APPLICATIONS OF UNIVERSALITY LIMITS TO ZEROS AND REPRODUCING KERNELS OF ORTHOGONAL POLYNOMIALS

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ABSTRACT. We apply universality limits to asymptotics of spacing of zeros $\{x_{kn}\}$ of orthogonal polynomials, for weights with compact support and for exponential weights. A typical result is

$$\lim_{n\to\infty} \left(x_{kn} - x_{k+1,n}\right) \tilde{K}_n \left(x_{kn}, x_{kn}\right) = 1$$

under minimal hypotheses on the weight, with $\tilde{K}_n$ denoting a normalized reproducing kernel. Moreover, for exponential weights, we derive asymptotics for the differentiated kernels

$$K_n^{(r,s)} \left(x, x\right) = \sum_{k=0}^{n-1} p_k^{(r)} \left(x\right) p_k^{(s)} \left(x\right).$$

1. INTRODUCTION AND RESULTS[1]

Let μ be a finite positive Borel measure on the real line, with all finite power moments. Then we may define orthonormal polynomials

$$p_n \left(x\right) = \gamma_n x^n + ..., \ \gamma_n > 0,$$

$n = 0, 1, 2, ...$ satisfying the orthonormality conditions

$$\int p_n p_m d\mu = \delta_{mn}.$$

The zeros of p_n are denoted

$$x_{nn} < x_{n-1,n} < x_{n-2,n} < ... < x_{1n}.$$

The universality limit of random matrix theory [4], [16] involves the reproducing kernel

$$K_n \left(x, y\right) = \sum_{k=0}^{n-1} p_k \left(x\right) p_k \left(y\right), \tag{1.1}$$

and its normalized cousin

$$\widetilde{K}_n \left(x, y\right) = w \left(x\right)^{1/2} w \left(y\right)^{1/2} K_n \left(x, y\right), \tag{1.2}$$

where, throughout,

$$w = \frac{d\mu}{dx}. \tag{1.3}$$

Date: May 30, 2007.

[1]Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353

In the bulk of the spectrum, the universality law has the form

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(x+\frac{a}{\widetilde{K}_n(x,x)}, x+\frac{b}{\widetilde{K}_n(x,x)}\right)}{\widetilde{K}_n(x,x)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{1.4}$$

Typically this holds uniformly for x in some subinterval of the interior of the support of μ and a, b in compact subsets of the real line, under appropriate hypotheses on μ. Of course, when $a = b$, we interpret $\frac{\sin\pi(a-b)}{\pi(a-b)}$ as 1. Some key references are [1], [2], [3], [4], [5], [6], [16], [20], and the forthcoming proceedings of the conference devoted to the 60th birthday of Percy Deift.

In this paper, we turn the subject around somewhat. Instead of establishing the universality limit under suitable hypotheses on μ, we show how universality limits obtained in [13], [14], [15] yield asymptotics of various quantities associated with orthogonal polynomials. In particular, they imply asymptotics of spacing between successive zeros of orthogonal polynomials - 'clock theorems' in the terminology of Barry Simon. Moreover, they do so under very weak hypotheses on the measure. It is easy to see why, from (1.4): the sin factor on the right-hand side changes sign every time $a-b$ increases by a unit. Furthermore, they yield asymptotics for the differentiated reproducing kernels

$$K_n^{(r,s)}(x,x) = \sum_{k=0}^{n-1} p_k^{(r)}(x)\, p_k^{(s)}(x). \tag{1.5}$$

We shall need the class of *regular measures* on $[-1,1]$, namely those measures μ supported on $[-1,1]$, satisfying

$$\lim_{n\to\infty} \gamma_n^{1/n} = 2.$$

This class was extensively studied in [25]. It is somewhat larger than the Nevai-Blumenthal class, which is defined in terms of the three term recurrence relation

$$xp_n(x) = A_{n+1}p_{n+1}(x) + B_n p_n(x) + A_n p_{n-1}(x).$$

Here $A_n = \frac{\gamma_{n-1}}{\gamma_n}$ and B_n is real. The Nevai-Blumenthal class $\mathcal{M}$ consists of those measures μ for which

$$\lim_{n\to\infty} A_n = \frac{1}{2} \text{ and } \lim_{n\to\infty} B_n = 0.$$

If $\mu' > 0$ a.e. in $[-1,1]$, then $\mu \in \mathcal{M}$, but there are pure jump and pure singularly continuous measures in $\mathcal{M}$ [21]. Our first result is:

Theorem 1.1

Let μ be a finite positive Borel measure on $[-1,1]$ that is regular. Let $\mathcal{K}$ be a compact subset of $(-1,1)$ such that μ is absolutely continuous in an open interval containing $\mathcal{K}$. Assume that $w = \mu'$ is positive and continuous at each point of $\mathcal{K}$.
(a) Let $k = k(n)$, $n \geq 1$, be such that as $n \to \infty$,

$$dist\,(x_{kn}, \mathcal{K}) = O\left(\frac{1}{n}\right). \tag{1.6}$$

Then

$$\lim_{n\to\infty} (x_{kn} - x_{k+1,n}) \frac{n}{\pi\sqrt{1-x_{k,n}^2}} = 1. \tag{1.7}$$

The limit holds uniformly for families of zeros that satisfy (1.6) uniformly, in particular for zeros lying in $\mathcal{K}$.
(b) For each $x \in \mathcal{K}$, there exists a sequence of zeros x_{kn}, where $k = k(n)$, satisfying

$$x_{kn} = x + O\left(\frac{1}{n}\right). \tag{1.8}$$

Note that it is permissible that $\mathcal{K}$ consists of a single point a, say. Then our hypothesis is that μ is regular in $(-1,1)$, absolutely continuous in $(a-\varepsilon, a+\varepsilon)$ for some $\varepsilon > 0$, while μ' is continuous and positive at a. The conclusion in this case is that if $k = k(n)$ satisfies as $n \to \infty$,

$$x_{kn} = a + O\left(\frac{1}{n}\right),$$

then (1.7) holds. Moreover, (b) shows that there are such zeros. Of course we could replace $\sqrt{1-x_{k,n}^2}$ by $\sqrt{1-x^2}$ for any $x \in [x_{k+1,n}, x_{kn}]$.

This result should be compared to the 'clock theorems' in [11], [22], [23] and earlier work of Freud [8, p. 266]. Freud assumed that w is bounded below a.e. in $(-1,1)$ by the square of a not identically vanishing polynomial, and that w is positive and continuous in a closed interval $\mathcal{K}$. Last and Simon assume conditions on the recurrence coefficients that imply that μ lies in a subset of the Nevai-Blumenthal class. More precisely, they assume that $A_n \to \frac{1}{2}, B_n \to 0$ and

$$\sum_n \left(|A_{n+1} - A_n| + |B_{n+1} - B_n|\right) < \infty.$$

They then establish a uniform version of Theorem 1.1 in each compact subinterval of $(-1,1)$. Their hypothesis implies, by a result of Dombrowski and Nevai [7] that μ is absolutely continuous in $(-1,1)$ and w is positive and continuous in $(-1,1)$. Our global assumption of regularity is hence more general in the special case that the support of μ is $[-1,1]$. However, when we do not assume that the support of μ is $[-1,1]$, then it is not more general.

We also note that while the class of regular measures is larger than the Nevai-Blumenthal class $\mathcal{M}$, there is no known example of a regular measure outside $\mathcal{M}$, with absolutely continuous component in some subinterval. Nevertheless, we believe that such an example exists.

Our next result concerns asymptotics of the zeros close to 1. We need the Jacobi weight

$$w^J(x) = (1-x)^\alpha (1+x)^\beta,$$

$\alpha, \beta > -1$. For the Jacobi measure, the universality limit at 1 takes the following form: uniformly for a, b in compact subsets of $(0, \infty)$,

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n^J \left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha(a,b). \tag{1.9}$$

Here the superscript J indicates quantities associated with w^J, and

$$\mathbb{J}_\alpha(u,v) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J_\alpha'(\sqrt{v}) - J_\alpha(\sqrt{v})\sqrt{u}J_\alpha'(\sqrt{u})}{2(u-v)}$$

is the Bessel kernel of order α, and J_α is the usual Bessel function of the first kind and order α. We denote the positive zeros of J_α by

$$0 < j_{\alpha,1} < j_{\alpha,2} < j_{\alpha,3} < \dots.$$

Theorem 1.2
Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Assume that for some $\rho>0$, μ is absolutely continuous in $\mathcal{K}=[1-\rho,1]$, and in $\mathcal{K}$, its absolutely continuous component has the form $w=hw^{(\alpha,\beta)}$, where $\alpha,\beta>-1$. Assume that $h(1)>0$ and h is continuous at 1. Then for each fixed $k\geq 1$,

$$\lim_{n\to\infty} n\sqrt{1-x_{kn}^2}=j_{\alpha,k} \tag{1.10}$$

and

$$\lim_{n\to\infty} n^2\left(x_{kn}-x_{k+1,n}\right)=\frac{1}{2}\left(j_{\alpha,k+1}^2-j_{\alpha,k}^2\right). \tag{1.11}$$

Next, we discuss exponential weights $w=W^2=e^{-2Q}$, where $Q:\mathbb{R}\to[0,\infty)$ is continuous, and all moments

$$\int_{\mathbb{R}} x^j W^2(x)\,dx,\ j=0,1,2,\dots,$$

are finite. Our class of exponential weights is:

Definition 1.3
Let $W=e^{-Q}$, where $Q:\mathbb{R}\to[0,\infty)$ satisfies the following conditions:
(a) Q' is continuous in $\mathbb{R}$ and $Q(0)=0$.
(b) Q' is non-decreasing in $\mathbb{R}$, and Q'' exists in $\mathbb{R}\setminus\{0\}$.
(c)

$$\lim_{|t|\to\infty} Q(t)=\infty.$$

(d) The function

$$T(t)=\frac{tQ'(t)}{Q(t)},\ t\neq 0,$$

is quasi-increasing in $(0,\infty)$, in the sense that for some $C>0$,

$$0<x<y\Rightarrow T(x)\leq CT(y).$$

We assume an analogous restriction for $y<x<0$. In addition, we assume that for some $\Lambda>1$,

$$T(t)\geq\Lambda \text{ in } \mathbb{R}\setminus\{0\}.$$

(e) There exists $C_1>0$ such that

$$\frac{Q''(x)}{|Q'(x)|}\leq C_1\frac{Q'(x)}{Q(x)}\ x\in\mathbb{R}\setminus\{0\}.$$

Then we write $W\in\mathcal{F}\left(C^2\right)$.

This class of weights is a special case of the class of weights considered in [12, p. 7]. There more general intervals than the real line were permitted, and we did not require Q'' to exist at every point except 0. Examples of weights in this class are $W=\exp(-Q)$, where

$$Q(x)=\begin{cases} Ax^{\alpha}, & x\in[0,\infty)\\ B|x|^{\beta}, & x\in(-\infty,0)\end{cases},$$

where $\alpha, \beta > 1$ and $A, B > 0$. More generally, if $\exp_k = \exp(\exp(\ldots\exp()))$ denotes the kth iterated exponential, we may take

$$Q(x) = \begin{cases} \exp_k(Ax^\alpha) - \exp_k(0), & x \in [0,\infty) \\ \exp_\ell\left(B|x|^\beta\right) - \exp_\ell(0), & x \in (-\infty, 0) \end{cases}$$

where $k, \ell \geq 1$, $\alpha, \beta > 1$.

A key descriptive role is played by the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n} < 0 < a_n,$$

defined for $n \geq 1$ by the equations

$$n = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx; \tag{1.12}$$

$$0 = \frac{1}{\pi}\int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x-a_{-n})(a_n-x)}}dx. \tag{1.13}$$

In the case where Q is even, $a_{-n} = -a_n$. The existence and uniqueness of these numbers is established in the monographs [12], [17], [24], but goes back to earlier work of Mhaskar, Saff, and Rakhmanov. One illustration of their role is the Mhaskar-Saff identity:

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[a_{-n},a_n]},$$

valid for $n \geq 1$ and all polynomials P of degree $\leq n$.

We also define,

$$\beta_n = \frac{1}{2}(a_n + a_{-n}) \text{ and } \delta_n = \frac{1}{2}(a_n + |a_{-n}|), \tag{1.14}$$

which are respectively the center, and half-length of the Mhaskar-Rakhmanov-Saff interval $\Delta_n = [a_{-n}, a_n]$. The linear transformation

$$L_n(x) = \frac{x - \beta_n}{\delta_n}$$

maps Δ_n onto $[-1,1]$. Its inverse is

$$L_n^{[-1]}(u) = \beta_n + u\delta_n.$$

For $0 < \varepsilon < 1$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n]. \tag{1.15}$$

Next, we define the equilibrium density

$$\sigma_n(x) = \frac{\sqrt{(x-a_{-n})(a_n-x)}}{\pi^2}\int_{a_{-n}}^{a_n} \frac{Q'(s)-Q'(x)}{s-x}\frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}, \; x \in \Delta_n. \tag{1.16}$$

It satisfies the equation for the equilibrium potential [12, p. 16]:

$$\int_{a_{-n}}^{a_n} \log\frac{1}{|x-s|}\sigma_n(s)\,ds + Q(x) = C, \; x \in \Delta_n, \tag{1.17}$$

and has total mass n :

$$\int_{a_{-n}}^{a_n} \sigma_n(s)\,ds = n. \tag{1.18}$$

When dealing with exponential weights, we assume that our measure μ is absolutely continuous and that $w = W^2 = e^{-2Q}$ (or later $w = \left(W^h\right)^2 = W^2 h^2$). The orthonormal polynomials, reproducing kernels, and zeros are denoted respectively by $\{p_n\}, \{K_n\}$, $\{x_{kn}\}$, so that, in particular,

$$\int_{-\infty}^{\infty} p_n p_m W^2 = \delta_{mn}.$$

Our first result for exponential weights is:

Theorem 1.4

Let $W = \exp(-Q) \in \mathcal{F}\left(C^2\right)$. Let $0 < \varepsilon < 1$. Then uniformly for $x_{kn} \in J_n(\varepsilon)$, we have as $n \to \infty$,

$$\lim_{n\to\infty} \left(x_{kn} - x_{k+1,n}\right) \sigma_n\left(x_{kn}\right) = 1. \tag{1.19}$$

In particular, if W is even, this holds uniformly for $|x_{kn}| \leq (1-\varepsilon) a_n$. Moreover, the zeros to the left and right of any point x satisfy (1.19).

Note that uniformly in n and $x \in J_n(\varepsilon)$,

$$\sigma_n(x) \sim \frac{n}{\delta_n}.$$

By this we mean that the ratio of the two sides is bounded above and below by positive constants independent of n and x. Note too that the proof works without change for a larger class of weights, namely the class $\mathcal{F}\left(lip\frac{1}{2}\right)$ in [12, p. 12]. However, the definition of that class is more implicit, so is omitted. One may restate (1.19) in an alternative form: uniformly for $x_{kn} \in J_n(\varepsilon)$, we have as $n \to \infty$,

$$\lim_{n\to\infty} \left(x_{kn} - x_{k+1,n}\right) \tilde{K}_n\left(x_{kn}, x_{kn}\right) = 1. \tag{1.20}$$

We shall also deal with weights

$$w = \left(W^h\right)^2 = W^2 h^2.$$

Their reproducing kernels will be denoted respectively by $K_n^h(x,t)$, and in normalized form by $\tilde{K}_n^h(x,t)$. The superscript h will also be used to indicate other quantities associated with this weight. Recall that a generalized Jacobi weight w has the form

$$w(x) = \prod_{j=1}^{N} |x - \alpha_j|^{\beta_j}, \tag{1.21}$$

where all $\{\alpha_j\}$ are distinct, and all $\beta_j > -1$.

Theorem 1.5

Let $W = \exp(-Q) \in \mathcal{F}\left(C^2\right)$. Let σ_n denote the equilibrium measure for Q, defined by (1.16). Let $h : \mathbb{R} \to [0,\infty)$ be a function that is square integrable over every finite interval. Assume that there is a generalized Jacobi weight w, a compact interval J, and $C > 0$ such that

$$h^2 \geq Cw \text{ in } J, \tag{1.22}$$

while

$$\lim_{r\to\infty} \frac{\log \|\log h\|_{L_\infty([0,r]\setminus J)}}{\log Q(r)} = 0, \tag{1.23}$$

with an analogous limit as $r \to -\infty$. Assume that $\mathcal{K}$ is a closed subset of $\mathbb{R}$ in which $\log h$ is uniformly continuous. Let $0 < \varepsilon < 1$. Then uniformly for a, b in compact subsets of the real line, and $x_{kn}^h \in J_n(\varepsilon) \cap \mathcal{K}$, we have

$$\lim_{n\to\infty} \left(x_{kn}^h - x_{k+1,n}^h\right) \sigma_n\left(x_{kn}^h\right) = 1. \tag{1.24}$$

Moreover, the zeros to the left and right of any point x satisfy (1.24).

Note that we can take $h(x) = |x|^\beta$, where $\beta > -\frac{1}{2}$, so that

$$W^h(x) = |x|^\beta W(x),$$

or more generally, may take

$$W^h(x) = \left(\prod_{k=1}^m |x - \alpha_k|^{\beta_k}\right) W(x)\, g(x),$$

where all $\{\alpha_k\}$ are distinct, all $\beta_k > -\frac{1}{2}$, and g is a positive continuous function, with $\log g$ uniformly continuous in the real line, and

$$\lim_{|x|\to\infty} \frac{\log\log g(x)}{\log|x|} = 0.$$

Our final result concerns the differentiated reproducing kernels $K_n^{(r,s)}$ defined by (1.5). When $r = s$, $K_n^{(r,s)}$ is the solution of an extremal problem, namely

$$1/K_n^{(r,r)}(x) = \inf\left\{\int_{\mathbb{R}} (PW)^2 / \left(P^{(r)}(x)\right)^2 : \deg(P) \le n-1\right\}.$$

In the special case $r = 0$, we obtain the classical Christoffel function, but the case of general r was used by Freud to establish Markov-Bernstein inequalities. Freud obtained estimates for $K_n^{(r,r)}$ for Freud weights, but not asymptotics [9]. We let

$$\tau_{r,s} = \begin{cases} 0, & r+s \text{ odd} \\ \frac{(-1)^{(r-s)/2}}{r+s+1}, & r+s \text{ even} \end{cases}. \tag{1.25}$$

Theorem 1.6
Let $W = \exp(-Q) \in \mathcal{F}\left(C^2\right)$ and σ_n be defined by (1.16). Let $0 < \varepsilon < 1$, $r, s \ge 0$. Then uniformly for $x \in J_n(\varepsilon)$, we have

$$\frac{W^2(x)\, K_n^{(r,s)}(x,x)}{(\sigma_n(x))^{r+s+1}} = \sum_{j=0}^r \binom{r}{j} \sum_{k=0}^s \binom{s}{k} \tau_{j,k} \pi^{j+k} \left(\frac{Q'(x)}{\sigma_n(x)}\right)^{r+s-j-k} + o(1). \tag{1.26}$$

If we restrict x to a compact subset of the real line, we may simplify this as

$$\frac{W^2(x)\, K_n^{(r,s)}(x,x)}{(\sigma_n(x))^{r+s+1}} = \tau_{r,s}\pi^{r+s} + o(1),$$

since in such a set $Q'(x)/\sigma_n(x) = o(1)$ as $n \to \infty$. More generally, this holds uniformly for $x \in J_n(\varepsilon_n)$, provided $\varepsilon_n \to 1$ as $n \to \infty$. For weights on a finite interval, an analogue of the above result was presented in [14].

This paper is organised as follows: in Section 2, we prove Theorem 1.1. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorems 1.4 and 1.5. Finally, in Section 5, we prove Theorem 1.6. In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurrences. We shall write $C = C(\alpha)$ or $C \ne C(\alpha)$

to respectively denote dependence on, or independence of, the parameter α. Given sequences $\{c_n\}$, $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1, C_2 such that for $n \geq 1$,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions and sequences of functions. $[x]$ denotes the greatest integer $\leq x$.

Acknowledgement
The authors thank the referees for helpful comments, in particular pointing out a gap in a proof of one result.

2. Proof of Theorem 1.1

In [14, Theorem 1.1], we showed that

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(x+\frac{a}{\tilde{K}_n(x,x)}, x+\frac{b}{\tilde{K}_n(x,x)}\right)}{\tilde{K}_n(x,x)} = \frac{\sin\pi(a-b)}{\pi(a-b)},$$

uniformly for $x \in \mathcal{K}$ and a, b in compact subsets of the real line. In Theorem 2.1 there, it was also shown that uniformly for $x \in \mathcal{K}$ and a in a bounded set,

$$\tilde{K}_n\left(x+\frac{a}{n}, x+\frac{a}{n}\right) \sim n.$$

Now let x_{kn} satisfy (1.6), where $k = k(n)$, so that for some $u_n \in \mathcal{K}$, and bounded sequence $\{\tilde{a}_n\}$,

$$x_{kn} = u_n + \frac{\tilde{a}_n}{\tilde{K}_n(u_n, u_n)}. \tag{2.1}$$

Because of the asymptotics for Christoffel functions in [14, Theorem 2.1], and continuity of w at each point of $\mathcal{K}$, we have

$$\tilde{K}_n(x_{kn}, x_{kn}) = \tilde{K}_n(u_n, u_n)(1+o(1)), \tag{2.2}$$

so we may also write

$$x_{kn} = u_n + \frac{a_n}{\tilde{K}_n(x_{kn}, x_{kn})},$$

where $\{a_n\}$ is bounded. We need the fundamental polynomial ℓ_{kn} of Lagrange interpolation that satisfies

$$\ell_{kn}(x_{jn}) = \delta_{jk}.$$

One well known representation of ℓ_{kn}, which follows from the Christoffel-Darboux formula, is

$$\ell_{kn}(x) = K_n(x_{kn}, x)/K_n(x_{kn}, x_{kn}). \tag{2.3}$$

Setting $x = x_{kn} + \frac{b}{\tilde{K}_n(x_{kn},x_{kn})}$, we obtain from (2.1) to (2.3), and the uniformity in a, b above, that

$$\begin{aligned} & \ell_{kn}\left(x_{kn} + \frac{b}{\tilde{K}_n\left(x_{kn}, x_{kn}\right)}\right) \\ = \quad & \left(1 + o\left(1\right)\right)\tilde{K}_n\left(u_n + \frac{\tilde{a}_n}{\tilde{K}_n\left(u_n, u_n\right)}, u_n + \frac{\tilde{a}_n + b + o\left(1\right)}{\tilde{K}_n\left(u_n, u_n\right)}\right)/\tilde{K}_n\left(u_n, u_n\right) \\ = \quad & \frac{\sin \pi b}{\pi b} + o\left(1\right), \end{aligned} \tag{2.4}$$

uniformly for b in compact subsets of the real line. We also used the continuity of w at each point of $\mathcal{K}$. Since $\frac{\sin \pi b}{\pi b}$ changes sign at $b = -1$, it follows that $x_{k+1,n}$, the zero of ℓ_{kn} closest on the left to x_{kn}, must satisfy

$$x_{k+1,n} = x_{kn} + \frac{\beta_n}{\tilde{K}_n\left(x_{kn}, x_{kn}\right)},$$

where $\beta_n \in (-\infty, 0)$, and

$$\liminf_{n\to\infty} \beta_n \geq -1.$$

In particular $\{\beta_n\}$ is bounded. We have to show that

$$\lim_{n\to\infty} \beta_n = -1. \tag{2.5}$$

Choose any subsequence of $\{\beta_n\}$ with some limit β, say. Necessarily $\beta \in [-1, 0]$. Since $\ell_{kn}\left(x_{k+1,n}\right) = 0$, we obtain from (2.4), as $n \to \infty$ through the subsequence, that

$$\frac{\sin \pi\beta}{\pi\beta} = 0,$$

so $\beta = -1$. As this is true for any subsequence, we obtain (2.5). That in turn gives

$$\lim_{n\to\infty} \left(x_{kn} - x_{k+1,n}\right)\tilde{K}_n\left(x_{kn}, x_{kn}\right) = 1.$$

Finally, from the asymptotics for Christoffel functions in [14, Theorem 2.1], and classical asymptotics for Christoffel functions for the Legendre weight,

$$\tilde{K}_n\left(x_{kn}, x_{kn}\right) = \frac{n}{\pi\sqrt{1 - x_{kn}^2}}\left(1 + o\left(1\right)\right).$$

(b) Fix $x \in \mathcal{K}$. By the universality limit,

$$\lim_{n\to\infty} \tilde{K}_n\left(x, x + \frac{b}{\tilde{K}_n\left(x, x\right)}\right)/\tilde{K}_n\left(x, x\right) = \frac{\sin \pi b}{\pi b},$$

uniformly for b in compact subsets of the real line. This has the consequence that $K_n\left(x, t\right)$ has sign changes at y_n and z_n, where $y_n \leq x < z_n$, and

$$y_n, z_n = x + O\left(\frac{1}{n}\right). \tag{2.6}$$

We assume that y_n and z_n are the closest sign changes to the left and right of x.

Next, we claim that as a function of t, $K_n\left(x, t\right)$ has at most one sign change in each interval $[x_{j+1,n}, x_{jn})$, $1 \leq j \leq n-1$. When x is not a zero of p_n or p_{n-1}, this was proved in [8, p. 19], where still more is shown: the zeros of

$$K_n\left(x, t\right)\left(x - t\right) = \frac{\gamma_{n-1}}{\gamma_n}\left(p_n\left(x\right)p_{n-1}\left(t\right) - p_{n-1}\left(x\right)p_n\left(t\right)\right)$$

strictly interlace those of p_n. When x is a zero of p_{n-1}, then $K_n(x,t)(x-t)$ is a multiple of $p_{n-1}(t)$, whose zeros interlace those of $p_n(t)$. Finally, if x is a zero of p_n, then $K_n(x,t)(x-t)$ is a multiple of $p_n(t)$, so the claim is trivially true.

Our claim and the relation (2.6) imply that in the interval $[y_n, z_n)$, there must be at least one zero of p_n. (For otherwise, some interval $[x_{j+1,n}, x_{j,n})$ contains 2 sign changes, namely y_n, z_n of $K_n(x,t)$.) In view of (2.6), that zero of p_n must satisfy (1.8). ■

We note that the proof can be substantially simplified in the case when x lies in the interior of $\mathcal{K}$.

3. Proof of Theorem 1.2

Throughout, we assume the hypotheses of Theorem 1.2. In this section, we let

$$w^J(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

We use the superscript J to denote quantities associated with w^J, such as $p_n^J, x_{kn}^J, K_n^J, \ell_{kn}^J$. The most difficult part of the proof is the asymptotic for the largest zero x_{1n}.

Lemma 3.1
(a) Let $A > 0$. Then uniformly for $a \in [0, A]$,

$$\lim_{n\to\infty} \lambda_n\left(1-\frac{a}{2n^2}\right)/\lambda_n^J\left(1-\frac{a}{2n^2}\right) = h(1). \tag{3.1}$$

Moreover, uniformly for $n \geq n_0(A)$ and $a \in [0, A]$,

$$\lambda_n\left(1-\frac{a}{2n^2}\right) \sim \lambda_n^J\left(1-\frac{a}{2n^2}\right) \sim n^{-(2\alpha+2)}. \tag{3.2}$$

(b) For each fixed $k \geq 1$,

$$\lim_{n\to\infty} n\sqrt{1-\left(x_{kn}^J\right)^2} = j_{\alpha,k} \tag{3.3}$$

and

$$\lim_{n\to\infty} n^2\left(x_{kn}^J - x_{k+1,n}^J\right) = \frac{1}{2}\left(j_{\alpha,k+1}^2 - j_{\alpha,k}^2\right). \tag{3.4}$$

(c) Given $\varepsilon \in (0,1)$, we have

$$\lim_{n\to\infty} \int_{-1}^{1-\varepsilon} \left(\ell_{1n}^J\right)^2 w^J / \int_{-1}^{1} \left(\ell_{1n}^J\right)^2 w^J = 0. \tag{3.5}$$

Proof
(a) This is Theorem 2.1 in [15].
(b) Let us write $x_{kn}^J = \cos\theta_{kn}^J$. It is known [26, pp. 192-3] that for each fixed k,

$$\lim_{n\to\infty} n\theta_{kn}^J = j_{\alpha,k}.$$

Then

$$\begin{aligned} n^2\left(1-\left(x_{kn}^J\right)^2\right) &= \left(n\sin\theta_{kn}^J\right)^2 \\ &= \left(n\theta_{kn}^J\right)^2(1+o(1)) = j_{\alpha,k}^2(1+o(1)). \end{aligned}$$

(c) The method of proof below is one that will later work also for ℓ_{1n}. The reader familiar with estimates for Jacobi polynomials could simplify the procedure below. We use the representation

$$\ell_{1n}^J(x) = \frac{K_n^J\left(x, x_{1n}^J\right)}{K_n^J\left(x_{1n}^J, x_{1n}^J\right)} = \frac{1}{K_n^J\left(x_{1n}^J, x_{1n}^J\right)} \frac{\gamma_{n-1}^J}{\gamma_n^J} \frac{p_{n-1}^J\left(x_{1n}^J\right) p_n^J(x)}{x - x_{1n}^J},$$

which follows from the Christoffel-Darboux formula. We also use the identity

$$\int_{-1}^{1} \left(\ell_{1n}^J\right)^2 w^J = 1/K_n^J\left(x_{1n}^J, x_{1n}^J\right).$$

Note that

$$\frac{\gamma_{n-1}^J}{\gamma_n^J} = \int_{-1}^{1} x p_{n-1}^J(x)\, p_n^J(x)\, w^J(x)\, dx \leq 1,$$

by Cauchy-Schwarz and orthonormality. For large enough n, $x_{1n}^J \geq 1 - \frac{\varepsilon}{2}$, so

$$\begin{aligned} & \int_{-1}^{1-\varepsilon} \left(\ell_{1n}^J\right)^2 w^J / \int_{-1}^{1} \left(\ell_{1n}^J\right)^2 w^J \\ \leq\ & \left(\frac{2}{\varepsilon}\right)^2 \frac{\left(p_{n-1}^J\left(x_{1n}^J\right)\right)^2}{K_n^J\left(x_{1n}^J, x_{1n}^J\right)} \int_{-1}^{1-\varepsilon} \left(p_n^J\right)^2 w^J \\ \leq\ & \frac{4\left(p_{n-1}^J\left(x_{1n}^J\right)\right)^2}{\varepsilon^2 K_n^J\left(x_{1n}^J, x_{1n}^J\right)}. \end{aligned} \tag{3.6}$$

Next, as w^J lies in the Nevai-Blumenthal class $\mathcal{M}$, we have [19, Theorem 2.1, p. 218]

$$\sup_{x \in [-1,1]} \frac{\left(p_{n-1}^J(x)\right)^2}{K_n^J(x, x)} \to 0, n \to \infty. \tag{3.7}$$

Then the result follows from (3.6). ■

Next, we establish a one-sided bound for $1 - x_{1n}$:

Lemma 3.2

$$\limsup_{n \to \infty} \frac{1 - x_{1n}}{1 - x_{1n}^J} \leq 1. \tag{3.8}$$

Proof

We use a well known extremal characterization of the largest zero:

$$x_{1n} = \sup_{\deg(P) \leq n-1} \int_{-1}^{1} x P^2(x)\, d\mu(x) / \int_{-1}^{1} P^2(x)\, d\mu(x).$$

This follows easily from the Gauss quadrature formula. Note that the maximizing polynomial is just $P = \ell_{1n}$. Then

$$1 - x_{1n} = \inf_{\deg(P) \leq n-1} \int_{-1}^{1} (1 - x)\, P^2(x)\, d\mu(x) / \int_{-1}^{1} P^2(x)\, d\mu(x). \tag{3.9}$$

Let $\eta \in \left(0, \frac{1}{2}\right)$ and choose $\varepsilon \in (0, \rho)$ such that

$$\frac{1}{1+\eta} \leq w / \left[h(1)\, w^J\right] \leq 1 + \eta \text{ in } [1 - \varepsilon, 1]. \tag{3.10}$$

Let $m = n - [\eta n]$ and choose

$$P = \ell_{1m}^J R,$$

where R has degree $\leq [\eta n]$, $0 \leq R \leq 1$ in $[-1, 1]$,

$$|R - 1| \leq n^{-2} \text{ in } \left[1 - \frac{\varepsilon}{2}, 1\right] \tag{3.11}$$

and for some $C_0 > 0$,

$$R \leq e^{-C_0 n} \text{ in } [-1, 1 - \varepsilon]. \tag{3.12}$$

For the construction of these, we may choose $R = 1 - P_n$, where P_n is the polynomial of Theorem 7.5 in [12, p. 172] with appropriate choice of parameters there. Then

$$\begin{aligned}
&\int_{-1}^{1} (1-x) P^2(x)\, d\mu(x) \\
\leq\ & e^{-2C_0 n} \sup_{x \in [-1, 1-\varepsilon]} (1-x) \left(\ell_{1m}^J(x)\right)^2 \int_{-1}^{1-\varepsilon} d\mu + (1+\eta)\, h(1) \int_{1-\varepsilon}^{1} (1-x) \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx \\
\leq\ & h(1) \int_{-1}^{1} (1-x) \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx \left\{e^{-C_0 n} + 1 + \eta\right\},
\end{aligned}$$

(3.13)

by the regularity of the Jacobi weight, which ensures [25, p. 68] that

$$\sup_{\deg(S) \leq n} \left(\|S\|_{L_\infty[-1,1]} / \int_{-1}^{1} |S|\, w^J \right)^{1/n} \leq 1 + o(1).$$

Next, by (3.10) and (3.11),

$$\begin{aligned}
\int_{-1}^{1} P^2 d\mu \quad &\geq \quad \frac{(1-n^{-2})^2}{1+\eta} h(1) \int_{1-\frac{\varepsilon}{2}}^{1} \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx \\
&\geq \quad \frac{1+o(1)}{1+\eta} h(1) \int_{-1}^{1} \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx,
\end{aligned} \tag{3.14}$$

by Lemma 3.1(c) above. Substituting this, and (3.13) into (3.9) gives

$$\begin{aligned}
&1 - x_{1n} \\
\leq\ & \left((1+\eta)^2 + o(1)\right) \int_{-1}^{1} (1-x) \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx \Big/ \int_{-1}^{1} \left(\ell_{1m}^J(x)\right)^2 w^J(x)\, dx \\
=\ & \left((1+\eta)^2 + o(1)\right) \left(1 - x_{1m}^J\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\limsup_{n \to \infty} \frac{1 - x_{1n}}{1 - x_{1n}^J} \\
\leq\ & (1+\eta)^2 \limsup_{n \to \infty} \frac{1 - x_{1m}^J}{1 - x_{1n}^J} \\
\leq\ & (1+\eta)^2 \left(\frac{1}{1-\eta}\right)^2,
\end{aligned}$$

by (b) of the lemma above, recall $m = n - [\eta n]$. As $\eta > 0$ is arbitrary, we obtain the result. ■

Next, we need:

Lemma 3.3
(a) Given $\varepsilon \in (0,1)$*, we have*

$$\lim_{n\to\infty} \int_{-1}^{1-\varepsilon} (\ell_{1n})^2 \, d\mu / \int_{-1}^{1} (\ell_{1n})^2 \, d\mu = 1. \tag{3.15}$$

(b)

$$\lim_{n\to\infty} \frac{1 - x_{1n}}{1 - x_{1n}^J} = 1. \tag{3.16}$$

Proof
(a) From the previous lemma and Lemma 3.1(b), it follows that for some $C > 0$,

$$x_{1n} \geq 1 - Cn^{-2}.$$

Then Lemma 3.1(a) gives

$$\begin{aligned}
\frac{p_{n-1}^2(x_{1n})}{K_n(x_{1n}, x_{1n})} &= 1 - \frac{K_{n-1}(x_{1n}, x_{1n})}{K_n(x_{1n}, x_{1n})} \\
&= 1 - \frac{\lambda_n(x_{1n})}{\lambda_{n-1}(x_{1n})} \\
&= 1 - \frac{\lambda_n^J(x_{1n})}{\lambda_{n-1}^J(x_{1n})}(1 + o(1)) \\
&= 1 - \frac{K_{n-1}^J(x_{1n}, x_{1n})}{K_n^J(x_{1n}, x_{1n})} + o(1) \\
&= \frac{\left(p_{n-1}^J(x_{1n})\right)^2}{K_n^J(x_{1n}, x_{1n})} + o(1) = o(1),
\end{aligned}$$

recall (3.7). Now we can repeat the argument of Lemma 3.1(c). Exactly as at (3.6), for large enough n,

$$\begin{aligned}
&\int_{-1}^{1-\varepsilon} (\ell_{1n})^2 \, d\mu / \int_{-1}^{1} (\ell_{1n})^2 \, d\mu \\
\leq\ & \frac{4p_{n-1}^2(x_{1n})}{\varepsilon^2 K_n(x_{1n}, x_{1n})},
\end{aligned} \tag{3.17}$$

and then the result follows.
(b) We have to show that

$$\limsup_{n\to\infty} \frac{1 - x_{1n}^J}{1 - x_{1n}} \leq 1. \tag{3.18}$$

Once this is established, Lemma 3.2 gives (3.16). To do this, we proceed much as in Lemma 3.2. Now

$$1 - x_{1n}^J = \inf_{\deg(P) \leq n-1} \int_{-1}^{1} (1 - x) P^2(x) w^J(x) \, dx / \int_{-1}^{1} P^2(x) w^J(x) \, dx. \tag{3.19}$$

Let $\eta \in (0,1)$ and choose $\varepsilon \in (0,\rho)$ such that (3.10) holds. Let $m = n - [\eta n]$ and choose

$$P = \ell_{1m} R,$$

where R has degree $\leq [\eta n]$, $0 \leq R \leq 1$ in $[-1,1]$, and satisfies (3.11) and (3.12). Then

$$\begin{aligned}
&\int_{-1}^{1} (1-x)\, P^2(x)\, w^J(x)\, dx \\
\leq\ & e^{-2C_0 n} \sup_{x\in[-1,1-\varepsilon]} (1-x)\left(\ell_{1m}(x)\right)^2 \int_{-1}^{1-\varepsilon} w^J + (1+\eta)\, h(1)^{-1} \int_{1-\varepsilon}^{1} (1-x)\left(\ell_{1m}(x)\right)^2 w(x)\, dx \\
\leq\ & h(1)^{-1} \int_{-1}^{1} (1-x)\left(\ell_{1m}(x)\right)^2 d\mu(x) \left\{e^{-C_0 n} + 1 + \eta\right\},
\end{aligned}$$

(3.20)

by the regularity of the measure μ, which ensures [25, p. 68] that

$$\sup_{\deg(S)\leq n} \left(\|S\|_{L_\infty[-1,1]} / \int_{-1}^{1} |S|\, d\mu \right)^{1/n} \leq 1 + o(1).$$

Also, by (3.10) and absolute continuity of μ in $[1-\varepsilon, 1]$,

$$\begin{aligned}
&\int_{-1}^{1} P^2(x)\, w^J(x)\, dx \\
\geq\ & \frac{\left(1-n^{-2}\right)^2}{1+\eta} h(1)^{-1} \int_{1-\varepsilon}^{1} \ell_{1m}^2\, d\mu \\
=\ & \frac{1+o(1)}{1+\eta} h(1)^{-1} \int_{1}^{1} \ell_{1m}^2\, d\mu,
\end{aligned}$$

by (a) of this lemma. Then substituting this and (3.20) in (3.19) gives

$$\begin{aligned}
&1 - x_{1n}^J \\
\leq\ & \left\{(1+\eta)^2 + o(1)\right\} \int_{-1}^{1} (1-x)\left(\ell_{1m}(x)\right)^2 d\mu(x) \Big/ \int_{-1}^{1} \left(\ell_{1m}(x)\right)^2 d\mu(x) \\
=\ & \left\{(1+\eta)^2 + o(1)\right\} (1 - x_{1m}).
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1-x_{1m}^J}{1-x_{1m}} \\
\leq\ & \left\{(1+\eta)^2 + o(1)\right\} \frac{1-x_{1m}^J}{1-x_{1n}^J} \\
\leq\ & \left\{(1+\eta)^2 + o(1)\right\} \left(\frac{1}{1-\eta}\right)^2,
\end{aligned}$$

by Lemma 3.1(b). Here as n runs through the positive integers, so does $m = m(n) = n - [\eta n]$. (Indeed the difference between $m(n)$ and $m(n+1)$ is at most 1). So

$$\limsup_{k\to\infty} \frac{1-x_{1k}^J}{1-x_{1k}} \leq \left(\frac{1+\eta}{1-\eta}\right)^2.$$

Since $\eta > 0$ is arbitrary, we obtain (3.18). ■

Proof of Theorem 1.2

We use the universality limit from [15]: uniformly for a, b in compact subsets of $(0, \infty)$, we have

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n \left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha (a, b). \tag{3.21}$$

Of course, we also have uniformly for such a, b

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n^J \left(1 - \frac{a}{2n^2}, 1 - \frac{b}{2n^2}\right) = \mathbb{J}_\alpha (a, b). \tag{3.22}$$

Now choose $b = b_n$ such that

$$1 - \frac{b_n}{2n^2} = x_{1n}^J.$$

Observe that by Lemma 3.1(b), as $n \to \infty$,

$$b_n = 2n^2 \left(1 - x_{1n}^J\right) \to j_{\alpha,1}^2.$$

Moreover, $\tilde{K}_n^J \left(1 - \frac{a}{2n^2}, 1 - \frac{b_n}{2n^2}\right) = \tilde{K}_n^J \left(1 - \frac{a}{2n^2}, x_{1n}^J\right)$ is a constant multiple of $\ell_{1n}^J \left(1 - \frac{a}{2n^2}\right)$, so vanishes only when

$$1 - \frac{a}{2n^2} = x_{kn}^J, \ k = 2, 3, ..., n,$$

where it has sign changes. Next, as a function of a,

$$\mathbb{J}_\alpha \left(a, j_{\alpha,1}^2\right) = \frac{J_\alpha (\sqrt{a})\, j_{\alpha,1} J_\alpha' (j_{\alpha,1})}{2\left(a - j_{\alpha,1}^2\right)}$$

vanishes only when $a = j_{\alpha,k}^2$, $k = 2, 3, ...$, where it has sign changes. In view of (3.21) and the previous lemma, also

$$\lim_{n\to\infty} \frac{1}{2n^2} \tilde{K}_n \left(1 - \frac{a}{2n^2}, x_{1n}\right) = \mathbb{J}_\alpha \left(a, j_{\alpha,1}^2\right)$$

uniformly for a in compact subsets of $(0, \infty)$. Let us write for some $a_n > 0$,

$$x_{2n} = 1 - \frac{a_n}{2n^2}.$$

Then $\tilde{K}_n \left(1 - \frac{a}{2n^2}, x_{1n}\right) \neq 0$ for $a < a_n$, and the above considerations, and the intermediate value theorem, show that

$$0 \leq \limsup_{n\to\infty} a_n \leq j_{\alpha,2}^2. \tag{3.23}$$

If some subsequence of $\{a_n\}$ converges to a number a, then as $K_n (x_{2n}, x_{1n}) = 0$, we obtain

$$0 = \mathbb{J}_\alpha \left(a, j_{\alpha,1}^2\right).$$

Thus necessarily $a = j_{\alpha,2}^2$. As this is true of every such subsequence, we have

$$\lim_{n\to\infty} a_n = j_{\alpha,2}^2$$

and hence

$$\lim_{n\to\infty} n^2 \left(1 - x_{2n}^2\right) = j_{\alpha,2}^2.$$

Repeating this argument by induction on k, and considering $\tilde{K}_n \left(1 - \frac{a}{2n^2}, x_{k-1,n}\right)$, shows that for each fixed k,

$$\lim_{n\to\infty} n^2 \left(1 - x_{kn}^2\right) = j_{\alpha,k}^2.$$

■

4. Proof of Theorems 1.4 and 1.5

This is similar to Theorem 1.1, but we provide the details. Obviously Theorem 1.4 is a special case of Theorem 1.5, so we prove the latter. In [13], we showed that

$$\lim_{n\to\infty} \frac{\tilde{K}_n^h\left(x+\frac{a}{\tilde{K}_n^h(x,x)}, x+\frac{b}{\tilde{K}_n^h(x,x)}\right)}{\tilde{K}_n^h(x,x)} = \frac{\sin\pi(a-b)}{\pi(a-b)}, \tag{4.1}$$

uniformly for $x \in J_n(\varepsilon)$ and a, b in compact subsets of the real line. Now let $x_{kn}^h \in J_n(\varepsilon)$. We need the fundamental polynomial ℓ_{kn}^h of Lagrange interpolation that satisfies

$$\ell_{kn}^h\left(x_{kn}^h\right) = 1$$

and vanishes at all other zeros of p_n^h. Recall the representation (2.3) for the fundamental polynomials. Choosing $a = 0$ and $x = x_{kn}^h$ in (4.1), we have

$$\frac{W\left(x_{kn}^h+\frac{b}{\tilde{K}_n^h\left(x_{kn}^h,x_{kn}^h\right)}\right)}{W\left(x_{kn}^h\right)}\ell_{kn}^h\left(x_{kn}^h+\frac{b}{\tilde{K}_n^h\left(x_{kn}^h,x_{kn}^h\right)}\right) = \frac{\sin\pi b}{\pi b}+o(1),$$

uniformly for b in compact subsets of the real line, and uniformly for $x_{kn}^h \in J_n(\varepsilon)$. Since $\frac{\sin\pi b}{\pi b}$ changes sign at $b = -1$, it follows that $x_{k+1,n}^h$, the zero of ℓ_{kn}^h closest on the left to x_{kn}^h, must satisfy

$$x_{k+1,n}^h = x_{kn}^h + \frac{\beta_n}{\tilde{K}_n^h\left(x_{kn}^h,x_{kn}^h\right)}$$

where $\beta_n \in (-\infty, 0)$, and

$$\liminf_{n\to\infty} \beta_n \geq -1.$$

In particular $\{\beta_n\}$ is bounded. We have to show that

$$\lim_{n\to\infty} \beta_n = -1. \tag{4.2}$$

Choose any subsequence of $\{\beta_n\}$ with some limit β, say. Necessarily $\beta \in [-1, 0]$. Since $\ell_{kn}^h\left(x_{k+1,n}^h\right) = 0$, we obtain, as $n \to \infty$ through the subsequence, that

$$\frac{\sin\pi\beta}{\pi\beta} = 0,$$

so $\beta = -1$. As this is true for any subsequence, we obtain (4.2). That in turn gives, as $n \to \infty$,

$$\left(x_{kn}^h - x_{k+1,n}^h\right)\tilde{K}_n^h\left(x_{kn}^h,x_{kn}^h\right) = -\beta_n \to 1.$$

Finally, from the asymptotics for Christoffel functions in [13, Lemma 2.2(a) and Theorem 4.1],

$$\tilde{K}_n^h\left(x_{kn}^h,x_{kn}^h\right) = \sigma_n\left(x_{kn}^h\right)(1+o(1)).$$

The fact that the zeros to the left and right of every fixed point satisfy (1.19), may be established as in the proof of Theorem 1.1(b). ■

5. Proof of Theorem 1.6

Throughout, we assume $W \in \mathcal{F}\left(C^2\right)$. The class $\mathcal{F}\left(C^2\right)$ is contained in the classes $\mathcal{F}\left(Lip\frac{1}{2}\right)$, $\mathcal{F}\left(lip\frac{1}{2}\right)$, $\mathcal{F}$ in [12], see p. 13 there. So we can apply estimates for all these classes from there. We define for $r > 0$ the square root factor

$$\rho_r(x) = \sqrt{\left|(x - a_{-r})(a_r - x)\right|},\ x \in \mathbb{R}. \tag{5.1}$$

Sometimes, instead of σ_n, we also use the density transformed to $[-1,1]$,

$$\sigma_n^*(t) = \frac{\delta_n}{n}\sigma_n\left(L_n^{[-1]}(t)\right),\ t \in [-1,1], \tag{5.2}$$

which has total mass 1. Recall that $L_n^{[-1]}(t) = \beta_n + \delta_n t$ is the linear transformation of $[-1,1]$ onto Δ_n, defined after (1.14).

Lemma 5.1
(a) Let $0 < \varepsilon < 1$. Then uniformly for $n \geq 1$ and $x \in J_n(\varepsilon)$,

$$\tilde{K}_n(x,x) = \sigma_n(x)(1 + o(1)) \sim \frac{n}{\delta_n}. \tag{5.3}$$

(b) For $n \geq 1$ and $x \in \Delta_n$,

$$\tilde{K}_n(x,x) = \lambda_n^{-1}\left(W^2, x\right)W^2(x) \leq C\frac{n}{\delta_n\sqrt{1 - L_n^2(x)}}. \tag{5.4}$$

(c) For $n \geq 1$ and $t \in (-1,1)$,

$$\sigma_n^*(t) \leq \frac{C}{\sqrt{1-t^2}}. \tag{5.5}$$

(d) For $n \geq 1$,

$$Q'(a_{\pm n}) \sim n\sqrt{\frac{T(a_{\pm n})}{\delta_n\left|a_{\pm n}\right|}}, \tag{5.6}$$

and for $x \in \Delta_n$,

$$\left|Q'(x)\right| \leq C\frac{n}{\rho_n(x)}. \tag{5.7}$$

(e) There exists $\varepsilon > 0$ such that for $n \geq 1$,

$$\frac{\delta_n T(a_{\pm n})}{\left|a_{\pm n}\right|} = O\left(n^{2-\varepsilon}\right), \tag{5.8}$$

and

$$Q(a_{\pm n}) \sim n\sqrt{\frac{\left|a_{\pm n}\right|}{\delta_n T(a_{\pm n})}} \geq Cn^{\varepsilon}. \tag{5.9}$$

(f) For $n \geq 1$ and polynomials P of degree $\leq n$,

$$\|PW\|_{L_\infty(\mathbb{R})} = \|PW\|_{L_\infty[a_{-n},a_n]}. \tag{5.10}$$

Proof
(a) The asymptotic is Theorem 1.25 in [12, p. 26]. For the second relation in (5.3), we use Theorem 5.2(b) in [12, Theorem 5.2, p. 110]: for any fixed $s \in (0,1)$,

$$\sigma_n(x) \sim \frac{n}{\rho_n(x)} \text{ in } \Delta_{sn} \tag{5.11}$$

uniformly in n, x. Since $\Delta_{sn} \supset J_n(\varepsilon)$ for some $s = s(\varepsilon) < 1$, and since

$$\rho_n(x) \geq \varepsilon \delta_n \text{ in } J_n(\varepsilon),$$

see [13, Lemma 2.1(e), (f)], we obtain the second relation in (5.3).
(b) In [12, Corollary 1.14(c), p. 20], it is shown that for all $x \in \Delta_n$,

$$\begin{aligned} & \lambda_n\left(W^2, x\right)/W^2(x) \\ \geq \quad & C\varphi_n(x) = C\frac{|x - a_{-2n}|\,|a_{2n} - x|}{n\sqrt{\left(|x - a_{-n}| + \left|a_{-n}\eta_{-n}\right|\right)\left(|x - a_n| + a_n\eta_n\right)}}, \end{aligned}$$

where

$$\eta_{\pm n} = \left(nT(a_{\pm n})\sqrt{\frac{|a_{\pm n}|}{\delta_n}}\right)^{-2/3}.$$

For $x \in [a_{-n}, a_n]$,

$$\begin{aligned} |a_{2n} - x| &= a_{2n} - a_n + a_n - x \\ &\geq C\frac{a_n}{T(a_n)} + a_n - x, \end{aligned}$$

see [12, (3.51), Lemma 3.11, p. 81]. Using [12, (3.39), Lemma 3.7, p. 76], we continue this as

$$\geq Ca_n\eta_n + |a_n - x|.$$

A similar inequality holds for $|x - a_{-2n}|$. Thus for $x \in [a_{-n}, a_n]$,

$$\begin{aligned} K_n^{-1}(x,x)\,W^2(x) &= \lambda_n\left(W^2, x\right)/W^2(x) \\ &\geq C\frac{\sqrt{\left(|x - a_{-n}| + \left|a_{-n}\eta_{-n}\right|\right)\left(|x - a_n| + a_n\eta_n\right)}}{n} \\ &\geq C\frac{\sqrt{|x - a_{-n}|\,|x - a_n|}}{n} = C\frac{\delta_n}{n}\sqrt{1 - L_n^2(x)}. \end{aligned}$$

(c) This follows directly from Theorem 6.1(b) in [12, p. 146].
(d) The first relation (5.6) is (3.17) of Lemma 3.4 in [12, p. 69]. The second relation (5.7) follows directly from Lemma 3.8(a) in [12, p. 77].
(e) The first relation (5.8) is (3.38) in Lemma 3.7 of [12, p. 76]. The second relation (5.9) is (3.18) in Lemma 3.4 of [12, p. 69], together with (5.8) above.
(f) This is classical, see for example [12, p. 95]. ■

We now prove a uniform bound on the reproducing kernel K_n in the plane:

Lemma 5.2
Let $0 < \varepsilon < 1, A > 0$. There exists C such that uniformly for $n \geq 1$, $r, s \in J_n(\varepsilon)$ and $|u|, |v| \leq A$,

$$W(r)\,W(s)\frac{\delta_n}{n}\left|K_n\left(r + iu\frac{\delta_n}{n}, s + iv\frac{\delta_n}{n}\right)\right| \leq C. \tag{5.12}$$

Proof
By Lemma 5.1(b), for all $x \in \Delta_n = [a_{-n}, a_n]$,

$$K_n(x,x)\,W^2(x) \leq C\frac{n}{\delta_n\sqrt{1 - L_n^2(x)}}.$$

By the Cauchy-Schwarz inequality, we obtain for $x, t \in \Delta_n$,

$$W^4(x) W^4(t) \left|K_n^4(x,t)\left(1-L_n^2(x)\right)\left(1-L_n^2(t)\right)\right| \leq C\left(\frac{n}{\delta_n}\right)^4. \tag{5.13}$$

Next, recall that if

$$V_n(x) = \int_{\Delta_n} \log \frac{1}{|x-t|} \sigma_n(t)\, dt,$$

then

$$V_n(x) + Q(x) = c_n,\ x \in \Delta_n. \tag{5.14}$$

Thus we may recast (5.13) as

$$e^{4(V_n(x)+V_n(t))} |S(x,t)| \leq C,\ \ x, t \in \Delta_n, \tag{5.15}$$

where

$$S(x,t) = \left(\frac{\delta_n}{n}\right)^4 e^{-8c_n} K_n^4(x,t)\left(1-L_n^2(x)\right)\left(1-L_n^2(t)\right)$$

is a polynomial in x, t of degree $\leq 4n-2$ in each variable. We now use the maximum principle for subharmonic functions. Fix t and let

$$f(z) = 4\left(V_n(z) + V_n(t)\right) + \log|S(z,t)|.$$

This function is subharmonic in $\mathbb{C} \backslash \Delta_n$, and approaches $-\infty$ as $z \to \infty$. By the maximum principle, and (5.15), we have for all complex z, and fixed $t \in \Delta_n$,

$$e^{4(V_n(z)+V_n(t))} |S(z,t)| = e^{f(z)} \leq C. \tag{5.16}$$

Repeating this argument with fixed z and now focusing on t, we see that (5.16) holds for all complex z and t. Taking 4th roots gives for all $z, t \in \mathbb{C}$, with $\operatorname{Re} z$, $\operatorname{Re} t \in \Delta_n$,

$$\begin{aligned} &\frac{\delta_n}{n} W(\operatorname{Re} z) W(\operatorname{Re} t) |K_n(z,t)| \left|1-L_n^2(z)\right|^{1/4} \left|1-L_n^2(t)\right|^{1/4} \\ \leq\ & C e^{V_n(\operatorname{Re} z) - V_n(z) + V_n(\operatorname{Re} t) - V_n(t)}. \end{aligned} \tag{5.17}$$

Now let us set $z = r + iu\frac{\delta_n}{n}$ and $t = s + iv\frac{\delta_n}{n}$, where $r, s \in J_n(\varepsilon)$ and $|u|, |v| \leq A$. We have

$$|1 \pm L_n(z)| \geq |1 \pm L_n(r)| = \left|\frac{a_{\pm n} - r}{\delta_n}\right| \geq \varepsilon,$$

with a similar inequality for $|1 \pm L_n(t)|$. Moreover,

$$\begin{aligned} & V_n(\operatorname{Re} z) - V_n(z) = V_n(r) - V_n\left(r + iu\frac{\delta_n}{n}\right) \\ =\ & \frac{1}{2}\int_{\Delta_n} \log\left[1 + \left(\frac{\delta_n}{n}\frac{u}{r-x}\right)^2\right] \sigma_n(x)\, dx \\ =\ & \frac{n}{2}\int_{-1}^{1} \log\left[1 + \left(\frac{u}{n\left(L_n(r) - \xi\right)}\right)^2\right] \sigma_n^*(\xi)\, d\xi, \end{aligned}$$

by the substitution $\xi = L_n(x)$. Using the bound Lemma 5.1(c) on σ_n^* and that $|L_n(r)| \le 1-\varepsilon$, $|u| \le A$, we can continue this as

$$\begin{aligned}
&\le Cn \sup_{|a|\le 1-\varepsilon} \int_{-1}^{1} \log\left[1+\left(\frac{A}{n(a-\xi)}\right)^2\right]\frac{d\xi}{\sqrt{1-\xi^2}} \\
&\le Cn \sup_{|a|\le 1-\varepsilon} \left\{ \begin{array}{c} \int_{-1+\varepsilon/2}^{1-\varepsilon/2} \log\left[1+\left(\frac{A}{n(a-\xi)}\right)^2\right] d\xi \\ +\log\left[1+\left(\frac{2A}{n\varepsilon}\right)^2\right]\int_{1-\frac{\varepsilon}{2}\le|\xi|<1}\frac{d\xi}{\sqrt{1-\xi^2}} \end{array}\right\} \\
&\le Cn \sup_{|a|\le 1-\varepsilon} \left\{\frac{1}{n}\int_{-\infty}^{\infty}\log\left[1+\frac{1}{\zeta^2}\right]d\zeta + \frac{C}{n^2}\right\} \le C.
\end{aligned}$$

A similar bound holds for $V_n(\operatorname{Re} t) - V_n(t)$. Then the result follows from (5.17). ■
We need one last estimate:

Lemma 5.3
Let $0<\varepsilon<1, A>0$. Uniformly for $n\ge 1$, $x\in J_n(\varepsilon)$ and $|a|\le A$,

$$W\left(x+\frac{a}{\tilde{K}_n(x,x)}\right)/W(x) = \exp\left(-a\frac{Q'(x)}{\tilde{K}_n(x,x)}\right)(1+o(1)). \tag{5.18}$$

Proof
First, if $x\in[-1,1]$, we have that Q' is bounded in $[-1,1]$ and Q is continuous there, while $\tilde{K}_n(x,x)\sim \frac{n}{\delta_n}\to\infty$, so both sides of (5.18) are $1+o(1)$. Now suppose $|x|\ge 1$. Then $\left|x+\frac{a}{\tilde{K}_n(x,x)}\right| \ge \frac{1}{2}$ for $n\ge n_0(A)$. We use a Taylor series expansion to second order:

$$\begin{aligned}
&Q\left(x+\frac{a}{\tilde{K}_n(x,x)}\right) \\
=\ &Q(x) + a\frac{Q'(x)}{\tilde{K}_n(x,x)} + \frac{a^2}{2}\frac{Q''(\xi)}{\tilde{K}_n(x,x)^2},
\end{aligned}$$

where ξ is between x and $x+\frac{a}{\tilde{K}_n(x,x)}$. Moreover, for n large enough, $|\xi|\ge\frac{1}{2}$. Here by Definition 1.3(e),

$$Q''(\xi) \le C\frac{Q'(\xi)^2}{Q(\xi)}.$$

Now if first $\xi\in\Delta_{\log n}\backslash\left[-\frac{1}{2},\frac{1}{2}\right]$, then by (5.6) and (5.8),

$$Q'(\xi)\le C(\log n)\,T(a_{\pm\log n})^{1/2} \le C(\log n)^2,$$

while $Q(\xi)$ is bounded below, so that

$$Q''(\xi)\le C(\log n)^4.$$

If instead $\xi\in J_n(\varepsilon)\backslash\Delta_{\log n}$, then by (5.7),

$$Q'(\xi)\le C\frac{n}{\sqrt{1-L_n^2(\xi)}} \le C\frac{n}{\delta_n},$$

while by (5.9),

$$Q(\xi)\ge Q(a_{\pm\log n})\ge C(\log n)^{C_0},$$

so that

$$Q''(\xi) \leq C\left(\frac{n}{\delta_n}\right)^2 (\log n)^{-2C_0}.$$

Then uniformly for the range of x, a considered, the above considerations show that

$$0 \leq \frac{a^2}{2}\frac{Q''(\xi)}{\tilde{K}_n(x,x)^2} \leq CQ''(\xi)\left(\frac{\delta_n}{n}\right)^2 = o(1).$$

We also use here that $\delta_n = O\left(n^{1/\Lambda}\right)$, where $\Lambda > 1$ is as in Definition 1.3(e) [13, Lemma 2.1(a)]. Thus we have shown that

$$\begin{aligned} & Q\left(x + \frac{a}{\tilde{K}_n(x,x)}\right) - Q(x) \\ = \quad & a\frac{Q'(x)}{\tilde{K}_n(x,x)} + o(1), \end{aligned}$$

and this is equivalent to (5.18). ■

Proof of Theorem 1.6

Since $\tilde{K}_n(x,x) \sim \frac{n}{\delta_n}$ uniformly for $x \in J_n(\varepsilon)$, we have uniformly for $|a|, |b| \leq A$,

$$\begin{aligned} & \left|K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right)\right| / K_n(x,x) \\ \sim \quad & \frac{\delta_n}{n} W^2(x)\left|K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right)\right|. \end{aligned}$$

By Lemma 5.2, this is uniformly bounded in n, a, b, x for complex a, b with $|a|, |b| \leq A$ and $x \in J_n(\varepsilon)$. Then also for this range of parameters, we have uniform boundedness of $\frac{Q'(x)}{\tilde{K}_n(x,x)}$ (recall (5.7)) and hence of

$$\exp\left(-(a+b)\frac{Q'(x)}{\tilde{K}_n(x,x)}\right) K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / K_n(x,x).$$

Note that this is an entire function of a, b. By Lemma 5.3, and then the universality limit (4.1) in the special case $h = 1$, we also have for real a, b in $[-A, A]$ and $x \in J_n(\varepsilon)$, that

$$\begin{aligned} & \exp\left(-(a+b)\frac{Q'(x)}{\tilde{K}_n(x,x)}\right) K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / K_n(x,x) \\ = \quad & (1 + o(1))\tilde{K}_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / \tilde{K}_n(x,x) \\ = \quad & \frac{\sin \pi(a-b)}{\pi(a-b)} + o(1). \end{aligned}$$

Because of the uniform boundedness in complex a, b, convergence continuation theorems imply that this convergence also takes place uniformly for complex a, b such that $|a|, |b| \leq A$. Equivalently as $\frac{Q'(x)}{\tilde{K}_n(x,x)}$ is bounded for such x, we obtain

$$\begin{aligned} & K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / K_n(x,x) \\ = \quad & \exp\left((a+b)\frac{Q'(x)}{\tilde{K}_n(x,x)}\right)\frac{\sin \pi(a-b)}{\pi(a-b)} + o(1), \end{aligned}$$

uniformly for $x \in J_n(\varepsilon)$ and complex a, b such that $|a|, |b| \le A$. We substitute in this the expansions

$$\frac{\sin(a-b)}{(a-b)} = \sum_{\ell,m=0}^{\infty} \frac{a^\ell b^m}{\ell! m!} \tau_{\ell,m},$$

and

$$\exp\left(a \frac{Q'(x)}{\tilde{K}_n(x,x)}\right) = \sum_{j=0}^{\infty} \frac{1}{j!} \left(a \frac{Q'(x)}{\tilde{K}_n(x,x)}\right)^j.$$

Gathering like powers, we obtain

$$\begin{aligned} & K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / K_n(x,x) \\ = & \sum_{r,s\ge 0} a^r b^s \sum_{\substack{j+\ell=r;\\ k+m=s}} \frac{\tau_{\ell,m}}{j!k!\ell!m!} \pi^{\ell+m} \left(\frac{Q'(x)}{\tilde{K}_n(x,x)}\right)^{j+k} + o(1) \\ (5.19) = & \sum_{r,s=0}^{\infty} \frac{a^r b^s}{r!s!} \sum_{\ell=0}^{r} \binom{r}{\ell} \sum_{m=0}^{s} \binom{s}{m} \tau_{\ell,m} \pi^{\ell+m} \left(\frac{Q'(x)}{\tilde{K}_n(x,x)}\right)^{r+s-\ell-m} + o(1). \end{aligned}$$

Next, we find a Taylor series expansion for the extreme left-hand side in (5.19). First note that

$$\begin{aligned} K_n(x+\alpha, x+\beta) &= \sum_{k=0}^{n-1} p_k(x+\alpha) p_k(x+\beta) \\ &= \sum_{k=0}^{n-1} \left(\sum_{r=0}^{\infty} \frac{p_k^{(r)}(x)}{r!} \alpha^r\right) \left(\sum_{s=0}^{\infty} \frac{p_k^{(s)}(x)}{s!} \beta^s\right) \\ &= \sum_{r,s=0}^{\infty} K_n^{(r,s)}(x,x) \frac{\alpha^r \beta^s}{r!s!}, \end{aligned}$$

recall the notation (1.5). The series all terminate, so the interchanges are justified. Hence,

$$\begin{aligned} & K_n\left(x + \frac{a}{\tilde{K}_n(x,x)}, x + \frac{b}{\tilde{K}_n(x,x)}\right) / K_n(x,x) \\ = & \sum_{r,s=0}^{\infty} \frac{K_n^{(r,s)}(x,x)}{K_n(x,x) r!s!} \left(\frac{a}{\tilde{K}_n(x,x)}\right)^r \left(\frac{b}{\tilde{K}_n(x,x)}\right)^s. \end{aligned}$$

Next, we compare this and (5.19), and recall that when a sequence of analytic functions converges uniformly, the Taylor series coefficients of functions in the sequence converge to those of the limit function. This gives for each fixed r, s, and $x \in J_n(\varepsilon)$,

$$\frac{K_n^{(r,s)}(x,x)}{K_n(x,x) \tilde{K}_n(x,x)^{r+s}} = \sum_{\ell=0}^{r} \binom{r}{\ell} \sum_{m=0}^{s} \binom{s}{m} \tau_{\ell,m} \pi^{\ell+m} \left(\frac{Q'(x)}{\tilde{K}_n(x,x)}\right)^{r+s-\ell-m} + o(1).$$

Finally, the asymptotic $\tilde{K}_n(x,x) = \sigma_n(x)(1+o(1))$ gives (1.26) for fixed x. To prove the uniformity in $x \in J_n(\varepsilon)$, we may proceed as follows: for $n \ge 1$, choose

$x_n \in J_n(\varepsilon)$. Our proof above, without any change, shows that the sequence of functions

$$\left\{K_n\left(x_n+\frac{a}{\tilde{K}_n(x_n,x_n)},x_n+\frac{b}{\tilde{K}_n(x_n,x_n)}\right)/K_n(x_n,x_n)-\exp\left((a+b)\frac{Q'(x_n)}{\tilde{K}_n(x_n,x_n)}\right)\frac{\sin\pi(a-b)}{\pi(a-b)}\right\}_{n=1}^{\infty}$$

is bounded uniformly for a,b in compact subsets of the plane, and converges uniformly for such a,b to 0. Hence individual Taylor series coefficients of the sequence also converge to 0. As this is true for any sequence $\{x_n\}$ in $J_n(\varepsilon)$, the uniformity in x follows. ■

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