b) = \frac{\tilde{K}_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{b}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\tilde{K}_n(\xi_n,\xi_n)} + o(1). \tag{4.10}$$

Lemma 4.3
(a) $\{f_n(u,v)\}_{n=1}^{\infty}$ *is uniformly bounded for* u,v *in compact subsets of the plane.*
(b) If $f(u,v)$ *is the locally uniform limit of some subsequence* $\{f_n(u,v)\}_{n\in\mathcal{S}}$ *of* $\{f_n(u,v)\}_{n=1}^{\infty}$*, then for each fixed real number* u*,* $f(u,\cdot)$ *is entire of exponential type. Moreover, for some* C_1 *and* C_2 *independent of* $u,v\in\mathbb{C}$*,*

$$|f(u,v)| \le C_1 e^{C_2(|\operatorname{Im} u|+|\operatorname{Im} v|)}. \tag{4.11}$$

(c) For each fixed real number u*,* $f(u,\cdot)$ *has only real zeros.*
Proof
(a) By our bound (4.8) and by Cauchy-Schwarz, we have

$$\begin{aligned} &\frac{1}{n}|K_n(\xi,t)|\,W_n^n(\xi)\,W_n^n(t) \\ \le\ &\left(\frac{1}{n}K_n(\xi,\xi)\,W_n^{2n}(\xi)\right)^{1/2}\left(\frac{1}{n}K_n(t,t)\,W_n^{2n}(t)\right)^{1/2} \le C \end{aligned}$$

for $\xi,t\in I$ and $n\ge 1$. By Lemma 4.1, applied separately in each variable, we then have for $\xi,t\in J_1$, and real a,b,

$$\frac{1}{n}\left|K_n\left(\xi+i\frac{a}{n},t+i\frac{b}{n}\right)\right|W_n^n(\xi)\,W_n^n(t)\le Ce^{C_2(|a|+|b|)}. \tag{4.12}$$

Because (3.8) is the only restriction on J_1 and J_2, we may relabel, and assume that (4.12) holds for $\xi,t\in J_2$, and real a,b. Let $A>0$. Note that for $n\ge n_0(A)$, for $\xi\in J_1$, and complex u,v with $|u|,|v|\le A$, we may then also recast (4.12) in the form

$$\frac{1}{n}\left|K_n\left(\xi+\frac{u}{n},\xi+\frac{v}{n}\right)\right|W_n^n\left(\xi+\frac{\operatorname{Re} u}{n}\right)W_n^n\left(\xi+\frac{\operatorname{Re} v}{n}\right)\le Ce^{C_2(|\operatorname{Im} u|+|\operatorname{Im} v|)}. \tag{4.13}$$

Here C_1 and C_2 do not depend on A. The threshhold n_0 is designed to ensure that $\xi+\frac{\operatorname{Re} u}{n},\xi+\frac{\operatorname{Re} v}{n}\in J_2$. Next, recall that

$$\tilde{K}_n(\xi,\xi)\sim n,$$

and by Lemma 4.2(a), uniformly for $\xi\in J_2$,

$$\begin{aligned} \frac{W_n^n\left(\xi+\frac{\operatorname{Re} u}{\tilde{K}_n(\xi,\xi)}\right)}{W_n^n(\xi)} &= e^{\Psi(\xi,n)\operatorname{Re} u+o(1)} \\ &= \left|e^{\Psi(\xi,n)u+o(1)}\right|. \end{aligned}$$

Thus (4.13) implies

$$|f_n(u,v)|\le C_1e^{C_2(|\operatorname{Im} u|+|\operatorname{Im} v|)},$$

for $n\ge n_0(A)$ and $|u|,|v|\le A$, where C_1,C_2 are independent of n,u,v,A (and of ξ).
(b) Now $\{f_n(u,v)\}_{n=1}^{\infty}$ is a normal family of two variables u,v. If $f(u,v)$ is

the locally uniform limit through the subsequence $\mathcal{S}$ of integers, we see that $f(u,v)$ is an entire function in u,v satisfying for all complex u,v,

$$|f(u,v)| \leq Ce^{C_2(|\operatorname{Im} u|+|\operatorname{Im} v|)}.$$

In particular, $f(u,v)$ is bounded for $u,v\in\mathbb{R}$, and is an entire function of exponential type in each variable.
(c) It is shown in [21, p. 19, proof of Theorem 3.1], that for each real ξ_n, $K_n(\xi_n,t)$ has only real simple zeros. Hence for real u, $f_n(u,v)$ has only real zeros as a function of v. Hurwitz's theorem shows that the same is true of $f(u,v)$. ■

Lemma 4.4
(a) Uniformly for $u\in\mathbb{R}$,

$$f(u,u)\sim 1. \tag{4.14}$$

(b) For all $a\in\mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(a,s)|^2\, ds \leq f(a,\bar{a}). \tag{4.15}$$

(c) For each $a\in\mathbb{R}$, $f(a,\cdot)$ has infinitely many real zeros.
Proof
(a) We have uniformly for a in compact subsets of the real line,

$$\begin{aligned}
&\frac{K_n\left(\xi_n+\frac{a}{\tilde{K}_n(\xi,\xi)},\xi_n+\frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi_n,\xi_n)}e^{\Psi(\xi_n,n)(2a)}\\
=\ &\frac{\tilde{K}_n\left(\xi_n+\frac{a}{\tilde{K}_n(\xi,\xi)},\xi_n+\frac{a}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi_n,\xi_n)}(1+o(1))\\
\geq\ & C_1(1+o(1)),
\end{aligned}$$

where C_1 is independent of the compact set in which a lies, and comes only from the upper and lower bounds on the Christoffel functions implicit in (4.8). From this we deduce that for all real a,

$$f(a,a)\geq C_1.$$

The corresponding upper bound is similar.
(b) We use the identity

$$K_n(s,\bar{s}) = \int |K_n(s,t)|^2\, d\mu_n(t),$$

valid for all complex s. Let $a\in\mathbb{C}$, and

$$s=\xi_n+\frac{a}{\tilde{K}_n(\xi_n,\xi_n)}.$$

Let $r > 0$. We drop most of the integral and make the substitution $t = \xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)}$:

$$\begin{aligned} 1 &\geq \int_{\xi_n-\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}}^{\xi_n+\frac{r}{\tilde{K}_n(\xi_n,\xi_n)}} \frac{|K_n(s,t)|^2}{K_n(s,\bar{s})}\mu_n'(t)\,dt \\ &= \int_{-r}^{r} \left| \frac{K_n\left(s,\xi_n+\frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{K_n(\xi_n,\xi_n)} \right|^2 \frac{K_n(\xi_n,\xi_n)}{K_n(s,\bar{s})} \frac{\mu_n'\left(\xi_n+\frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu_n'(\xi_n)} dy \\ &= \int_{-r}^{r} \frac{|f_n(a,y)|^2}{f_n(a,\bar{a})} \left| e^{-\Psi(\xi_n,n)(2a+2y-a-\bar{a})} \right| \frac{\mu_n'\left(\xi_n+\frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu_n'(\xi_n)} dy \\ &= \int_{-r}^{r} \frac{|f_n(a,y)|^2}{f_n(a,\bar{a})}(1+o(1))dy. \end{aligned}$$

Here we have used Lemma 4.2(a). As $n \to \infty$ through a subsequence, the last right-hand side has lim inf at least

$$\int_{-r}^{r} \frac{|f(a,y)|^2}{f(a,\bar{a})} dy,$$

by Fatou's Lemma. Finally, let $r \to \infty$.

(c) We note first that $f(a,\cdot)$ is non-constant, and moreover, is not a polynomial. Indeed, it belongs to $L_2(\mathbb{R})$ and satisfies $f(a,a) \neq 0$. It also lies in the Cartwright class, because of (a), and is real along the real axis. We can then write [30, p. 130]

$$f(a,z+a) = f(a,a) \lim_{R\to\infty} \prod\nolimits_{b:|b|<R \text{ and } f(a,b+a)=0} \left(1-\frac{z}{b}\right).$$

■

5. Proof of Theorem 1.2

It follows from Lemma 4.3(b) that for each real a, $f(a,\cdot)$ is entire of exponential type σ_a, say. We first show that σ_a is independent of a. We note that σ_a *does possibly* depend on $\{\xi_n\}$ and the subsequence $\mathcal{S}$.

Lemma 5.1

For $a \in \mathbb{R}$, let $n(f(a,\cdot),r)$ denote the the number of zeros of $f(a,\cdot)$ in the ball center 0, radius r, counting multiplicity. Then for any real a, we have as $r \to \infty$,

$$n(f(a,\cdot),r) - n(f(0,\cdot),r) = O(1). \tag{5.1}$$

Consequently,

$$\sigma_a = \sigma_0 = \sigma, \text{ say.} \tag{5.2}$$

Moreover, for all $a \in \mathbb{R}$, $f(a,\cdot) \in L^2_\sigma$.

Proof

Let K_n denote the reproducing kernel for μ_n. We use the following basic property of

$$\psi_n(\xi,t) = \left(\frac{\gamma_{n,n-1}}{\gamma_{n,n}}\right)^{-1} K_n(\xi,t)(\xi-t) = p_{n,n}(\xi)\, p_{n,n-1}(t) - p_{n,n-1}(\xi)\, p_{n,n}(t).$$

For real ξ, with $p_{n,n-1}(\xi)\, p_{n,n}(\xi) \neq 0$, $\psi_n(\xi,t)$ has, as a function of t, simple zeros in each of the intervals

$$(x_{nn}, x_{n-1,n}), (x_{n-1,n}, x_{n-2,n}), ..., (x_{2n}, x_{1n}).$$

There is a single remaining zero, and this lies outside $[x_{nn}, x_{1n}]$. When $p_{n,n-1}(\xi)\, p_{n,n}(\xi) = 0$, $\psi_n(\xi,t)$ is a multiple of $p_{n,n}$ or $p_{n,n-1}$. As the zeros of the latter polynomials interlace, we see that in this case, there is a simple zero in each of the intervals

$$[x_{nn}, x_{n-1,n}), [x_{n-1,n}, x_{n-2,n}), ..., [x_{2n}, x_{1n}).$$

For all this, see [21, proof of Theorem 3.1, p. 19]. It follows that whatever ξ is, the number j of zeros of $K_n(\xi,t)$ in $[x_{mn}, x_{kn}]$ satisfies

$$|j-(m-k)| \leq 1.$$

Consider now $K_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)$ and $K_n\left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n,\xi_n)}\right)$ as a function of t. In any fixed interval $[-r,r]$, it follows that the difference between the number of zeros of these two functions is at most 2. Hence the same is true of $f_n(a,\cdot)$ and $f_n(0,\cdot)$. Letting $n \to \infty$ through $\mathcal{S}$, we see that (5.1) holds. Then (5.2) follows from (3.17). Finally, $f(a,\cdot) \in L_2(\mathbb{R})$, by (4.15), so also $f(a,\cdot) \in L^2_\sigma$. ■

In the sequel, σ denotes the type of $f(a,\cdot)$ for all real a.

Lemma 5.2
(a) For all $a \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin\sigma(a-s)}{\sigma(a-s)}\right)^2 ds \leq \frac{1}{f(a,a)} - \frac{\pi}{\sigma}. \tag{5.3}$$

(b)

$$\sigma \geq \pi \sup_{a\in\mathbb{R}} f(a,a) \geq \pi. \tag{5.4}$$

Proof
(a) The left-hand side in (5.3) equals

$$\frac{1}{f(a,a)^2}\int_{-\infty}^{\infty} f(a,s)^2\, ds - \frac{2}{f(a,a)}\int_{-\infty}^{\infty} f(a,s)\frac{\sin\sigma(a-s)}{\sigma(a-s)} ds + \int_{-\infty}^{\infty}\left(\frac{\sin\sigma(a-s)}{\sigma(a-s)}\right)^2 ds$$
$$\leq \frac{1}{f(a,a)} - 2\frac{\pi}{\sigma} + \frac{\pi}{\sigma},$$

by Lemma 4.4(b), and the identities (3.20) and (3.21). Recall that $f(a,\cdot) \in L^2_\sigma$, so (3.20) is applicable.
(b) Since the left-hand side of (5.3) is nonnegative, we obtain for all real a,

$$\sigma \geq \pi f(a,a).$$

As $f(0,0) = 1$, we then obtain (5.4). ■

Recall from Section 3, the Gauss type quadrature formula, with nodes $\{t_{jn}\} = \{t_{jn}(\xi_n)\}$ including the point ξ_n:

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P(t)\, d\mu(t),$$

for all polynomials P of degree $\leq 2n-3$. Recall that we order the nodes as

$$... < t_{-2,n} < t_{-1,n} < t_{0,n} = \xi_n < t_{1,n} < t_{2,n} < ... \,.$$

and write

$$t_{jn} = \xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n,\xi_n)}. \tag{5.5}$$

We need a Markov-Stieltjes inequality:

Lemma 5.3
Let $1 \leq k < \ell \leq n$. Let $B \in \mathbb{R}$. Then

$$\sum_{j=k+1}^{\ell-1} \lambda_n(t_{jn}) e^{-Bt_{jn}} \leq \int_{t_{kn}}^{t_{\ell n}} e^{-Bt} d\mu_n(t) \leq \sum_{j=k}^{\ell} \lambda_n(t_{jn}) e^{-Bt_{jn}}. \tag{5.6}$$

Proof
We begin by assuming that

$$\int e^{At} d\mu_n(t) \tag{5.7}$$

is finite for all real A. Now let $B \geq 0$. By the classical Posse-Markov-Stieltjes inequality [21, (5.10), p. 33],

$$\sum_{j:t_{jn}<t_{\ell n}} \lambda_n(t_{jn}) e^{Bt_{jn}} \leq \int_{-\infty}^{t_{\ell n}} e^{Bt} d\mu_n(t) \leq \sum_{j:t_{jn}\leq t_{\ell n}} \lambda_n(t_{jn}) e^{Bt_{jn}}.$$

A similar inequality holds if $B < 0$. Indeed, consider the reflected measure $d\mu_n^-(t) = d\mu_n(-t)$. The quadrature points for $d\mu_n^-$ including $-\xi_n$ will be $\{-t_{jn}\}$. Let us assume that there are n quadrature points $\{t_{jn}\}$ (the case of $n-1$ points requires trivial changes). Applying the inequality above to μ_n^-, making a substitution, and then taking account of our ordering, gives, with $B > 0$,

$$\sum_{j:t_{jn}>t_{n+1-\ell,n}} \lambda_n(t_{jn}) e^{-Bt_{jn}} \leq \int_{t_{n+1-\ell,n}}^{\infty} e^{-Bt} d\mu_n(t) \leq \sum_{j:t_{jn}\geq t_{n+1-\ell,n}} \lambda_n(t_{jn}) e^{-Bt_{jn}}.$$

Setting $k = n + 1 - \ell$ gives

$$\sum_{j:t_{jn}>t_{kn}} \lambda_n(t_{jn}) e^{-Bt_{jn}} \leq \int_{t_{kn}}^{\infty} e^{-Bt} d\mu_n(t) \leq \sum_{j:t_{jn}\geq t_{kn}} \lambda_n(t_{jn}) e^{-Bt_{jn}}.$$

Now let $\ell > k$ and subtract this last inequality for k and ℓ: for $B > 0$,

$$\sum_{j=k+1}^{\ell-1} \lambda_n(t_{jn}) e^{-Bt_{jn}} \leq \int_{t_{kn}}^{t_{\ell n}} e^{-Bt} d\mu_n(t) \leq \sum_{j=k}^{\ell} \lambda_n(t_{jn}) e^{-Bt_{jn}}.$$

For $B \leq 0$, the same inequality follows from the first Markov-Stieltjes inequality above. Thus (5.6) is valid for all real B, provided we assume the convergence of all the integrals in (5.7). We now drop that condition by a limiting argument. Throughout this argument, n, k, ℓ are fixed. Let $\varepsilon > 0$ and

$$d\omega_\varepsilon(t) = e^{-\varepsilon t^2} d\mu_n(t).$$

Then the analogue of (5.7) holds for ω_ε and so the analogue of (5.6) holds for ω_ε. Let us denote the quadrature points and Christoffel numbers for ω_ε respectively by $\{t_{jn\varepsilon}\}$ and $\{\lambda_{n\varepsilon}(t_{jn\varepsilon})\}$. We must show that as $\varepsilon \to 0+$,

$$t_{jn\varepsilon} \to t_{jn} \text{ and } \lambda_{n\varepsilon}(t_{jn\varepsilon}) \to \lambda_n(t_{jn}).$$

To see that this is indeed the case, we note that for each $0 \leq j \leq 2n$,

$$\lim_{\varepsilon \to 0+} \int t^j d\omega_\varepsilon(t) = \int t^j d\mu_n(t).$$

Hence from the well known determinantal representation for orthogonal polynomials involving power moments [21, (1.6), p. 57], [46, p. 15], the orthogonal polynomials for ω_ε of degree k, $0 \leq k \leq n$, converge to those of μ_n as $\varepsilon \to 0+$. ■

Lemma 5.4
(a) $f(0,z)$ has zeros $\{\rho_j\}_{j\neq 0}$, with

$$\ldots \leq \rho_{-2} \leq \rho_{-1} < 0 < \rho_1 \leq \rho_2 \leq \ldots \tag{5.8}$$

and for $j = \pm 1, \pm 2, \pm 3, \ldots$

$$\rho_j = \lim_{n\to\infty, n\in\mathcal{S}} \rho_{jn}.$$

There are no other zeros of f. We also set $\rho_0 = 0$.
(b) Given $\ell > k$, we have

$$\sum_{j=k+1}^{\ell-1} \frac{1}{f(\rho_j, \rho_j)} \leq \rho_\ell - \rho_k \leq \sum_{j=k}^{\ell} \frac{1}{f(\rho_j, \rho_j)}. \tag{5.9}$$

(c) The zeros $\{\rho_j\}$ are at most double zeros of $f(0,z)$, and there exist C_1, C_2 such that for all j,

$$C_1 \leq \rho_j - \rho_{j-2} \leq C_2. \tag{5.10}$$

The constants are independent of j. Moreover, zeros are repeated in the sequence $\{\rho_j\}$ according to their multiplicity.

Proof

(a) We know that $f_n(0,s) = \left[K_n\left(\xi_n, \xi_n + \frac{s}{\tilde{K}_n(\xi_n,\xi_n)}\right)/K(\xi_n,\xi_n)\right] e^{s\Psi(\xi_n,n)}$ has simple zeros at $s = \rho_{jn}$, $j \neq 0$, and no other zeros. Moreover as $n \to \infty$ through $\mathcal{S}$, this sequence converges to $f(0,z)$, uniformly for z in compact sets. As $f(0,0) = 1$, the function $f(0,z)$ is not identically zero. In particular, as $n \to \infty$ through our subsequence $\mathcal{S}$, we obtain that necessarily $\rho_{jn} \to \rho_j$, the jth (possibly multiple) zero of $f(0,z)$. There can be no other zeros because of Hurwitz' Theorem.

(b) We use the Markov-Stieltjes inequality (5.6) above, with

$$B = \tilde{K}_n(\xi_n,\xi_n)\, 2\Psi(\xi_n,n)$$

Recall that $\Psi(\xi_n,n)$ is defined by (3.13). From (5.5), we deduce that for all j,

$$Bt_{jn} = B\xi_n + \rho_{jn} 2\Psi(\xi_n,n).$$

We multiply (5.6) by $K_n(\xi_n,\xi_n)$ and cancel $e^{-B\xi_n}$ from both sides. We also make the substitution $t = \xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)} \Rightarrow Bt = B\xi_n + 2\Psi(\xi_n,n)\, y$ in the integral. We deduce that

$$\begin{aligned} \sum_{j=k+1}^{\ell-1} \frac{1}{f_n(\rho_{jn},\rho_{jn})} &\leq \int_{\rho_{kn}}^{\rho_{\ell,n}} e^{-2\Psi(\xi_n,n)y} \frac{\mu_n'\left(\xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu_n'(\xi_n)} dy \\ &\leq \sum_{j=k}^{\ell} \frac{1}{f_n(\rho_{jn},\rho_{jn})}. \end{aligned} \tag{5.11}$$

Here by Lemma 4.2, and (a) of this lemma, for fixed ℓ and k,

$$\begin{aligned} &\int_{\rho_{kn}}^{\rho_{\ell n}} e^{-2\Psi(\xi,n)y} \frac{\mu_n'\left(\xi_n + \frac{y}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\mu_n'(\xi_n)} dy \\ = \; & \rho_\ell - \rho_k + o(1), \end{aligned}$$

as $n \to \infty$ through $\mathcal{S}$. Thus letting $n \to \infty$ through $\mathcal{S}$ in (5.11), and taking account of the uniform convergence of $f_n(\cdot,\cdot)$ to $f(\cdot,\cdot)$, gives (5.9).

(c) From (b),

$$\frac{1}{f(\rho_{k+1},\rho_{k+1})} \leq \rho_{k+2} - \rho_k \leq \sum_{j=k}^{k+2} \frac{1}{f(\rho_j,\rho_j)}.$$

Since $f(t,t)$ is bounded above and below by positive constants for real t, (5.10) follows. Of course, we also deduce $\rho_{j+2} \neq \rho_j$, so there are at most double zeros. Since the $\{\rho_{jn}\}$ are simple zeros of f_n, it follows that ρ_k can only be a double zero of $f(0,\cdot)$ if it appears twice in the $\{\rho_j\}$. ■

Next, we deduce:

Lemma 5.5
Let

$$\Lambda = \sup_{x\in\mathbb{R}} f(x,x). \tag{5.12}$$

For each real a, $f(a,\cdot)$ is entire of exponential type $\sigma = \pi\Lambda$.

Proof
Because of Lemma 5.1, it suffices to show that $f(0,\cdot)$ is entire of exponential type $\sigma = \Lambda\pi$. To do this, we use (b) of the previous lemma. We have for each $\ell > k$,

$$\sum_{j=k+1}^{\ell-1} \frac{1}{f(\rho_j,\rho_j)} \leq \rho_\ell - \rho_k. \tag{5.13}$$

Since $f(\rho_j,\rho_j) \leq \Lambda$ for each j, we obtain

$$\ell - k - 1 \leq \Lambda(\rho_\ell - \rho_k). \tag{5.14}$$

Next, recall that $\{\rho_j\}_{j\neq 0}$ are all the zeros of f. Moreover, each zero is at most a double zero, and is repeated in the sequence $\{\rho_j\}$ if it is a double zero. Thus the total number of zeros of $f(0,\cdot)$ in $[\rho_k,\rho_\ell]$ is $\ell-k+1$ or $\ell-k+2$ or $\ell-k+3$ if 0 does not belong to $[k,\ell]$, and $\ell-k$ or $\ell-k+1$ or $\ell-k+2$ if it does. Thus the total number of zeros of $f(0,\cdot)$ in $[\rho_k,\rho_\ell]$, is at most

$$(\ell-k-1)+4 \leq \Lambda(\rho_\ell-\rho_k)+4.$$

Recall that $n(f(0,\cdot),r)$ denotes the number of zeros of $f(0,\cdot)$ in $[-r,r]$ (or equivalently in the ball center 0, radius r). In view of (5.10), we can choose ρ_k a bounded distance from r, and ρ_ℓ a bounded distance from $-r$. We obtain that $n(f(0,\cdot),r)$ is at most the number of zeros in $[\rho_k,\rho_\ell]$ plus $O(1)$, and hence at most $\Lambda(\rho_\ell-\rho_k)+O(1)$. So

$$n(f(0,\cdot),r) \leq 2\Lambda r + O(1).$$

Then by (3.17),

$$\frac{\sigma}{\pi} = \lim_{r\to\infty} \frac{n(f(0,\cdot),r)}{2r} \leq \Lambda.$$

Together with our lower bound $\sigma \geq \pi\Lambda$ from Lemma 5.2(b), we obtain the result. ■

Proof of Theorem 1.2
Since $\tilde{K}_n(\xi_n,\xi_n) \sim n$, our hypothesis (1.11), with its uniformity in a, implies also that for all real a,

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}, \xi_n + \frac{a}{\tilde{K}_n(\xi_n,\xi_n)}\right)}{\tilde{K}_n(\xi_n,\xi_n)} = 1,$$

and hence (cf. (4.10)), for all real a,

$$\lim_{n\to\infty} f_n(a,a) = 1.$$

So
$$f(a,a)=1 \text{ for all real } a.$$
Hence
$$\Lambda=\sup_{x\in\mathbb{R}} f(x,x)=1.$$
By Lemma 5.5, for each fixed a, $f(a,\cdot)$ is entire of exponential type $\sigma=\pi$. By Lemma 5.2(a), we then obtain, for each real a,
$$\begin{aligned}&\int_{-\infty}^{\infty}\left(f(a,s)-\frac{\sin\pi(s-a)}{\pi(s-a)}\right)^2 ds\\ =\;&\int_{-\infty}^{\infty}\left(\frac{f(a,s)}{f(a,a)}-\frac{\sin\pi(s-a)}{\pi(s-a)}\right)^2 ds=0.\end{aligned}$$
So for real a and s,
$$\lim_{n\to\infty,n\in\mathcal{S}} f_n(a,s)=f(a,s)=\frac{\sin\pi(s-a)}{\pi(s-a)}.$$
By analytic continuation,
$$\lim_{n\to\infty,n\in\mathcal{S}} f_n(a,b)=\frac{\sin\pi(a-b)}{\pi(a-b)},$$
uniformly for a,b in compact subsets of the plane. (Recall that the left-hand side is uniformly bounded for a,b in such sets). As the limit function is independent of the subsequence $\mathcal{S}$, we obtain
$$\lim_{n\to\infty} f_n(a,b)=\frac{\sin\pi(a-b)}{\pi(a-b)},$$
again with the appropriate uniformity in a,b. Finally, using (4.10) again, and as $\{\xi_n\}$ can be any sequence in J, we obtain the conclusion (1.7) of Theorem 1.2, uniformly for $\xi\in J$, as well as the limit (1.13). ■

6. Proof of Theorem 1.1

In this section, we show that the hypotheses of Theorem 1.1 imply those of Theorem 1.2.

Proof of Theorem 1.1
Let I be an open interval containing J in which ν_W is absolutely continuous while ν_W' and Q' are continuous and $\nu_W'>0$. In particular, this implies that I lies in supp$[\nu_W]$. For $n\geq 1$, we let
$$W_n=W_{|\bar{I}}.$$
It is known [45, Theorem 1.6(e), p. 196] that the equilibrium measure ν_n for W_n satisfies
$$\nu_n=\hat{\nu}_W,$$

where $\hat{\nu}_W$ is the balayage measure of ν_W onto I. This balayage measure is obtained by sweeping out $(\nu_W)_{|\mathbb{R}\backslash I}$ onto I, and adding it to $(\nu_W)_{|I}$. Thus

$$\hat{\nu}_W = (\nu_W)_{|I} + \left(\widehat{\nu_{W|\mathbb{R}\backslash I}}\right).$$

Moreover, $\left(\widehat{\nu_{W|\mathbb{R}\backslash I}}\right)$ is absolutely continuous and its density is infinitely differentiable in the interior of I [45, (4.47), p. 122], [52, p. 9, (2.28)]. Hence $\nu_n = \hat{\nu}_W$ is absolutely continuous in I, and its density ν_n' is bounded in J, and of course this holds uniformly in n. Since $Q_n' = Q'$, our hypothesis that Q' is continuous in J shows that $\{Q_n'\}$ are equicontinuous in J. Totik [50, Theorem 1.2, p. 326] proved that

$$\lim_{n\to\infty} n^{-1}\lambda_n^{-1}\left(hW^{2n}, x\right) hW^{2n}(x) = \nu_W'(x)\,, \tag{6.1}$$

uniformly in a neighborhood of J, say in I. It follows that uniformly in n and $x \in I$,

$$C_1 \leq \lambda_n^{-1}(x)\, W^{2n}(x)/n \leq C_2.$$

Finally, the asymptotic (6.1), and the continuity of ν_W' also give

$$\lim_{n\to\infty} \frac{\lambda_n\left(hW^{2n}, \xi + \frac{a}{n}\right)}{\lambda_n\left(hW^{2n}, \xi\right)} \frac{\left(hW^{2n}\right)(\xi)}{\left(hW^{2n}\right)\left(\xi + \frac{a}{n}\right)} = 1,$$

uniformly for a in compact subsets of the real line, and ξ in a neighborhood of J. As h is continuous (and uniformly so in the region desired), (1.11) follows. So we have verified all the hypotheses of Theorem 1.2, and that theorem gives the result. ■

7. Fixed Exponential Weights

In [32], the authors substituted first order asymptotics for orthogonal polynomials for fixed exponential weights into the Christoffel-Darboux formula, and used a Markov-Bernstein inequality to control the tail. This led to universality in the bulk for a class of exponential weights considered in [31].

In this section, we show how universality for fixed exponential weights can be deduced from Theorem 1.2. One definite advantage over the method of [32] is that one does not need pointwise asymptotics for orthogonal polynomials, so one may treat a more general class of weights. We begin by recalling the result of [32].

Definition 7.1
Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0, \infty)$ satisfies the following conditions:
(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) Q'' exists and is positive in $\mathbb{R}\backslash\{0\}$;
(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t)=\frac{tQ'(t)}{Q(t)},\ t\neq 0,$$

is quasi-increasing in $(0,\infty)$*, in the sense that for some* $C>0$,

$$0<x<y\Rightarrow T(x)\leq CT(y).$$

We assume, with an analogous definition, that T is *quasi-decreasing in* $(-\infty,0)$*. In addition, we assume that for some* $\Lambda>1$,

$$T(t)\geq\Lambda \text{ in } \mathbb{R}\backslash\{0\}.$$

(e) There exists $C_1>0$ *such that*

$$\frac{Q''(x)}{|Q'(x)|}\leq C_1\frac{Q'(x)}{Q(x)} \text{ a.e. } x\in\mathbb{R}\backslash\{0\}.$$

Then we write $W\in\mathcal{F}\left(C^2\right)$.

Examples of weights in this class are $W=\exp(-Q)$, where

$$Q(x)=\left\{\begin{array}{ll} Ax^{\alpha}, & x\in[0,\infty) \\ B|x|^{\beta}, & x\in(-\infty,0)\end{array}\right.,$$

where $\alpha,\beta>1$ and $A,B>0$. More generally, if $\exp_k=\exp(\exp(...\exp()))$ denotes the kth iterated exponential, we may take

$$Q(x)=\left\{\begin{array}{ll} \exp_k(Ax^{\alpha})-\exp_k(0), & x\in[0,\infty) \\ \exp_\ell\left(B|x|^{\beta}\right)-\exp_\ell(0), & x\in(-\infty,0)\end{array}\right.$$

where $k,\ell\geq 1$, $\alpha,\beta>1$.

A key descriptive role is played by the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n}<0<a_n,$$

defined for $n\geq 1$ by the equations

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n}\frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx; \tag{7.1}$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n}\frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx. \tag{7.2}$$

In the case where Q is even, $a_{-n}=-a_n$. The existence and uniqueness of these numbers is established in the monographs [31], [42], [45], but goes back to earlier work of Mhaskar, Rakhmanov, and Saff. On $[a_{-n},a_n]$, the orthonormal polynomials $p_n\left(W^2,x\right)$ behave much like Szegő polynomials on $[-1,1]$.

We also define,

$$\beta_n=\frac{1}{2}(a_n+a_{-n}) \text{ and } \delta_n=\frac{1}{2}(a_n+|a_{-n}|), \tag{7.3}$$

which are respectively the center, and half-length of the Mhaskar-Rakhmanov-Saff interval

$$\Delta_n = [a_{-n}, a_n]. \tag{7.4}$$

The linear transformation

$$L_n(x) = \frac{x-\beta_n}{\delta_n} \tag{7.5}$$

maps Δ_n onto $[-1,1]$. Its inverse

$$L_n^{[-1]}(u) = \beta_n + u\delta_n$$

maps $[-1,1]$ onto Δ_n. For $0<\varepsilon<1$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n}+\varepsilon\delta_n, a_n - \varepsilon\delta_n]. \tag{7.6}$$

We let $p_n\left(W^2,x\right)$ denote the nth orthonormal polynomial for W^2, so that

$$\int p_n\left(W^2,x\right)p_m\left(W^2,x\right)W^2(x)\,dx = \delta_{mn}.$$

Moreover, we let

$$K_n\left(W^2,x,t\right) = \sum_{j=0}^{n-1} p_j\left(W^2,x\right)p_j\left(W^2,t\right)$$

and

$$\tilde{K}_n\left(W^2,x,t\right) = W(x)\,W(t)\,K_n\left(W^2,x,t\right).$$

Thus, in this section, the parameter W^2 inside p_n or K_n is used to distingush these fixed weight quantities from the corresponding quantities for the varying weights W_n^{2n}.

The first result of [32] was:

Theorem 7.2
Let $W=\exp(-Q)\in\mathcal{F}\left(C^2\right)$. Let $0<\varepsilon<1$. Then uniformly for a,b in compact subsets of the real line, and $x\in J_n(\varepsilon)$, we have as $n\to\infty$,

$$\lim_{n\to\infty}\frac{\tilde{K}_n\left(W^2, x+\frac{a}{\tilde{K}_n(W^2,x,x)}, x+\frac{b}{\tilde{K}_n(W^2,x,x)}\right)}{\tilde{K}_n\left(W^2,x,x\right)} = \frac{\sin\pi(b-a)}{\pi(b-a)}. \tag{7.7}$$

In particular, if W is even, this holds uniformly for $|x|\le(1-\varepsilon)\,a_n$.

In [32], we also established universality for weights of the form hW^2, when h does not grow or decay too rapidly at $\pm\infty$.

In this section, we shall deduce universality for a more general class of weights than in Theorem 7.2. Our class is [31, pp. 10-11]:

Definition 7.3
Let $I=(c,d)$ be an open interval containing 0 in its interior. Let $W=\exp(-Q)$, where $Q:I\to[0,\infty)$ satisfies the following properties:
(a) Q' is continuous in I and $Q(0)=0$.

(b) Q' is non-decreasing in I;
(c)

$$\lim_{t\to c+} Q(t) = \lim_{t\to d-} Q(t) = \infty. \tag{7.8}$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)},\ t \neq 0,$$

is quasi-increasing in $(0,d)$, and quasi-decreasing in $(c,0)$. In addition, we assume that for some $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } I\setminus\{0\}.$$

(e) There exists $\varepsilon_0 \in (0,1)$ such that for $y \in I\setminus\{0\}$,

$$T(y) \sim T\left(y\left[1 - \frac{\varepsilon_0}{T(y)}\right]\right).$$

(f) For every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in I\setminus\{0\}$,

$$\int_{x-\frac{\delta|x|}{T(x)}}^{x+\frac{\delta|x|}{T(x)}} \frac{Q'(s) - Q'(x)}{s-x} ds \leq \varepsilon \left|Q'(x)\right|. \tag{7.9}$$

Then we write $W \in \mathcal{F}(dini)$.

Note that [32, p. 13] $\mathcal{F}\left(C^2\right) \subset \mathcal{F}(dini)$. The term dini refers to the Dini type condition in (7.9). In particular, Definition 7.3 does not assume pointwise estimates for Q''. We shall deduce the following result from Theorem 1.2:

Theorem 7.4
Let $W = \exp(-Q) \in \mathcal{F}(dini)$. Let $0 < \varepsilon < 1$. Then uniformly for a, b in compact subsets of the real line, and $x \in J_n(\varepsilon)$, we have as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(W^2, x + \frac{a}{\tilde{K}_n(W^2,x,x)}, x + \frac{b}{\tilde{K}_n(W^2,x,x)}\right)}{\tilde{K}_n(W^2,x,x)} = \frac{\sin \pi(b-a)}{\pi(b-a)}. \tag{7.10}$$

In particular, if W is even, this holds uniformly for $|x| \leq (1-\varepsilon) a_n$.

Remarks
(a) Using the techniques of [32, Theorem 1.3], one can extend this to weights of the form hW^2, where h does not grow or decay too rapidly.
(b) Theorem 7.4 implies asymptotics for spacing of zeros of orthogonal polynomials as in [33].

We shall apply Theorem 1.2 with

$$Q_n(x) = \frac{1}{n} Q\left(L_n^{[-1]}(x)\right),\ x \in L_n(I); \tag{7.11}$$

$$W_n(x) = \exp(-Q_n(x)),\ x \in L_n(I); \tag{7.12}$$

$$d\mu_n(x) = W_n^{2n}(x)\,dx. \tag{7.13}$$

Observe that with the notation (7.5),

$$W_n^{2n} = W^2 \circ L_n^{[-1]}. \tag{7.14}$$

We emphasize that I in this section is used in a different sense to that in Theorem 1.2. There I was the interval in which all $\{\mu_n\}$ are absolutely continuous, and in which the Christoffel functions admitted a uniform bound. Here, to accord with [31], I is the (possibly unbounded) interval of orthogonality of W^2. We shall fix

$$0 < \varepsilon' < \varepsilon < 1$$

and let

$$I' = \left[-1+\varepsilon', 1-\varepsilon'\right] \text{ and } J' = \left[-1+\varepsilon, 1-\varepsilon\right]. \tag{7.15}$$

These intervals will play respectively the roles of I and J of Theorem 1.2. We shall verify the hypotheses of Theorem 1.2 in a series of lemmas:

Lemma 7.5
Let ν_n denote the equilibrium measure of $W_{n|I'}$ for $n \geq 1$. Let $J_2 \subset (I')^o$. Then $\{\nu_n'\}$ are positive and uniformly bounded in J_2.
Proof
We use estimates for equilibrium densities from [31] together with properties of balayage measures. The equilibrium measure $\sigma_n(t)\,dt$ for W_n is a measure of total mass n, with support on the Mhaskar-Rakhmanov-Saff interval $\Delta_n = [a_{-n}, a_n]$, such that

$$V^{\sigma_n}(x) + Q(x) = C_n,\ x \in \Delta_n.$$

Here C_n is an equilibrium constant. The contracted density

$$\sigma_n^*(x) = \frac{\delta_n}{n}\sigma\left(L_n^{[-1]}(x)\right)$$

has support on $[-1,1]$ and total mass 1, and has the property that

$$V^{\sigma_n^*}(x) + Q_n(x) = C_n^*,\ x \in [-1,1].$$

Again, C_n^* is an equilibrium constant. For further orientation, see [31, pp. 16-17]. To obtain the equilibrium measure ν_n for $(W_n)_{|I'}$, we use Theorem 1.6(e) in [45, p. 196]. We have

$$d\nu_n(t) = \hat{\sigma}_n^*(t)\,dt,$$

where $\hat{\sigma}_n^*(t)\,dt$ denotes the balayage measure of $\sigma_n^*(t)\,dt$ onto I'. Moreover, this balayage measure is obtained by sweeping out $(\sigma_n^*(t)dt)_{|[-1,1]\backslash I'}$ onto I', and adding it to $(\sigma_n^*(t)\,dt)_{|I'}$. Thus

$$\hat{\sigma}_n^* = (\sigma_n^*)_{|I'} + \left(\widehat{\sigma_{n|[-1,1]\backslash I'}^*}\right). \tag{7.16}$$

Now we apply estimates for σ_n^* from [31, (6.11), Theorem 6.1, p. 146]:

$$C_1\sqrt{1-t^2} < \sigma_n^*(t) \le \frac{C_2}{\sqrt{1-t^2}},\ t \in (-1,1),\ n \ge 1. \tag{7.17}$$

There the upper and lower bounds in (7.17) were proved in stronger forms, and for the slightly larger class of weights $\mathcal{F}(Dini)$, which satisfy a less restrictive Dini condition than (7.9). In particular, then,

$$0 < (\sigma_n^*)_{|I'}(t) \le C,\ t \in J_2,\ n \ge 1.$$

Next, we need the formula [45, (4.47), p. 122], [52, (2.28), p. 9], valid for $t \in I'$:

$$\begin{aligned}&\left(\widehat{\sigma^*_{n|[-1,1]\backslash I'}}\right)(t)\\ =\ & \frac{1}{\pi\sqrt{(1-\varepsilon')^2 - t^2}}\int_{[-1,1]\backslash I'} \frac{\sqrt{s^2-(1-\varepsilon')^2}}{|t-s|}\sigma_n^*(s)\,ds.\end{aligned}$$

Since the interval $[-1,1]\backslash I'$ is independent of n, our upper bound (7.17) shows also that

$$\left(\widehat{\sigma^*_{n|[-1,1]\backslash I'}}\right)(t) \le C,\ t \in J_2,\ n \ge 1.$$

Now (7.16) gives the result. ■

Lemma 7.6
Let $J_2 \subset (I')^o$.
(a) $\{Q_n'\}_{n=1}^\infty$ are uniformly bounded in I'.
(b) For each fixed $a > 0$,

$$\sup_{t\in J_2, |h|\le a}\left|Q_n'(t) - Q_n'\left(t+\frac{h}{n}\right)\right| \to 0 \text{ as } n \to \infty. \tag{7.18}$$

Proof
(a) We use the bound [31, (3.40), Lemma 3.8, p.77]

$$\left|Q'(x)\right| \le \frac{Cn}{\sqrt{\delta_n(a_n - x)}},\ x \in [0, a_n).$$

This readily yields

$$\left|Q'(x)\right| \le C(\varepsilon')\frac{n}{\delta_n},\ x \in \left[0, \max\left\{a_n - \varepsilon'\delta_n, 0\right\}\right].$$

A similar bound holds for negative x, and we deduce that
(7.19)

$$\left|Q'(x)\right| \le C\frac{n}{\delta_n},\ x \in L_n^{[-1]}\left[-1+\varepsilon', 1-\varepsilon'\right] = \left[a_{-n} + \varepsilon'\delta_n, a_n - \varepsilon'\delta_n\right].$$

Then, recalling (7.11),

$$\left|Q_n'(t)\right| = \frac{\delta_n}{n}\left|Q'\left(L_n^{[-1]}(t)\right)\right| \le C,\ t \in \left[-1+\varepsilon', 1-\varepsilon'\right] = I'.$$

(b) This is the most technical estimate in this section. We first establish the estimate

$$\frac{T(t)}{|t|}=\frac{|Q'(t)|}{Q(t)}=o\left(\frac{n}{\delta_n}\right),\ t\in[a_{-n}+\varepsilon\delta_n,a_n-\varepsilon\delta_n]\setminus[-\eta,\eta]. \tag{7.20}$$

Here $\eta>0$ is any fixed positive number. Indeed for $t\in[a_{-n}+\varepsilon\delta_n,a_n-\varepsilon\delta_n]\setminus[a_{-\log n},a_{\log n}]$, we have by (7.19) and the monotonicity of Q that

$$\frac{T(t)}{|t|}\leq C\frac{n}{\delta_n Q(a_{\pm\log n})}=o\left(\frac{n}{\delta_n}\right),$$

since $Q(a_{\pm\log n})\to\infty$ as $n\to\infty$. This latter limit follows from (7.8) and the fact that $a_{\log n}\to d$, $a_{-\log n}\to c$ as $n\to\infty$ [31, Theorem 2.4(iii), p. 41]. Next, for $t\in[a_{-\log n},a_{\log n}]\setminus[-\eta,\eta]$, we have that Q is bounded below, so

$$\frac{T(t)}{|t|}\leq C\left|Q'(a_{\pm\log n})\right|\leq C(\log n)^2=o\left(\frac{n}{\delta_n}\right).$$

Here we used (3.17) in [31, p. 69] and (3.38) in [31, p. 76] for the upper estimate on $|Q'(a_{\pm\log n})|$. Also, if $\Lambda>1$ is as in Definition 7.3, we used the fact that δ_n increases with n and that [31, (3.30), Lemma 3.5(c), p. 72]

$$\delta_n=O\left(n^{1/\Lambda}\right).$$

So (7.20) is established.

Let us now fix small $\rho>0$. By (7.9) of Definition 7.3, we can choose $\alpha>0$ so small that for all $X\in I\setminus\{0\}$,

$$\int_{X-\frac{2\alpha|X|}{T(X)}}^{X+\frac{2\alpha|X|}{T(X)}}\frac{Q'(s)-Q'(X)}{s-X}ds\leq\rho\left|Q'(X)\right|. \tag{7.21}$$

Suppose that $a>0$, $n\geq1$ and $x,y\in I'$ with $x<y\leq x+\frac{a}{n}$. Let

$$X=L_n^{[-1]}(x)\text{ and }Y=L_n^{[-1]}(y).$$

Then

$$0\leq Y-X=\delta_n(y-x)\leq a\frac{\delta_n}{n}.$$

Consequently for $n\geq n_0(a,\alpha)$, we have, by (7.20), as long as $X,Y\notin[-\eta,\eta]$,

$$Y-X\leq\alpha\frac{|X|}{T(X)}. \tag{7.22}$$

The threshold n_0 does not depend on n,x, or y. We use the fact Q' is monotone increasing, and the integral

$$\int_Y^{Y+\frac{1}{2}(Y-X)}\frac{ds}{s-X}=\log\frac{3}{2}$$

to deduce that

$$\begin{aligned} Q_n'(y)-Q_n'(x) &= \frac{\delta_n}{n}\left(Q'(Y)-Q'(X)\right) \\ &= \frac{\delta_n}{n\log\frac{3}{2}}\int_Y^{Y+\frac{1}{2}(Y-X)}\frac{Q'(Y)-Q'(X)}{s-X}ds \\ &\leq \frac{\delta_n}{n\log\frac{3}{2}}\int_X^{X+2\alpha\frac{|X|}{T(X)}}\frac{Q'(s)-Q'(X)}{s-X}ds \\ &\leq \frac{\delta_n}{n\log\frac{3}{2}}\rho\left|Q'(X)\right|. \end{aligned}$$

In the next to last line, we used (7.22) and the monotonicity of Q', and in the last line, we used (7.21). Finally our bound on Q' from (7.19) gives for $n\geq n_0$ and $x,y\in I'$ with $x<y\leq x+\frac{a}{n}$

$$0<Q_n'(y)-Q_n'(x)\leq C\rho,$$

as long as also $X,Y\notin[-\eta,\eta]$. It is crucial here that C is independent of n,x,y,ρ, so we may choose ρ as small as we please provided $n\geq n_0(\rho)$. Finally, if $X,Y\in[-\eta,\eta]$, we can use the boundedness of Q' in $[-\eta,\eta]$ to deduce that

$$Q_n'(y)-Q_n'(x)=\frac{\delta_n}{n}\left(Q'(Y)-Q'(X)\right)\leq C\frac{\delta_n}{n}=o(1).$$

The case where one of X,Y belongs to $[-\eta,\eta]$, and the other does not, may be handled by considering X,η and Y,η. Thus uniformly for $x,y\in I'$, with $x<y\leq x+\frac{a}{n}$, we have

$$Q_n'(y)-Q_n'(x)=o(1).$$

The range $x-\frac{a}{n}\leq y\leq x$ is similar. So we have (7.18). ■

Lemma 7.7
(a) For some $C_1,C_2>0$, and for $n\geq 1$ and $\xi\in I'$, we have

$$C_1\leq\lambda_n^{-1}(\xi)\,W_n^{2n}(\xi)/n\leq C_2. \tag{7.23}$$

(b) Uniformly for a in compact subsets of the real line, and $\xi\in I'$,

$$\lim_{n\to\infty}\frac{\lambda_n\left(\xi+\frac{a}{n}\right)}{\lambda_n(\xi)}\frac{W_n^{2n}(\xi)}{W_n^{2n}\left(\xi+\frac{a}{n}\right)}=1. \tag{7.24}$$

Proof
(a) Let $0<\alpha<1$. By Corollary 1.14 in [31, p. 20], we have uniformly for $n\geq 1$ and $x\in[a_{-\alpha n},a_{\alpha n}]$,

$$\lambda_n\left(W^2,x\right)\sim\varphi_n(x)\,W^2(x).$$

Here

$$\varphi_n(x)=\frac{|x-a_{-2n}|\,|a_{2n}-x|}{n\sqrt{\left[|x-a_{-n}|+|a_{-n}|\,\eta_{-n}\right]\left[|x-a_n|+a_n\eta_n\right]}},$$

and

$$\eta_{\pm n}=\left[nT\left(a_{\pm n}\right)\sqrt{\frac{\left|a_{\pm n}\right|}{\delta_n}}\right]^{-2/3}=o\left(\delta_n\right).$$

If α is close enough to 1, it follows from [31, (3.50), Lemma 3.11, page 81] that

$$L_n^{[-1]}\left(I'\right)=L_n^{[-1]}\left(\left[-1+\varepsilon',1-\varepsilon\right]\right)\subset\left[a_{-\alpha n},a_{\alpha n}\right]. \tag{7.25}$$

Moreover, for $n\geq 1$ and $x\in\left[a_{-n}+\varepsilon'\delta_n,a_n-\varepsilon'\delta_n\right]=L_n^{[-1]}\left(I'\right)$, we have

$$\varphi_n\left(x\right)\sim\frac{\delta_n}{n}.$$

Thus for $n\geq 1$ and $x\in L_n^{[-1]}\left(I'\right)$,

$$\lambda_n\left(W^2,x\right)\sim\frac{\delta_n}{n}W^2\left(x\right). \tag{7.26}$$

Next,

$$\begin{aligned}\lambda_n\left(\xi\right) &= \lambda_n\left(W_n^{2n},\xi\right)\\ &= \inf_{\deg(P)\leq n-1}\frac{\int\left(P^2W_n^{2n}\right)\left(t\right)dt}{P^2\left(\xi\right)}\\ &= \inf_{\deg(P)\leq n-1}\frac{\int P^2\left(t\right)\exp\left(-2Q\circ L_n^{[-1]}\left(t\right)\right)dt}{P^2\left(\xi\right)}\\ &= \inf_{\deg(R)\leq n-1}\frac{\int R^2\left(s\right)\exp\left(-2Q\left(s\right)\right)\delta_n^{-1}ds}{R^2\left(L_n^{[-1]}\left(\xi\right)\right)}\\ &= \delta_n^{-1}\lambda_n\left(W^2,L_n^{[-1]}\left(\xi\right)\right). \qquad (7.27)\end{aligned}$$

Then (7.23) follows from (7.26).

(b) Let $0<\alpha<1$. By Theorem 1.25 in [31, p. 26], we have as $n\to\infty$

$$\lambda_n\left(W^2,x\right)/W^2\left(x\right)=\sigma_n^{-1}\left(x\right)\left(1+o\left(1\right)\right),$$

uniformly for $x\in\left[a_{-\alpha n},a_{\alpha n}\right]$. Then (7.27) and (7.14) show that uniformly for $\xi\in I'$,

$$\begin{aligned}\lambda_n\left(\xi\right)/W_n^{2n}\left(\xi\right) &= \delta_n^{-1}\sigma_n^{-1}\left(L_n^{[-1]}\left(\xi\right)\right)\left(1+o\left(1\right)\right)\\ &= n^{-1}\sigma_n^{*-1}\left(\xi\right)\left(1+o\left(1\right)\right). \qquad (7.28)\end{aligned}$$

It is shown in Theorem 6.2 in [31, p. 147] that $\{\sigma_n^*\}$ are equicontinuous in each compact subset of $(-1,1)$. Then the desired conclusion (7.24) follows from (7.28). To deal with the possibility that $\xi+\frac{a}{n}$ lies outside I', we use the arbitrariness of $\varepsilon\in(0,1)$ in (7.15). ■

Proof of Theorem 7.4

By Theorem 1.2, we have universality for the varying weights $\left\{W_n^{2n}\right\}$ at each

$\xi \in J'$, uniformly with respect to ξ. Indeed, the four hypotheses of Theorem 1.2 were established in Lemmas 7.5, 7.6, and 7.7 (except that we established (7.18) rather than equicontinuity of $\{Q_n'\}$. As noted after Theorem 1.2, this is what we used in the proof of Theorem 1.2. The orthogonal polynomials $p_n(x) = p_n\left(W_n^{2n}, x\right)$ are related to those for W^2 by the identity

$$p_n(x) = p_n\left(W_n^{2n}, x\right) = \delta_n^{1/2} p_n\left(W^2, L_n^{[-1]}(x)\right).$$

This is easily established by a substitution in the orthonormality relation for $\{p_n(x)\}$. Hence the reproducing kernel $K_n(x,t) = K_n\left(W_n^{2n}, x, t\right)$ for W_n^{2n} is related to the reproducing kernel $K_n\left(W^2, x, t\right)$ for W^2 by the identity

$$K_n(x,t) = \delta_n K_n\left(W^2, L_n(x), L_n(t)\right).$$

Then the result follows from Theorem 1.2. ∎

Acknowledgement
The authors thank the referee for a very careful reading of the manuscript, and for suggestions to improve the presentation.

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