

UNIVERSALITY IN THE BULK HOLDS CLOSE TO GIVEN POINTS

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ABSTRACT. Let μ be a measure with compact support. Assume that ξ is a Lebesgue point of μ and that μ' is positive and continuous at ξ . Let $\{A_n\}$ be a sequence of positive numbers with limit ∞ . We show that one can choose $\xi_n \in [\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n}]$ such that

$$\lim_{n \rightarrow \infty} \frac{K_n \left(\xi_n, \xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n(\xi_n, \xi_n)} = \frac{\sin \pi a}{\pi a},$$

uniformly for a in compact subsets of the plane. Here K_n is the n th reproducing kernel for μ , and \tilde{K}_n is its normalized cousin. Thus universality in the bulk holds on a sequence close to ξ , without having to assume that μ is a regular measure. Similar results are established for sequences of measures.

1. INTRODUCTION¹

Although they have much older roots, the theory of random matrices rose to prominence in the 1950's, when Wigner found them an indispensable tool in analysing scattering theory for neutrons off heavy nuclei. The mathematical context of the unitary case may be briefly described as follows. Let $\mathcal{M}(n)$ denote the space of n by n Hermitian matrices $M = (m_{ij})_{1 \leq i, j \leq n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$\begin{aligned} P^{(n)}(M) &= cw(M) dM \\ &= cw(M) \left(\prod_{j=1}^n dm_{jj} \right) \left(\prod_{j < k} d(\operatorname{Re} m_{jk}) d(\operatorname{Im} m_{jk}) \right). \end{aligned}$$

Here $w(M)$ is a function defined on $\mathcal{M}(n)$, and c is a normalizing constant. The most important case is

$$w(M) = \exp(-2n \operatorname{tr} Q(M)),$$

involving the trace tr , for appropriate functions Q defined on $\mathcal{M}(n)$. In particular, the choice

$$Q(M) = M^2,$$

leads to the Gaussian unitary ensemble (apart from scaling) that was considered by Wigner. One may identify $P^{(n)}$ above with a probability density

Date: August 28, 2009.

¹Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353

on the eigenvalues $x_1 \leq x_2 \leq \dots \leq x_n$ of M ,

$$P^{(n)}(x_1, x_2, \dots, x_n) = c \left(\prod_{j=1}^n w(x_j) \right) \left(\prod_{i < j} (x_i - x_j)^2 \right).$$

See [3, p. 102 ff.]. Again, c is a normalizing constant.

It is at this stage that orthogonal polynomials arise [3], [14]. Let μ be a finite positive Borel measure with compact support and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \dots, \gamma_n > 0,$$

$n = 0, 1, 2, \dots$, satisfying the orthonormality conditions

$$\int p_j p_k d\mu = \delta_{jk}.$$

Throughout we use μ' to denote the Radon-Nikodym derivative of μ . The n th reproducing kernel for μ is

$$(1.1) \quad K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

and the normalized kernel is

$$(1.2) \quad \tilde{K}_n(x, y) = \mu'(x)^{1/2} \mu'(y)^{1/2} K_n(x, y).$$

When

$$\mu'(x) = e^{-2nQ(x)} dx,$$

there is the basic formula for the probability distribution $P^{(n)}$ [3, p.112]:

$$P^{(n)}(x_1, x_2, \dots, x_n) = \frac{1}{n!} \det \left(\tilde{K}_n(x_i, x_j) \right)_{1 \leq i, j \leq n}.$$

One may use this to compute a host of statistical quantities - for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One particularly important quantity is the m -point correlation function for $M(n)$ [3, p. 112]:

$$\begin{aligned} R_m(x_1, x_2, \dots, x_m) &= \frac{n!}{(n-m)!} \int \dots \int P^{(n)}(x_1, x_2, \dots, x_n) dx_{m+1} dx_{m+2} \dots dx_n \\ &= \det \left(\tilde{K}_n(x_i, x_j) \right)_{1 \leq i, j \leq m}. \end{aligned}$$

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, and ξ in the interior of the support of $\{\mu\}$, and real a_1, a_2, \dots, a_m , we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{1}{\tilde{K}_n(\xi, \xi)^m} R_m \left(\xi + \frac{a_1}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi, \xi)}, \dots, \xi + \frac{a_m}{\tilde{K}_n(\xi, \xi)} \right) \\ &= \det \left(\frac{\sin \pi (a_i - a_j)}{\pi (a_i - a_j)} \right)_{1 \leq i, j \leq m}. \end{aligned}$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin \pi(a_i - a_j)}{\pi(a_i - a_j)}$ as 1. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{\tilde{K}_n \left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{b}{\tilde{K}_n(\xi, \xi)} \right)}{\tilde{K}_n(\xi, \xi)} = \frac{\sin \pi(a - b)}{\pi(a - b)}.$$

Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. The term universal is quite justified: the limit on the right-hand side of (1.3) is independent of ξ , but more importantly is independent of the underlying measure. It is noteworthy that there are several other contexts in which this same universality limit arises: the orthogonal and symplectic cases [4], and the rather different context of random matrices with independently distributed entries [19], [22]

Typically, the limit (1.3) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n , we can also establish its validity for complex a, b , that is,

$$(1.4) \quad \lim_{n \rightarrow \infty} \frac{K_n \left(\xi + \frac{a}{K_n(\xi, \xi)}, \xi + \frac{b}{K_n(\xi, \xi)} \right)}{K_n(\xi, \xi)} = \frac{\sin \pi(a - b)}{\pi(a - b)}.$$

There are a variety of methods to establish (1.4). Perhaps the deepest methods are the Riemann-Hilbert methods, which yield far more than universality. See [2], [3], [4], [8], [12], [13], [16] for references.

Inspired by the 60th birthday conference for Percy Deift, the author came up with a new comparison method to establish universality. Let μ be a measure supported on $(-1, 1)$, that is regular in the sense of Stahl, Ullmann and Totik [20], so that

$$\lim_{n \rightarrow \infty} \gamma_n^{1/n} = 2.$$

Regularity is a weak global condition, that is satisfied if $\mu' > 0$ a.e. in the support of μ . Let μ be absolutely continuous in a neighborhood of some given $\xi \in (-1, 1)$ and assume that μ' is positive and continuous at ξ . Then [10] we established (1.4).

This result was soon extended to a far more general setting by Findley, Simon and Totik [5], [17], [23]. In particular, when μ is a measure with compact support that is regular, and $\log \mu'$ is integrable in a subinterval of the support (c, d) , then Totik established that the universality (1.4) holds a.e. in (c, d) . Totik used the method of polynomial pullbacks to go first from one to finitely many intervals, and then used the latter to approximate general compact sets. In contrast, Simon used the theory of Jost functions.

The drawback of this comparison method is that it requires regularity of the measure μ . Although the latter is a weak global condition, it is nevertheless most probably an unnecessary restriction. To circumvent this, the author developed a different method, based on classical complex analysis such as normal families, and the theory of entire functions of exponential

type. Here is a typical result: let μ be a measure with compact support, and assume that μ' is absolutely continuous near ξ , while μ' is bounded above and below by positive constants in that neighborhood. Then the universality (1.4) is equivalent to *universality along the diagonal*, that is, for all real a ,

$$(1.5) \quad \lim_{n \rightarrow \infty} \frac{K_n \left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{a}{\tilde{K}_n(\xi, \xi)} \right)}{K_n(\xi, \xi)} = 1.$$

Because $\tilde{K}_n(\xi, \xi)$ grows roughly like n in this context, we may also reformulate this as

$$(1.6) \quad \lim_{n \rightarrow \infty} \frac{K_n \left(\xi + \frac{a}{n}, \xi + \frac{a}{n} \right)}{K_n(\xi, \xi)} = 1,$$

yet in this passage, we need uniformity for a in compact sets. By contrast, we can allow (1.5) to hold just for a sequence of values of a with a finite limit point.

The equivalence of (1.4) and (1.6) is useful because it is far easier to analyze the Christoffel function

$$\lambda_n(x) = \frac{1}{K_n(x, x)}$$

than the general kernel $K_n(x, y)$. Indeed, there is the classical extremum property

$$\lambda_n(x) = \inf_{\deg(P) \leq n-1} \frac{\int P^2 d\mu}{P^2(x)},$$

which permits comparison of the Christoffel function for different measures. Unfortunately, so far this equivalence has not led to an explicit extension of the results of Simon, Totik and Findley. Primarily, this is because there is no known method to estimate the ratio in the left-hand side of (1.5) or (1.6) that does not first give limits for the Christoffel functions, and all known methods for the latter require regularity of the measure. However, the method has been useful in other contexts [1], [9], [18].

In this paper, we shall show that if we weaken the formulation a little, then we can establish universality for varying points close to a given point, without assuming regularity. Here is a typical result: we shall let μ^s denote the singular part of a measure μ .

Theorem 1.1

Let μ be a measure with compact support. Assume that ξ lies in that support, and

$$(1.7) \quad \lim_{h \rightarrow 0^+} \frac{1}{2h} \int_{\xi-h}^{\xi+h} d\mu^s = 0$$

while μ' is continuous at ξ , and

$$(1.8) \quad \mu'(\xi) > 0.$$

Let $\{A_n\}$ be a sequence of positive numbers with

$$(1.9) \quad \lim_{n \rightarrow \infty} A_n = \infty.$$

We can choose $\xi_n \in [\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n}]$ such that

$$(1.10) \quad \lim_{n \rightarrow \infty} \frac{K_n \left(\xi_n, \xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n(\xi_n, \xi_n)} = \frac{\sin \pi a}{\pi a},$$

uniformly for a in compact subsets of the plane.

Remarks

(a) Note that (1.7) is a Lebesgue point type condition on the singular part of μ . We need continuity of μ' at ξ , rather than a Lebesgue point type condition, because of the need to vary ξ_n close to ξ .

(b) Essentially, ξ_n is chosen so as to maximize $K_n(t, t)$ in $[\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n}]$.

(c) We have not shown the full limit (1.4) with both parameters a, b . Our proof actually shows that every subsequential limit of the normal family

$\left\{ \frac{K_n \left(\xi + \frac{a}{\tilde{K}_n(\xi, \xi)}, \xi + \frac{b}{\tilde{K}_n(\xi, \xi)} \right)}{K_n(\xi, \xi)} \right\}$ has the form

$$\sum_{j=-\infty}^{\infty} \alpha_j \frac{\sin \pi (a - j)}{\pi (a - j)} \frac{\sin \pi (b - j)}{\pi (b - j)}$$

where $\{\alpha_j\}$ is a bounded sequence with $\alpha_0 = 1$. See Lemma 3.5 for a precise statement.

We shall also prove a generalization of Theorem 1.1 for sequences of measures. For $n \geq 1$, let μ_n be a measure with support on the real line, and with at least the first $2n$ power moments finite. Let K_n denote the n th reproducing kernel corresponding to μ_n , so that

$$\int K_n(x, t) P(t) d\mu_n(t) = P(x)$$

for all polynomials P of degree $\leq n - 1$, and all x . Then under appropriate bounds on K_n , we have a similar result:

Theorem 1.2

For $n \geq 1$, let μ_n be a measure with support on the real line, for which the power moments $\int x^j d\mu_n(x)$, $0 \leq j \leq 2n - 2$, are finite. Let K_n denote the n th reproducing kernel for the measure μ_n , and \tilde{K}_n its normalized cousin. Let $\{A_n\}$ be a sequence of real numbers with limit ∞ . Assume that there exist $C_1, C_2, C_3, C_4, C_5 > 0$ with the following properties: given $A > 0$, there exists n_0 such that for $n \geq n_0$ and $|z|, |v| \leq A$,

$$(1.11) \quad \left| K_n \left(\xi + \frac{v}{n}, \xi + \frac{z}{n} \right) \right| \leq C_1 n e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)},$$

for $x \in [-A, A]$,

$$(1.12) \quad K_n \left(\xi + \frac{x}{n}, \xi + \frac{x}{n} \right) \geq C_3 n;$$

for $x \in [-A, A]$,

$$(1.13) \quad C_4 \leq \mu'_n \left(\xi + \frac{x}{n} \right) \leq C_5;$$

$$(1.14) \quad \lim_{n \rightarrow \infty} n \int_{\xi - \frac{A}{n}}^{\xi + \frac{A}{n}} d\mu_n^s = 0;$$

uniformly for $x \in [\xi - \frac{A}{n}, \xi + \frac{A}{n}]$,

$$(1.15) \quad \lim_{n \rightarrow \infty} n \int_{-\frac{A}{n}}^{\frac{A}{n}} |\mu'_n(x+t) - \mu'_n(x)| dt = 0.$$

Then we can choose $\xi_n \in [\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n}]$ such that (1.10) holds.

In the sequel, C, C_1, C_2, \dots denote positive constants independent of n, x, t, z and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. For $x \geq 0$, we let $[x]$ denote the greatest integer $\leq x$. For sequences $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that for all n ,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions, and sequences of functions.

This paper is organised as follows: in Section 2, we present the ideas of proof. In Section 3, we prove the results.

2. THE IDEAS OF PROOF

Most of the ideas of proof come from [11]. However, the details are sufficiently different to require proof, and we aim to keep this paper self contained. In this section, we shall give the ideas of proof of Theorem 1.2. Let us assume its hypotheses.

Step 1: Define a normal family

We can assume that the sequence $\{A_n\}$ with limit ∞ grows as slowly as we please. Having determined a sufficiently slow growth, choose $\xi_n \in [\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n}]$ such that

$$K_n(\xi_n, \xi_n) = \max \left\{ K_n(t, t) : t \in \left[\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n} \right] \right\}.$$

Then let

$$(2.1) \quad f_n(z, v) = \frac{K_n \left(\xi_n + \frac{z}{\bar{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{v}{\bar{K}_n(\xi_n, \xi_n)} \right)}{K_n(\xi_n, \xi_n)}.$$

We use the bounds (1.11) and (1.12) to show that for $n \geq 1$ and $|z|, |v| \leq A_n$,

$$(2.2) \quad |f_n(v, z)| \leq C_1 e^{C_2(|\operatorname{Im} v| + |\operatorname{Im} z|)}.$$

Thus $\{f_n\}$ is a normal family in each variable. Let $f(z, v)$ denote the limit of some subsequence $\{f_n\}_{n \in \mathcal{S}}$. It is entire of exponential type in each variable z and v .

Step 2: Finer bounds for f on a half-line

We use the choice of ξ_n to show that for all x in at least one of the intervals $(-\infty, 0]$ or $[0, \infty)$, we have

$$f(x, x) \leq 1.$$

Step 3: Some basic inequalities

If σ is the exponential type of $f(a, \cdot)$, we can show that σ is independent of a , using interlacing properties of zeros of K_n . From elementary properties of the reproducing kernel K_n , and scaling, and taking limits, we can show that for all $a \in \mathbb{C}$,

$$(2.3) \quad \int_{-\infty}^{\infty} |f(a, s)|^2 ds \leq f(a, \bar{a}).$$

Using the fact that $\frac{\sin \sigma(s-t)}{\pi(s-t)}$ is a reproducing kernel for the Paley-Wiener space, consisting of all entire functions of exponential type $\leq \sigma$ that are also square integrable on the real line, we show that

$$(2.4) \quad \int_{\mathbb{R}} \left(f(a, s) - \frac{\sin \sigma(s-a)}{\sigma(s-a)} \right)^2 ds \leq \frac{\sigma}{\pi} - f(a, a).$$

From this we deduce

$$(2.5) \quad \sigma \geq \pi \sup_{x \in \mathbb{R}} f(x, x) \geq \pi.$$

Step 4 Use of the Markov-Stieltjes Inequalities

For the converse inequality to (2.5), we use Markov-Stieltjes inequalities, and a classical formula relating exponential type of entire functions and their zero distribution, to obtain

$$\sigma \leq \pi \sup_{x \in [0, \infty)} f(x, x)$$

and also

$$\sigma \leq \pi \sup_{x \in (-\infty, 0]} f(x, x).$$

Together with Step 2, this gives the upper bound $\sigma \leq \pi$, and hence $\sigma = \pi$. Then (2.4) becomes

$$(2.6) \quad \int_{\mathbb{R}} \left(f(a, s) - \frac{\sin \pi(s-a)}{\pi(s-a)} \right)^2 ds \leq 1 - f(a, a).$$

Setting $a = 0$, and using $f(0, 0) = \lim_{n \rightarrow \infty, n \in \mathcal{S}} f_n(0, 0) = 1$, gives

$$f(0, s) = \frac{\sin \pi s}{\pi s}.$$

Since the right-hand side is independent of the subsequence, we obtain Theorem 1.2, and then Theorem 1.1 follows easily. The details are presented in the next two sections.

3. PROOF OF THEOREM 1.2

Throughout this section, we assume the hypotheses of Theorem 1.2.

Lemma 3.1

(a) *There exist C_1, C_2, C_3, C_4, C_5 and an increasing sequence $\{A_n^*\}$ with limit ∞ such that for $|z|, |v| \leq A_n^*$ and $x \in [-A_n^*, A_n^*]$,*

$$(3.1) \quad \left| K_n \left(\xi + \frac{v}{n}, \xi + \frac{z}{n} \right) \right| \leq C_1 n e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)},$$

for $x \in [-A_n^*, A_n^*]$,

$$(3.2) \quad K_n \left(\xi + \frac{x}{n}, \xi + \frac{x}{n} \right) \geq C_3 n;$$

$$(3.3) \quad C_4 \leq \mu_n' \left(\xi + \frac{x}{n} \right) \leq C_5;$$

$$(3.4) \quad \lim_{n \rightarrow \infty} n \int_{\xi - \frac{A_n^*}{n}}^{\xi + \frac{A_n^*}{n}} d\mu_n^s = 0;$$

uniformly for $x \in \left[\xi - \frac{A_n^*}{n}, \xi + \frac{A_n^*}{n} \right]$,

$$(3.5) \quad \lim_{n \rightarrow \infty} n \int_{-\frac{A_n^*}{n}}^{\frac{A_n^*}{n}} |\mu_n'(x+t) - \mu_n'(x)| dt = 0.$$

(b) *There exists an increasing sequence $\{A_n^\#\}$ with $A_n^\# \leq \min\{A_n, A_n^*\}$ and with limit ∞ such that for f_n defined by (2.1) and $|z|, |v| \leq A_n^\#$,*

$$(3.6) \quad |f_n(z, v)| \leq C_1 e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)}.$$

(c) *Let f be the limit of a subsequence $\{f_n\}_{n \in \mathcal{S}}$. Then f is entire of exponential type in each variable, and for all complex z, v ,*

$$(3.7) \quad |f(z, v)| \leq C_1 e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)}.$$

(d) *For $t \in (-\infty, 0]$ or $t \in [0, \infty)$, or both,*

$$(3.8) \quad f(t, t) \leq 1.$$

Moreover, for some $C_1 > 1$ and all $t \in \mathbb{R}$,

$$(3.9) \quad C_1^{-1} \leq f(t, t) \leq C_1.$$

Proof

(a) This follows easily from our hypotheses (1.11) - (1.15).

(b) Our bounds (3.1) - (3.3) show that for some $C_0 > 1$, and $x \in [-A_n^*, A_n^*]$,

$$(3.10) \quad C_0 n \geq \tilde{K}_n \left(\xi + \frac{x}{n}, \xi + \frac{x}{n} \right) \geq C_0^{-1} n.$$

Then for $|z|, |v| \leq A_n^*/C_0$, and $x \in [-A_n^*, A_n^*]$,

$$(3.11) \quad \left| K_n \left(\xi + \frac{v}{\tilde{K}_n \left(\xi + \frac{x}{n}, \xi + \frac{x}{n} \right)}, \xi + \frac{z}{\tilde{K}_n \left(\xi + \frac{x}{n}, \xi + \frac{x}{n} \right)} \right) \right| \leq C_1 n e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)}.$$

Of course, we have a new C_1 and C_2 on the right. Now define $A_n^\#$ by

$$A_n^\# = \min \{ A_n/2, A_n^*/(2C_0^2) \},$$

where C_0 is as in (3.10). Then choose $\xi_n \in \left[\xi - \frac{A_n^\#}{n}, \xi + \frac{A_n^\#}{n} \right] \subset \left[\xi - \frac{A_n}{n}, \xi + \frac{A_n}{n} \right]$ such that

$$K_n(\xi_n, \xi_n) = \max \left\{ K_n(t, t) : t \in \left[\xi - \frac{A_n^\#}{n}, \xi + \frac{A_n^\#}{n} \right] \right\}.$$

Let $|z|, |v| \leq A_n^\#$, and write

$$\begin{aligned} \xi_n + \frac{z}{\tilde{K}_n(\xi_n, \xi_n)} &= \xi + \frac{z_1}{\tilde{K}_n(\xi_n, \xi_n)}; \\ \xi_n + \frac{v}{\tilde{K}_n(\xi_n, \xi_n)} &= \xi + \frac{v_1}{\tilde{K}_n(\xi_n, \xi_n)}. \end{aligned}$$

Here

$$\begin{aligned} |z_1| &= \left| z + (\xi_n - \xi) \tilde{K}_n(\xi_n, \xi_n) \right| \\ &\leq A_n^\# + \frac{A_n^\#}{n} C n \\ &\leq A_n^*/(2C_0^2) + A_n^*/(2C_0) \leq A_n^*/C_0. \end{aligned}$$

A similar estimate holds for v_1 . Then our bounds (3.10) and (3.11) give

$$\begin{aligned} |f_n(z, v)| &= \left| \frac{K_n \left(\xi + \frac{z_1}{\tilde{K}_n(\xi_n, \xi_n)}, \xi + \frac{v_1}{\tilde{K}_n(\xi_n, \xi_n)} \right)}{K_n(\xi_n, \xi_n)} \right| \\ &\leq C_1 e^{C_2(|\operatorname{Im} z_1| + |\operatorname{Im} v_1|)} \\ &= C_1 e^{C_2(|\operatorname{Im} z| + |\operatorname{Im} v|)}. \end{aligned}$$

(c) This follows directly from (b), which in particular, shows that $\{f_n\}$ is a normal family.

(d) Let us suppose $\xi_n \geq \xi$. The case $\xi_n < \xi$ is similar. We shall show that

$$(3.12) \quad t \in \left[-\frac{A_n^\#}{C}, 0 \right] \Rightarrow f_n(t, t) \leq 1.$$

Choose such a t and write

$$\xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} = \xi + \frac{t_1}{n},$$

so that

$$t_1 = t \frac{n}{\tilde{K}_n(\xi_n, \xi_n)} + n(\xi_n - \xi) \leq 0 + A_n^\#,$$

and

$$t_1 \geq -\frac{A_n^\#}{C}C + 0 = -A_n^\#.$$

Then

$$f_n(t, t) = \frac{K_n\left(\xi + \frac{t_1}{n}, \xi + \frac{t_1}{n}\right)}{K_n(\xi_n, \xi_n)} \leq 1$$

by choice of ξ_n . So we have (3.12). Using that, or its alternative, and passing to our subsequence, gives $f(t, t) \leq 1$ in at least one of $(-\infty, 0]$, $[0, \infty)$. The bound (3.9) on the whole real line follows easily from (3.1) and (3.2). ■

In the sequel, f denotes the subsequential limit from the above lemma. It follows from the bound (3.7) that for each real a , $f(a, \cdot)$ is entire of exponential type $\sigma_a \geq 0$, say. We first show that σ_a is independent of a . We denote the zeros of p_n by $\{x_{jn}\}_{j=1}^n$, ordered in decreasing size.

Lemma 3.2

For $a \in \mathbb{R}$, let $n(f(a, \cdot), [c, d])$ denote the the number of zeros of $f(a, \cdot)$ in $[c, d]$, counting multiplicity.

(a) Then for any real a , we have for $r > 0$,

$$(3.13) \quad |n(f(a, \cdot), [0, r]) - n(f(0, \cdot), [0, r])| \leq 4.$$

The same assertion holds for $[-r, 0]$.

(b) For all real a ,

$$(3.14) \quad \sigma_a = \sigma_0 = \sigma, \text{ say.}$$

Proof

(a) We use a basic property of

$$L_n(t, \xi) = (t - \xi) K_n(t, \xi) = \left(\frac{\gamma_{n-1}}{\gamma_n} \right) (p_n(t) p_{n-1}(\xi) - p_{n-1}(t) p_n(\xi)).$$

For real ξ , with $p_{n-1}(\xi) p_n(\xi) \neq 0$, $L_n(\xi, t)$ has, as a function of t , simple zeros in each of the $n - 1$ intervals

$$(x_{nn}, x_{n-1,n}), (x_{n-1,n}, x_{n-2,n}), \dots, (x_{2n}, x_{1n}).$$

There is a single remaining zero, and this lies outside $[x_{nn}, x_{1n}]$ [6, proof of Theorem 3.1, p. 19]. When $p_{n-1}(\xi) p_n(\xi) = 0$, $L_n(\xi, t)$ is a multiple of p_n

or p_{n-1} . As the zeros of the latter polynomials interlace, we see that in this case, there is a simple zero in each of the intervals

$$[x_{nn}, x_{n-1,n}), [x_{n-1,n}, x_{n-2,n}), \dots, [x_{2n}, x_{1n}).$$

Again, see [6, proof of Theorem 3.1, p. 19]. It follows that whatever is ξ , the number j of zeros of $K_n(t, \xi)$ in $[x_{mn}, x_{kn}]$ satisfies

$$|j - (m - k)| \leq 1.$$

Consider now

$$f_n(a, t) = K_n \left(\xi_n + \frac{a}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} \right) / K_n(\xi_n, \xi_n)$$

and

$$f_n(0, t) = K_n \left(\xi_n, \xi_n + \frac{t}{\tilde{K}_n(\xi_n, \xi_n)} \right) / K_n(\xi_n, \xi_n)$$

as functions of t . In any fixed interval $[0, r]$, it follows that the difference between the number of zeros of these two functions is at most 2. Letting $n \rightarrow \infty$ through \mathcal{S} , we see that (3.13) holds. Indeed, as $f(a, z)$ has only real zeros, the same must be true of $f(a, \cdot)$ and Hurwitz' Theorem gives the result.

(b) Recall that an entire function g belongs to the *Cartwright class* if it is of exponential type $\sigma(g) \geq 0$ and

$$(3.15) \quad \int_{-\infty}^{\infty} \frac{\log^+ |g(t)|}{1+t^2} dt < \infty.$$

Here $\log^+ s = \max\{0, \log s\}$. For such functions, that are real on the real axis, and have all real zeros, it is known that

$$(3.16) \quad \lim_{r \rightarrow \infty} \frac{n(g, [0, r])}{r} = \lim_{r \rightarrow \infty} \frac{n(g, [-r, 0])}{r} = \frac{\sigma(g)}{\pi}.$$

See [7, p. 66]. Applying this to $f(a, \cdot)$ gives the result: recall that $f(a, \cdot)$ is bounded on the real axis, so trivially lies in the Cartwright class. ■

Lemma 3.3

(a) For all complex u ,

$$(3.17) \quad \int_{-\infty}^{\infty} |f(u, s)|^2 ds \leq f(u, \bar{u}).$$

(b) For all $a \in \mathbb{R}$,

$$(3.18) \quad \int_{-\infty}^{\infty} \left(f(a, s) - \frac{\sin \sigma(a-s)}{\pi(a-s)} \right)^2 ds \leq \frac{\sigma}{\pi} - f(a, a).$$

(c)

$$(3.19) \quad \sigma \geq \pi \sup_{a \in \mathbb{R}} f(a, a) \geq \pi.$$

Proof

(a) Let

$$(3.20) \quad v = \xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}.$$

There is the identity

$$K_n(v, \bar{v}) = \int |K_n(v, t)|^2 d\mu(t).$$

Let $r > 0$. We drop most of the integral:

(3.21)

$$\begin{aligned} 1 &\geq \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(v, t)|^2}{K_n(v, \bar{v})} \mu'_n(t) dt \\ &= \mu'_n(\xi_n) \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(v, t)|^2}{K_n(v, \bar{v})} dt + \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} \frac{|K_n(v, t)|^2}{K_n(v, \bar{v})} (\mu'_n(t) - \mu'_n(\xi_n)) dt \\ &=: I_1 + I_2. \end{aligned}$$

Here by Cauchy-Schwarz and the upper bound (3.1), for $t \in \left[\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}\right]$

$$\frac{|K_n(v, t)|^2}{K_n(v, \bar{v})} \leq K_n(t, t) \leq Cn.$$

Moreover, by (3.3), $\mu'_n(\xi_n) \geq C_4$. Then

$$\begin{aligned} |I_2| &\leq Cn \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} |\mu'_n(t) - \mu'_n(\xi_n)| dt \\ &\leq Cr \frac{\tilde{K}_n(\xi_n, \xi_n)}{r} \int_{\xi_n - \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}}^{\xi_n + \frac{r}{\tilde{K}_n(\xi_n, \xi_n)}} |\mu'_n(t) - \mu'_n(\xi_n)| dt \\ &\rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

by (3.5). Next, the substitution $t = \xi_n + \frac{y}{\tilde{K}_n(\xi_n, \xi_n)}$ gives

$$\begin{aligned} I_1 &= \int_{-r}^r \left| \frac{K_n\left(\xi_n + \frac{u}{\tilde{K}_n(\xi_n, \xi_n)}, \xi_n + \frac{y}{\tilde{K}_n(\xi_n, \xi_n)}\right)}{K_n(\xi_n, \xi_n)} \right|^2 \frac{K_n(\xi_n, \xi_n)}{K_n(v, \bar{v})} dy \\ &= \int_{-r}^r |f_n(u, y)|^2 \frac{dy}{f_n(u, \bar{u})}. \end{aligned}$$

As $n \rightarrow \infty$ through \mathcal{S} , the last right-hand side has \liminf at least

$$\int_{-r}^r \frac{|f(u, y)|^2}{f(u, \bar{u})} dy,$$

by Fatou's Lemma. Substituting into (3.21) gives

$$1 \geq \int_{-r}^r \frac{|f(u, y)|^2}{f(u, \bar{u})} dy.$$

Now let $r \rightarrow \infty$.

(a) The left-hand side in (3.18) equals

$$(3.22) \quad \int_{-\infty}^{\infty} f(a, s)^2 ds - 2 \int_{-\infty}^{\infty} f(a, s) \frac{\sin \sigma(a-s)}{\pi(a-s)} ds + \int_{-\infty}^{\infty} \left(\frac{\sin \sigma(a-s)}{\pi(a-s)} \right)^2 ds.$$

We showed in (b) that the first term is bounded by $f(a, a)$. Since $f(a, \cdot)$ is of exponential type $\leq \sigma$, and square integrable on the real line, it belongs to the classical Paley-Wiener space PW_σ , the set of all such functions satisfying these last two conditions. Moreover, $\frac{\sin \sigma(a-s)}{\pi(a-s)}$ is the reproducing kernel for this classical Paley-Wiener space [21, Cor. 1.10.5, p. 95]. Hence the second term equals

$$-2f(a, a).$$

Finally, this same reproducing kernel relation applied to the third term shows that it equals $\frac{\sigma}{\pi}$.

(c) Since the left-hand side of (3.18) is nonnegative, we obtain for all real a ,

$$\sigma \geq \pi f(a, a).$$

As $f(0, 0) = 1$, we then obtain (3.19). ■

Recall the Gauss type quadrature formula, with nodes $\{t_{jn}\}$ including the point ξ_n :

$$\sum_j \lambda_n(t_{jn}) P(t_{jn}) = \int P(t) d\mu(t),$$

for all polynomials P of degree $\leq 2n - 2$ [6, Theorem 3.2, p. 21]. The $\{t_{jn}\}$ are the zeros of $L_n(t, \xi_n) = (t - \xi_n) K_n(t, \xi_n)$, and moreover, if $j \neq k$, $K_n(t_{jn}, t_{kn}) = 0$. Recall too that λ_n is the n th Christoffel function for μ_n ,

$$\lambda_n(t_{jn}) = \frac{1}{K_n(t_{jn}, t_{jn})}.$$

Let us order the nodes as

$$\dots < t_{-2,n} < t_{-1,n} < t_{0,n} = \xi_n < t_{1,n} < t_{2,n} < \dots$$

and write

$$(3.23) \quad t_{jn} = \xi_n + \frac{\rho_{jn}}{\tilde{K}_n(\xi_n, \xi_n)} \Leftrightarrow \rho_{jn} = \tilde{K}_n(\xi_n, \xi_n) (t_{jn} - \xi_n).$$

Lemma 3.4

(a) For each fixed j , as $n \rightarrow \infty$ through \mathcal{S} ,

$$(3.24) \quad \rho_{jn} \rightarrow \rho_j,$$

where $\rho_0 = 0$ and

$$\dots \leq \rho_{-2} \leq \rho_{-1} < 0 < \rho_1 \leq \rho_2 \leq \dots .$$

(b) The function $f(0, z)$ has (possibly multiple) zeros at $\rho_j, j \neq 0$, and no other zeros.

(c)

$$(3.25) \quad \sup_{x \in \mathbb{R}} f(x, x) = 1.$$

Moreover, for each real a , $f(a, \cdot)$ is entire of exponential type $\sigma = \pi$.

Proof

(a), (b) We know that $f_n(0, z) = K_n\left(\xi_n, \xi_n + \frac{z}{\bar{K}_n(\xi_n, \xi_n)}\right) / K(\xi_n, \xi_n)$ has simple zeros at $\rho_{jn}, j \neq 0$, and no other zeros. Moreover as $n \rightarrow \infty$ through our subsequence, this sequence converges to $f(0, z)$, uniformly for z in compact sets, and $f(0, z)$ is not identically 0. The result then follows by Hurwitz' theorem, provided we actually know that $f(0, z)$ has infinitely many positive and negative zeros. For then, necessarily the smallest positive zero of $f_n(0, \cdot)$ must converge to the smallest positive zero of $f(0, \cdot)$, and so on. To show the existence of infinitely many zeros, we recall from Lemma 3.3 that the exponential type of $f(0, \cdot)$ is $\sigma \geq \pi > 0$, and then (3.16) gives the result. (c) We already know that $f(0, \cdot)$ is entire of exponential type $\sigma \geq \pi$. We also know from Lemma 3.1, that in one of the half-lines containing 0, that $f(\cdot, \cdot) \leq 1$. Let us assume that

$$f(t, t) \leq 1 \text{ for } t \in [0, \infty).$$

Let us consider the zero distribution of $f(0, \cdot)$, using the Markov-Stieltjes inequalities [6, p. 33]: for each $1 \leq k \leq \ell \leq n$,

$$\sum_{j=k+1}^{\ell-1} \lambda_n(t_{jn}) \leq \int_{t_{kn}}^{t_{\ell n}} d\mu(t) \leq \sum_{j=k}^{\ell} \lambda_n(t_{jn}).$$

Now assume that $t_{\ell n}, t_{kn}$ lie in $\left[\xi_n - \frac{A_n^\#}{n}, \xi_n + \frac{A_n^\#}{n}\right]$. Then by the substitution $t = \xi_n + \frac{s}{\bar{K}_n(\xi_n, \xi_n)}$, we obtain

$$\sum_{j=k+1}^{\ell-1} \frac{K_n(\xi_n, \xi_n)}{K_n(t_{jn}, t_{jn})} \leq \int_{\rho_{kn}}^{\rho_{\ell n}} \frac{d\mu\left(\xi_n + \frac{s}{\bar{K}_n(\xi_n, \xi_n)}\right)}{\mu'(\xi_n)} \leq \sum_{j=k}^{\ell} \frac{K_n(\xi_n, \xi_n)}{K_n(t_{jn}, t_{jn})}.$$

Next, for each fixed j , as $n \rightarrow \infty$ through \mathcal{S} ,

$$\frac{K_n(t_{jn}, t_{jn})}{K_n(\xi_n, \xi_n)} = f_n(\rho_{jn}, \rho_{jn}) \rightarrow f(\rho_j, \rho_j).$$

In this limit, we use the locally uniform convergence of f_n to f , and that $\rho_{jn} \rightarrow \rho_j$. Next, for the given k and ℓ , we have for large enough $n \in \mathcal{S}$,

$[t_{kn}, t_{\ell n}] \subset \left[\xi_n - \frac{A_n^*}{n}, \xi_n + \frac{A_n^*}{n} \right]$. Then

$$\begin{aligned} & \int_{\rho_{kn}}^{\rho_{\ell n}} \frac{d\mu_n^s \left(\xi_n + \frac{s}{\bar{K}_n(\xi_n, \xi_n)} \right)}{\mu'(\xi_n)} \\ &= K_n(\xi_n, \xi_n) \int_{t_{kn}}^{t_{\ell n}} d\mu_n^s \\ &\leq Cn \int_{\xi_n - \frac{A_n^*}{n}}^{\xi_n + \frac{A_n^*}{n}} d\mu_n^s \rightarrow 0, \end{aligned}$$

as $n \rightarrow \infty$ through \mathcal{S} , by (3.4). Also,

$$\begin{aligned} & \int_{\rho_{kn}}^{\rho_{\ell n}} \left| \frac{\mu_n'(\xi_n + \frac{s}{\bar{K}_n(\xi_n, \xi_n)})}{\mu'(\xi_n)} - 1 \right| ds \\ &= K_n(\xi_n, \xi_n) \int_{t_{kn}}^{t_{\ell n}} |\mu_n'(t) - \mu'(\xi_n)| dt \\ &\leq Cn \int_{\xi_n - \frac{A_n^*}{n}}^{\xi_n + \frac{A_n^*}{n}} |\mu_n'(t) - \mu'(\xi_n)| dt \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ through \mathcal{S} , by (3.5). Combining all the above, gives for each fixed k, ℓ ,

$$(3.26) \quad \sum_{j=k+1}^{\ell-1} \frac{1}{f(\rho_j, \rho_j)} \leq \rho_k - \rho_\ell \leq \sum_{j=k}^{\ell} \frac{1}{f(\rho_j, \rho_j)}.$$

In particular as f is bounded above and below, for some C_2 independent of j ,

$$C_1 \leq \rho_{j+2} - \rho_j \leq C_2,$$

so $f(0, \cdot)$ has at most double zeros. Moreover, because $\{\rho_{jn}\}$ are simple zeros of $f_n(0, \cdot)$, ρ_k can only be a double zero of $f(0, \cdot)$ if it is repeated in the sequence $\{\rho_j\}$.

Now assume that $\rho_k > 0$. As $f(\rho_j, \rho_j) \leq 1$ for all j , with $\rho_j > 0$, we obtain from (3.26),

$$k - \ell - 1 \leq \rho_k - \rho_\ell.$$

Then, in the interval $[\rho_\ell, \rho_k]$, the total multiplicity of zeros of $f(0, \cdot)$, namely $k - \ell + 1$ or $k - \ell + 2$ or $k - \ell + 3$, is at most $\rho_\ell - \rho_k + 4$. Recall that $n(f(0, \cdot), [0, r])$ denotes the number of zeros of $f(0, \cdot)$ in $[0, r]$. In view of the fact that $C_1 \leq \rho_{j+2} - \rho_j \leq C_2$ and there are infinitely many $\{\rho_j\}$, we can choose ρ_ℓ a bounded distance from r , and ρ_k a bounded distance from 0, lying to the right of 0. We obtain that $n(f(0, \cdot), [0, r])$ is at most the number of zeros in $[\rho_k, \rho_\ell]$ plus $O(1)$, and hence at most $\rho_\ell - \rho_k + O(1)$. So

$$n(f(0, \cdot), [0, r]) \leq r + O(1).$$

Then, recalling (3.16),

$$\frac{\sigma}{\pi} = \lim_{r \rightarrow \infty} \frac{n(f(0, \cdot), [0, r])}{r} \leq 1.$$

But we also know from (3.19) that

$$\sigma \geq \pi \sup_{x \in \mathbb{R}} f(x, x) \geq \pi.$$

Thus $\sup_{x \in \mathbb{R}} f(x, x) = 1$ and $\sigma = \pi$. ■

Lemma 3.5

(a)

$$(3.27) \quad f(0, s) = \frac{\sin \pi s}{\pi s}.$$

(b) For all complex u, v ,

$$(3.28) \quad f(u, v) = \sum_{j=-\infty}^{\infty} f(j, j) \frac{\sin \pi(u-j)}{\pi(u-j)} \frac{\sin \pi(v-j)}{\pi(v-j)}.$$

Proof

(a) By (3.18), with $\sigma = \pi$,

$$(3.29) \quad \int_{-\infty}^{\infty} \left(f(a, s) - \frac{\sin \pi(a-s)}{\pi(a-s)} \right)^2 ds \leq 1 - f(a, a).$$

Choosing $a = 0$ gives the result.

(b) Now for each real u , $f(u, \cdot)$ is of exponential type $\leq \pi$, and square integrable on the real axis. As such, it admits the cardinal series expansion [21, p. 91]

$$f(u, v) = \sum_{j=-\infty}^{\infty} f(u, j) \frac{\sin \pi(v-j)}{\pi(v-j)}.$$

In turn, the same is true of $f(\cdot, j)$, so we have the double series

$$f(u, v) = \sum_{j=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} f(k, j) \frac{\sin \pi(u-k)}{\pi(u-k)} \right] \frac{\sin \pi(v-j)}{\pi(v-j)}.$$

We claim now that for $j \neq k$,

$$(3.30) \quad f(j, k) = 0.$$

Once we have this, we obtain the result (3.28) for all complex u, v , by analytic continuation.

To see (3.30), we observe first that for $j \neq k$,

$$K_n(t_{jn}, t_{kn}) = 0.$$

Indeed this follows by substitution in the Christoffel-Darboux formula. Hence also for such j, k

$$f_n(\rho_{jn}, \rho_{kn}) = 0.$$

Letting $n \rightarrow \infty$ through the subsequence \mathcal{S} , gives

$$f(\rho_j, \rho_k) = 0.$$

Finally, as $f(0, s) = \frac{\sin \pi s}{\pi s}$, $\rho_j = j$ for all j . So we obtain (3.30). ■

Proof of Theorem 1.2

Since the limit of the subsequence $\{f_n(0, s)\}_{n \in \mathcal{S}}$ is independent of the subsequence, the limit through the full sequence of positive integers follows. ■

4. PROOF OF THEOREM 1.1

We shall begin by dealing with the singular part of μ , and obtaining a lower bound on $K_n(x, x)$. This has been dealt with in the literature, but we need the following form. Because the hypotheses are different, we shall use a measure ν rather than μ . Its reproducing kernel will be denoted by K_n^ν .

Lemma 4.1

Let ν be a measure with compact support and with infinitely many points in the support. Let ξ lie in the support. Assume that a.e. in a neighborhood of ξ , $\nu' \leq C$, and

$$\lim_{h \rightarrow 0^+} \frac{1}{h} \int_{\xi-h}^{\xi+h} d\nu^s = 0.$$

Then there exists $C_1 > 0$ with the following property: let $r > 0$. For $n \geq n_0(r)$,

$$(4.1) \quad K_n^\nu(x, x) \geq C_1 n \text{ for } |x - \xi| \leq \frac{r}{n}.$$

Remark

We emphasize that C_1 does not depend on r .

Proof

Let us fix $r \geq 1$. We estimate above

$$\lambda_n^\nu(x) = \frac{1}{K_n^\nu(x, x)} = \inf_{\deg(P) \leq n-1} \frac{\int P^2 d\nu}{P^2(x)}.$$

By a translation and dilation, we may assume that the support of ν lies in $[-1, 1]$ and $\xi = 0$. By hypothesis, for some $\eta, B > 0$,

$$(4.2) \quad \nu' \leq B \text{ a.e. in } [-\eta, \eta].$$

Our hypothesis on ν^s ensures the existence of $\delta = \delta(r)$ such that

$$(4.3) \quad 0 \leq h \leq \delta \Rightarrow \nu^s[-h, h] \leq \frac{h}{r}.$$

We shall use the reproducing kernel K_n^T for the classical Chebyshev weight $\frac{dt}{\sqrt{1-t^2}}$ on $[-1, 1]$. It is well known, and follows from the Christoffel-Darboux formula (see e.g. [15, p. 92]) that

$$|K_n^T(x, t)| \leq \frac{C_1 n}{1 + n|x - t|} \quad x, t \in [-1, 1];$$

$$K_n^T(x, x) \geq C_2 n \quad x \in [-1, 1].$$

Now let $|x| \leq \frac{r}{n}$. Note that $|x| \leq \frac{1}{\sqrt{n}}$ for $n \geq r^2$. Then for such n ,

$$\begin{aligned} \lambda_n^\nu(x) &\leq \int \left(\frac{K_n^T(x, t)}{K_n^T(x, x)} \right)^2 d\nu(t) \\ &\leq C \int \left(\frac{1}{1 + n|x - t|} \right)^2 d\nu(t) \\ &\leq C \int_{\{t: |t-x| \geq \frac{1}{\sqrt{n}}\}} \frac{1}{n} d\nu(t) + C \int_{\{t: |t-x| < \frac{1}{\sqrt{n}}\}} \left(\frac{1}{1 + n|x - t|} \right)^2 B dt \\ &\quad + C \int_{\{t: |t-x| < \frac{1}{\sqrt{n}}\}} \left(\frac{1}{1 + n|x - t|} \right)^2 d\nu^s(t) \\ (4.4) \quad &\leq \frac{C}{n} + C \int_{-\frac{2}{\sqrt{n}}}^{\frac{2}{\sqrt{n}}} \left(\frac{1}{1 + n|x - t|} \right)^2 d\nu^s(t), \end{aligned}$$

by (4.2). It is important that all the constants in the last right-hand side do not depend on r , though the threshold for n does. We now estimate the integral involving ν^s . Let $I_j = [2^{-j-1}, 2^{-j}]$, $j \geq 0$. We see that if dist denotes distance from a point to a set,

$$\begin{aligned} &\int_0^{\frac{2}{\sqrt{n}}} \left(\frac{1}{1 + n|x - t|} \right)^2 d\nu^s(t) \\ &\leq \sum_{j \geq \log_2 \frac{2}{\sqrt{n}}} \frac{1}{(1 + n \text{dist}(x, I_j))^2} \nu^s(I_j) \\ &\leq \frac{1}{r} \sum_{j \geq \log_2 \frac{2}{\sqrt{n}}} \frac{2^{-j}}{(1 + n \text{dist}(x, I_j))^2}, \end{aligned}$$

since $\nu^s(I_j) \leq \nu^s([0, 2^{-j}]) \leq \frac{1}{r} 2^{-j}$, by (4.3), for $n \geq n_0(r)$ and all $j \geq \log_2 \frac{2}{\sqrt{n}}$. Now if $\text{dist}(x, I_j) \geq 2^{-j-3}$, we have for $t \in I_j$,

$$\begin{aligned} &\frac{1 + n|x - t|}{1 + n \text{dist}(x, I_j)} \\ &\leq \frac{1 + n \text{dist}(x, I_j) + n 2^{-j-1}}{1 + n \text{dist}(x, I_j)} \leq 5. \end{aligned}$$

Thus,

$$\begin{aligned}
 & \frac{1}{r} \sum_{j \geq \log_2 \frac{2}{\sqrt{n}}, \text{dist}(x, I_j) \geq 2^{-j-3}} \frac{2^{-j}}{(1 + n \text{dist}(x, I_j))^2} \\
 & \leq \frac{2}{r} \sum_{j \geq \log_2 \frac{2}{\sqrt{n}}, \text{dist}(x, I_j) \geq 2^{-j-3}} \int_{I_j} \left(\frac{5}{1 + n|x-t|} \right)^2 dt \\
 & \leq C \int_{-\infty}^{\infty} \left(\frac{1}{1 + n|x-t|} \right)^2 dt \leq \frac{C}{n}.
 \end{aligned}$$

Note that $r > 1$, so we may take C independent of r . We now dealing with the remaining j , for which $\text{dist}(x, I_j) < 2^{-j-3}$. For such j ,

$$2^{-j-2} \leq 2^{-j-1} - 2^{-j-3} \leq x \leq 2^{-j} + 2^{-j-3} \leq 2^{-j+1}.$$

Thus there are at most four such j , and for each such j , $2^{-j} \leq 4x$, so

$$\begin{aligned}
 & \frac{1}{r} \sum_{j \geq \log_2 \frac{2}{\sqrt{n}}, \text{dist}(x, I_j) < 2^{-j-3}} \frac{2^{-j}}{(1 + n \text{dist}(x, I_j))^2} \\
 & \leq \frac{1}{r} 4 \cdot 4x \leq \frac{16}{n},
 \end{aligned}$$

recall $|x| \leq \frac{r}{n}$. It is here, and only here, that we need the $\frac{1}{r}$ term. As a similar estimate holds over $\left[-\frac{2}{\sqrt{n}}, 0\right]$, we have shown that for $n \geq n_0(r)$,

$$\int_{-\frac{2}{\sqrt{n}}}^{\frac{2}{\sqrt{n}}} \left(\frac{1}{1 + n|x-t|} \right)^2 d\nu^s(t) \leq \frac{C}{n},$$

with C independent of r . Together with (4.4), this gives the result. ■

Proof of Theorem 1.1

We show that the hypotheses of Theorem 1.2 hold with $\mu_n = \mu$, $n \geq 1$. Firstly, as $\mu' \geq C$ in a neighborhood of ξ , there is a neighborhood J of ξ such that

$$K_n(t, t) \leq Cn \text{ for } n \geq 1 \text{ and } t \in J.$$

See, for example, [15, p. 116, Theorem 20]. It is then an easy consequence of Bernstein's growth inequality for polynomials bounded on an interval that (1.11) holds for $n \geq n_0$ and $|z|, |v| \leq A$. See [11, Lemma 5.2, pp. 383-385]. Next, (1.12) was established in Lemma 4.1. Next, we assumed (1.14) for $\mu_n = \mu$. Finally, (1.15) follows easily from the assumed continuity of μ' at ξ . ■

Acknowledgement

The author thanks the organizers of OPSFA 10 for the wonderful conference, and the kind invitation.

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