UNIVERSALITY TYPE LIMITS FOR BERGMAN ORTHOGONAL POLYNOMIALS

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ABSTRACT. We establish universality type limits for Bergman orthogonal polynomials on simply connected regions in the complex plane with smooth boundary.

1. INTRODUCTION AND RESULTS

Let G be a bounded simply connected domain in the complex plane, bounded by a Jordan curve Γ. Let μ be a finite positive Borel measure on G. We may define, for $n \geq 0$, orthonormal polynomials

$$p_n(z) = \kappa_n z^n + ..., \kappa_n > 0$$

satisfying

$$\int_G p_n \overline{p_m} d\mu = \delta_{mn}.$$

We shall usually assume that μ is *regular* in the sense of Stahl and Totik [27, p. 60], so that

$$(1.1) \qquad \lim_{n \to \infty} \kappa_n^{1/n} = \frac{1}{cap(G)}.$$

Here $cap(G)$ denotes the logarithmic capacity of G. Moreover, we shall also assume that μ is absolutely continuous with respect to *planar* Lebesgue measure dA near given points on ∂G. In this sense, the polynomials $\{p_n\}$ fall within the framework of *Bergman polynomials*. Throughout, we let

$$w(z) = \frac{d\mu}{dA}(z), \text{ a.e. } z \in G.$$

Moreover, for $u \in \partial G$, we define

$$w(u) = \lim_{z \to u, z \in G} w(z),$$

whenever the limit is defined. We note that when $w > 0$ a.e. on G, then it follows from Widom's criterion for regularity, that μ is regular in the sense of Stahl and Totik [27, pp. 106-107].

Date: November 3, 2009.

There is a well developed theory of Bergman polynomials - their basic properties, and the asymptotic behavior, including that of their zeros [6], [7], [8], [12], [18], [19], [23]. In describing these, the conformal map ϕ of the exterior of Γ, namely

$$D = \mathbb{C} \backslash \bar{G}$$

onto the exterior of the unit ball plays a key role. We say that the curve Γ *is of class $C(p, \alpha)$* if the parametrization of the curve is p times continuously differentiable, with $p-$th derivative satisfying a Lipschitz condition of order $\alpha \in (0, 1)$.

Classical results of Suetin give asymptotics for p_n when μ is absolutely continuous on G, with $w = 1$ there. We shall denote the corresponding polynomials by $\left\{ p_n^C \right\}$, where the C stands for Carleman. It is known [28, Theorem 1.3, p. 21] that if $\Gamma \in C(1, \alpha)$, where $\alpha > \frac{1}{2}$, then uniformly for $z \in \partial G$, and even locally uniformly in $\bar{D}$,

$$(1.2) \qquad p_n^C(z) = \sqrt{\frac{n+1}{\pi}} \phi'(z) \phi(z)^n \left\{ 1 + O\left(\frac{1}{n^{2\alpha-1}} \right) \right\}.$$

The $n-$th reproducing kernel for μ is

$$(1.3) \qquad K_n(z, u) = \sum_{j=0}^{n-1} p_j(z) \overline{p_j(u)},$$

and its normalized cousin is

$$(1.4) \qquad \tilde{K}_n(z, u) = w(z)^{1/2} w(u)^{1/2} K_n(z, u).$$

Recall that $w = \frac{d\mu}{dA}$ or its limit on the boundary ∂G.

In formulating our result, we need the notion of the convex hull $Co(K)$ of a set K, as well as its boundary $\partial Co(K)$. If $J \subset \partial G$, a ∂G neighborhood of J means a relatively open subset J_1 of ∂G containing J. That is, $J_1 \supset J$ and $J_1 = U \cap \partial G$ for some open subset U of the plane. Our main result is:

Theorem 1.1
Let G be a bounded simply connected set, and assume that $\Gamma = \partial G$ is of class $C(1, \alpha)$, with $\alpha \in \left(\frac{1}{2}, 1 \right)$. Let $J \subset \partial G$ be compact, and let some ∂G neighborhood of J also lie in $\partial Co(G)$. Let μ be a finite positive Borel measure on G that is regular in the sense of Stahl and Totik. Assume that μ is absolutely continuous with respect to planar Lebesgue measure in an open subset of G whose boundary contains a ∂G neighborhood of J. Assume moreover, that w is positive and continuous at each point of J. Then uniformly for $z \in J$ and a, b in compact subsets of the plane, we have

$$(1.5) \qquad \lim_{n \to \infty} \frac{K_n\left(z + \frac{a}{n}, z + \frac{b}{n} \right)}{K_n(z, z)} = H\left(a\phi'(z)\overline{\phi(z)} + b\overline{\phi'(z)}\phi(z) \right),$$

where

(1.6)
$$H\left(t\right) = \begin{cases} 2\frac{e^t(t-1)+1}{t^2}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

Remarks

(a) Universality limits for measures with support on the real line take the form

$$\lim_{n \to \infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi, \xi\right)} = \frac{\sin \pi\left(a-b\right)}{\pi\left(a-b\right)}.$$

Here ξ is in the "bulk" of the support, that is, ξ lies in the interior of $\mathrm{supp}[\mu]$. See [2], [3], [4], [5], [9], [11], [13], [14], [15], [17], [25], [26], [30]. For measures supported on the unit circle, the analogous formulation is [10]

$$\lim_{n \to \infty} \frac{1}{n} \tilde{K}_n\left(e^{i\left(\theta + \frac{2\pi a}{n}\right)}, e^{i\left(\theta + \frac{2\pi b}{n}\right)}\right) = e^{i\pi(a-b)} \frac{\sin \pi\left(a-b\right)}{\pi\left(a-b\right)}$$

or

$$\lim_{n \to \infty} \frac{K_n\left(z\left(1 + \frac{i2\pi a}{n}\right), z\left(1 + \frac{i2\pi \bar{b}}{n}\right)\right)}{K_n\left(z, z\right)} = e^{i\pi(a-b)} \frac{\sin \pi\left(a-b\right)}{\pi\left(a-b\right)},$$

uniformly for a, b in compact subsets of the complex plane and $z = e^{i\theta}, \theta \in J$.

(b) Note that the increment inside the reproducing kernel in (1.5) is $\frac{a}{n}$ or $\frac{b}{n}$. By contrast, under the hypotheses of Theorem 1.1, $K_n\left(z, z\right)$ grows like n^2, and

$$\lim_{n \to \infty} \frac{K_n\left(z + \frac{a}{K_n(z,z)}, z + \frac{b}{K_n(z,z)}\right)}{K_n\left(z, z\right)} = 1.$$

(c) The restriction that $J \subset \partial G \cap \partial Co\left(G\right)$ is a severe geometric restriction - basically requiring that G is "locally convex" in some neighborhood of J. It is likely that there is some payoff between the generality of the measure μ and the geometry of the domain G. In particular, for Carleman polynomials, there is no need for a geometric restriction, as shown in Section 2. We use the convexity in constructing, for each $z \in J$, a polynomial R_z, such that $R_z\left(z\right) = 1$ and $|R_z| < 1$ in $\bar{G}\backslash\{z\}$. To allow for more general sets J, we need a definition:

Definition 1.2 *Let $S \subset \mathbb{C}$ be bounded.*
*(a) We say that $a \in S$ is a **peak polynomial point** for S if there exists a polynomial R_a such that*

(1.7)
$$R_a\left(a\right) = 1$$

and

$$|R_a| < 1 \text{ in } S\backslash\{a\}.$$

*(b) Let $J \subset S$. We say that J is a **uniform peak polynomial set** for S if each point $a \in J$ is a peak polynomial point for S, and moreover, the*

degree of R_a is bounded above independent of $a \in J$, and in addition, for any $\delta > 0$, there exists $r < 1$ independent of a, such that

$$(1.8) \qquad |R_a(z)| < r \text{ for } a \in J \text{ and } z \in G \text{ with } |z - a| \geq \delta.$$

Remark
Peak polynomial points have been investigated (without using that name) by Nagy and Totik [20]. Uniform peak polynomial sets have been discussed by Andrievskii and Pritsker [1]. In particular, they derived a necessary and sufficient condition involving circles that lie in $\mathbb{C} \backslash G$ touching ∂G at only one point. This overlaps with results of Totik [20], [31].

Theorem 1.3
Assume the hypotheses of Theorem 1.1, except the restriction that some ∂G neighborhood of J also lies in $\partial \mathrm{Co}(G)$. Assume instead that some ∂G neighborhood of J is a uniform peak polynomial set for $\bar{G}$. Then the conclusion of Theorem 1.1 remains valid.

Remark
Totik [31] has recently established asymptotics for Christoffel functions associated with Bergman polynomials over a region that consists of finitely many domains. Totik does not require a geometric condition on the boundary, but does require a locally C^2 boundary. This should lead to a version of Theorem 1.3 without geometric conditions, but with a locally smoother boundary.

Corollary 1.4
Assume the hypotheses of Theorem 1.1 or 1.3. Let r, s be non-negative integers and

$$(1.9) \qquad K_n^{(r,s)}(z,z) = \sum_{k=0}^{n-1} p_k^{(r)}(z) \overline{p_k^{(s)}(z)}.$$

Then uniformly for $z \in J$,

$$(1.10) \qquad \lim_{n \to \infty} \frac{K_n^{(r,s)}(z,z)}{n^{r+s} K_n(z,z)} = \frac{2 \left(\phi'(z) \overline{\phi(z)} \right)^r \left(\overline{\phi'(z)} \phi(z) \right)^s}{r+s+2}.$$

In the sequel $C, C_1, C_2, \ldots$ denote constants independent of n, z, u, v, s, t. The same symbol does not necessarily denote the same constant in different occurences. We shall write $C = C(\alpha)$ or $C \neq C(\alpha)$ to respectively denote dependence on, or independence of, the parameter α. Given measures μ^*, $\mu^{\#}$, we use $K_n^*, K_n^{\#}$ and $p_n^*, p_n^{\#}$ to denote respectively their reproducing kernels and orthonormal polynomials. Similarly superscripts $*, \#$ are used to distinguish other quantities associated with them. The superscript C (for

Carleman) denotes quantities associated with the Legendre weight 1 on G. For $z \in \mathbb{C}$ and $\delta > 0$, we set

$$B(z, \delta) = \{t : |t - z| < \delta\}.$$

The distance from a point z to a set of complex numbers J is denoted $dist(z, J)$. For such a set J, we set

$$J(\delta) = \{z \in \mathbb{C} : dist(z, J) \leq \delta\}.$$

For $n \geq 1$ and $M > 0$, we also let

$$G_n(M) = \left\{ z \in \bar{D} : dist(z, \partial G) \leq \frac{M}{n} \right\}$$

In particular, $G_n(M)$ contains ∂G. $[x]$ denotes the greatest integer $\leq x$. We denote the nth Christoffel function for the measure μ by

$$(1.11) \qquad \lambda_n(z) = 1/K_n(z, z) = \min_{\deg(P) \leq n-1} \int_G |P|^2 \, d\mu / |P(z)|^2.$$

As in [13], the main idea in this paper is a localization principle based on a comparison inequality (Lemma 4.1 below). Suppose that μ, μ^* are measures with $\mu \leq \mu^*$ in G, and let the superscript $*$ indicate quantities associated with the measure μ^*. Then for $z, t \in \mathbb{C}$,

$$|K_n(z, t) - K_n^*(z, t)| / K_n(z, z)$$
$$\leq \left(\frac{K_n(t, t)}{K_n(z, z)} \right)^{1/2} \left[1 - \frac{K_n^*(z, z)}{K_n(z, z)} \right]^{1/2}$$
$$= \left(\frac{\lambda_n(z)}{\lambda_n(t)} \right)^{1/2} \left[1 - \frac{\lambda_n(z)}{\lambda_n^*(z)} \right]^{1/2}.$$

The paper is organised as follows. In Section 2, we prove some of the results for the weight 1 on G, that is for Carleman polynomials. In Section 3, we present some asymptotics for Christoffel functions. In Section 4, we prove our localization principle, and hence Theorems 1.1, 1.3 and Corollary 1.4.

2. Carleman Polynomials

For the Legendre weight (or normalized Lebesgue measure) on G, recall that we use the superscript C to distinguish the orthonormal polynomials and related quantities. We shall use the asymptotics (1.2):

Theorem 2.1
Let G be a bounded simply connected set, and assume that $\Gamma = \partial G$ is of class $C(1, \alpha)$, with $\alpha \in \left(\frac{1}{2}, 1 \right)$. Let $M > 0$.

(a) *Uniformly for* $z, t \in G_n(M)$, *with* $z \neq t$,
(2.1)

$$K_n^C(z,t) = \frac{\phi'(z)\overline{\phi'(t)}}{\pi}\left\{-(n+1)\frac{\left[\phi(z)\overline{\phi(t)}\right]^n}{1-\phi(z)\overline{\phi(t)}} + \frac{1-\left[\phi(z)\overline{\phi(t)}\right]^{n+1}}{\left[1-\phi(z)\overline{\phi(t)}\right]^2}\right\} + O\left(n^{3-2\alpha}\right).$$

(b) *Uniformly for* $z \in \partial G$,

$$(2.2) \qquad\qquad K_n^C(z,z) = \frac{\left|\phi'(z)\right|^2}{\pi}\frac{n(n+1)}{2} + O\left(n^{3-2\alpha}\right).$$

Moreover, uniformly for $z \in G_n(M)$,

$$(2.3) \qquad\qquad K_n^C(z,z) = O\left(n^2\right).$$

(c) *Uniformly for* $z \in \partial G$ *and* a, b *in compact subsets of the plane,*

$$(2.4) \qquad \lim_{n \to \infty}\frac{K_n^C\left(z+\frac{a}{n}, z+\frac{b}{n}\right)}{K_n^C(z,z)} = H\left(a\phi'(z)\overline{\phi(z)} + \overline{b\phi'(z)}\phi(z)\right).$$

Proof

Let $v \in G_n(M)$. We can write $v = z + \frac{a}{n}$, where $z \in \partial G$ and $|a| \leq M$. Then

$$\phi(v) - \phi(z) = O(|v-z|) = O\left(\frac{1}{n}\right),$$

for $n \geq n_0(M)$. Here we have used the fact that $\Gamma = \partial G$ is of class $C(1,\alpha)$, while ϕ' is continuous and non-zero on ∂G, so that every point v in $G_n(M)$ may be joined to a point z in ∂G, by a path of length $O(|v-z|)$. As $|\phi(z)| = 1$, we obtain

$$|\phi(v)| \leq 1 + \frac{C}{n},$$

and hence

$$(2.5) \qquad\qquad \sup_{v \in G_n(M)}\sup_{1 \leq k \leq n}\left|\phi^k(v)\right| \leq C.$$

We also note that then if P is a polynomial of degree $\leq n$, the Walsh-Bernstein inequality, namely

$$|P(z)| \leq |\phi(z)|^n \sup_{\partial G}|P|$$

gives

$$(2.6) \qquad\qquad \sup_{G_n(M)}|P| \leq C\sup_{\partial G}|P|.$$

(a) From (1.2), uniformly for $z, t \in G_n(M)$,

$$\begin{aligned}K_n^C(z,t) &= \frac{\phi'(z)\overline{\phi'(t)}}{\pi}\sum_{k=0}^{n-1}(k+1)\left[\phi(z)\overline{\phi(t)}\right]^k\left\{1+O\left(\frac{1}{k^{2\alpha-1}}\right)\right\}\\ &= \frac{\phi'(z)\overline{\phi'(t)}}{\pi}\sum_{k=0}^{n-1}(k+1)\left[\phi(z)\overline{\phi(t)}\right]^k + O\left(\sum_{k=0}^{n-1}k^{2-2\alpha}\right)\end{aligned}$$

recall (2.5). Using the identity

$$\sum_{k=0}^{n-1} (k+1)\, u^k = -(n+1)\frac{u^n}{1-u} + \frac{1-u^{n+1}}{(1-u)^2},\ u \neq 1,$$

(this is easily obtained by differentiating the identity for a finite geometric series), and the fact that $\alpha > \frac{1}{2}$, we obtain (2.1).

(b) This is similar to (a), but easier.

(c) Suppose first that $z + \frac{a}{n}$ and $z + \frac{b}{n} \in \bar{D}$, so that $\phi\left(z + \frac{a}{n}\right)$ and $\phi\left(z + \frac{b}{n}\right)$ are well defined. We have for some path γ in $\bar{D}$ from z to $z + \frac{a}{n}$,

$$\phi\left(z + \frac{a}{n}\right) - \phi(z) - \frac{a}{n}\phi'(z) = \int_\gamma \left(\phi'(t) - \phi'(z)\right) dt = o\left(\frac{1}{n}\right),$$

uniformly for $z \in \partial G$, and a in compact subsets of the complex plane, because of the continuity of ϕ'. Let

$$\Delta = a\phi'(z)\overline{\phi(z)} + \overline{b\phi'(z)}\phi(z).$$

Then, as $|\phi(z)| = 1$,

$$\phi\left(z + \frac{a}{n}\right)\overline{\phi\left(z + \frac{b}{n}\right)} = 1 + \frac{\Delta}{n} + o\left(\frac{1}{n}\right);$$

$$\left[\phi\left(z + \frac{a}{n}\right)\overline{\phi\left(z + \frac{b}{n}\right)}\right]^n = e^{\Delta + o(1)}.$$

Substituting into (2.1) gives, as long as $\Delta \neq 0$, and n is so large that $z + \frac{a}{n}, z + \frac{b}{n} \in G_n(M)$,

$$K_n^C\left(z + \frac{a}{n}, z + \frac{b}{n}\right)$$

$$= \frac{\phi'\left(z + \frac{a}{n}\right)\overline{\phi'\left(z + \frac{b}{n}\right)}}{\pi}\left\{\frac{(n+1)\, n e^{\Delta + o(1)}}{\Delta + o(1)} + \frac{n^2\left(1 - e^{\Delta + o(1)}\right)}{\Delta^2 + o(1)}\right\}$$

and so using continuity of ϕ', and (2.2),

$$\frac{K_n^C\left(z + \frac{a}{n}, z + \frac{b}{n}\right)}{K_n^C(z, z)} = 2\left\{\frac{e^\Delta}{\Delta} + \frac{1 - e^\Delta}{\Delta^2}\right\} + o(1)$$

$$= H(\Delta) + o(1).$$

We still have to show the convergence when $\Delta = 0$, or when at least one of $z + \frac{a}{n}$ and $z + \frac{b}{n} \notin \bar{D}$, that is at least one is in G. For this we use convergence continuation theorems, and uniform boundedness. In view of Cauchy-Schwarz, (2.6) and (2.2), we have for all $z \in \partial G$,

$$\sup_{u,v \in G_n(M)} \left|K_n^C(u,v)\right| \leq \sup_{u \in G_n(M)} \left|K_n^C(u,u)\right| \leq Cn^2 \leq CK_n^C(z,z).$$

The maximum modulus principle then gives

$$\sup_{u,v \in G \cup G_n(M)} \left| K_n^C(u,v) \right| \leq C K_n^C(z,z).$$

Thus $\left\{ \frac{K_n^C\left(z+\frac{a}{n}, z+\frac{b}{n}\right)}{K_n^C(z,z)} \right\}_n$ is uniformly bounded for a,b in compact subsets of the plane, and for $z \in \partial G$. It is hence a normal family in the complex variables a,b. Inasmuch as for $\Delta \neq 0$, it has the limit $H\left(a\phi'(z)\overline{\phi(z)} + \overline{b\phi'(z)}\phi(z)\right)$, which is entire in a,b, convergence continuation theorems give the result, even when $\Delta = 0$. $\blacksquare$

Recall the notation (1.9).

Corollary 2.2
Let r,s be non-negative integers. Then uniformly for $z \in \partial G$,

$$(2.7) \qquad \lim_{n\to\infty} \frac{K_n^{C(r,s)}(z,z)}{n^{r+s} K_n(z,z)} = \frac{2\left(\phi'(z)\overline{\phi(z)}\right)^r \left(\overline{\phi'(z)}\phi(z)\right)^s}{r+s+2}.$$

Proof
Taylor series expansion shows that

$$(2.8) \qquad \frac{K_n^C\left(z+\frac{a}{n}, z+\frac{b}{n}\right)}{K_n^C(z,z)} = \sum_{r,s=0}^{\infty} \frac{a^r \bar{b}^s}{r!s!} \frac{K_n^{C(r,s)}(z,z)}{n^{r+s} K_n^C(z,z)}.$$

Next, the Maclaurin series for H is

$$H(t) = \sum_{k=0}^{\infty} \frac{2t^k}{(k+2)\,k!}$$

so if $w = \phi'(z)\overline{\phi(z)}$,

$$H\left(a\phi'(z)\overline{\phi(z)} + \overline{b\phi'(z)}\phi(z)\right)$$

$$= \sum_{k=0}^{\infty} \frac{2}{(k+2)\,k!} \sum_{r=0}^{k} \binom{k}{r} (aw)^r \left(\overline{bw}\right)^{k-r}$$

$$= 2\sum_{r=0}^{\infty} \frac{(aw)^r}{r!} \sum_{k=r}^{\infty} \frac{\left(\overline{bw}\right)^{k-r}}{(k+2)(k-r)!}$$

$$(2.9) \qquad = 2\sum_{r,s=0}^{\infty} \frac{(aw)^r}{r!} \frac{\left(\overline{bw}\right)^s}{s!\,(r+s+2)}.$$

The uniform convergence in a,b in (2.4) as $n \to \infty$, implies that corresponding Taylor series coefficients also converge. Then (2.8) and (2.9) give (2.7). The uniformity in z follows from Cauchy's estimates and the uniform convergence in $z \in \partial G$. $\blacksquare$

3. Christoffel functions

The methods used to prove the following result for Christoffel functions are well known for orthogonal polyonomials over intervals and curves [16], [21], [22], [29]. For Bergman polynomials, there are far fewer results [7], [8]. One difficulty in its formulation is that if $z \in \partial G$, and $a \in \mathbb{C}$, we cannot be sure that $z + \frac{a}{n} \in \partial G$ for large enough n. So instead we consider sequences $\{z_n\} \subset \partial G$. Recall that λ_n and λ_n^C denote respectively the Christoffel functions for the measure μ and for the measure with derivative 1 on G (the Carleman case).

Theorem 3.1
Let G, J and μ be as in Theorem 1.1 or 1.3. Let $M > 0$.
(a) Uniformly for $z_0 \in J$ and sequences $\{z_n\} \subset \partial G$ with $|z_n - z_0| < \frac{M}{n}$, $n \geq 1$,

$$(3.1) \qquad \lim_{n \to \infty} n^2 \lambda_n(z_n) = \frac{|\phi'(z_0)|^2}{\pi} w(z_0).$$

(b) There exists $\eta > 0$ and n_0 such that uniformly for $n \geq n_0$ and $z \in J(\eta) \cap \partial G$,

$$(3.2) \qquad \lambda_n(z) \sim \frac{1}{n^2}.$$

Remarks
(a) The notation $\sim$ means that the ratio of the two quantities is bounded above and below by positive constants independent of n, z and a.
(b) We emphasize that we are assuming that w is continuous in J when regarded as a function defined on G.
(c) Totik [31, Theorem 1.4] has recently proved a generalization of Theorem 3.1, where the region G may consist of finitely many components, and moreover, there is no geometric condition on the boundary. The decription of the limit involves a Green's function rather than a conformal map. However, instead it is assumed that J is contained in a C^2 arc, so the curve is assumed to be locally smoother than in our case.
We shall assume the hypotheses of Theorem 1.1, and at the end, indicate the simple changes required for the hypotheses of Theorem 1.3. We first need:

Lemma 3.2
Assume the hypotheses of Theorem 1.1 and that J_1 is a ∂G neighborhood of J such that $J_1 \subset \partial G \cap \partial Co(G)$. There exists for $z \in J_1$, a quadratic polynomial R_z with the following properties:
(i) $R_z(z) = 1$;
(ii) $|R_z| \leq 1$ in G;

(iii) Given $\delta > 0$, there exists $\eta \in (0,1)$ such that $|R_z| \leq \eta$ in $G\backslash B\left(z, \frac{\delta}{2}\right)$.
Here η is independent of $z \in J_1$.

Proof

By a translation and rotation of G, we may assume that $\bar{G}$ lies in the closed
left-half plane, and that the z for which R_z must be constructed is $z = 0$.
Thus the imaginary axis is a supporting line for $Co\left(\bar{G}\right)$. By a dilation, we
may also assume that

$$\bar{G} \subset \left\{z : |z| \leq \frac{1}{13}\sin\frac{\pi}{8}\right\}.$$

We shall see that

$$R_0\left(t\right) = 1 + t + 2t^2$$

does the job. Let $t = re^{i\theta}$, where $\theta \in \left[\frac{\pi}{2}, \pi\right]$ and $0 \leq r \leq \frac{1}{13}\sin\frac{\pi}{8}$. It suffices
to consider such t, since $|R\left(\bar{t}\right)| = |R\left(t\right)|$. A simple calculation shows that

$$\begin{aligned}
&|R_0\left(t\right)|^2 \\
&= \quad 1 + r\left\{2\cos\theta + r\left[1 + 4\cos 2\theta\right] + 4r^2\cos\theta + 4r^3\right\}.
\end{aligned}$$

If first, $\theta \in \left[\frac{\pi}{2}, \frac{5\pi}{8}\right]$, then as $\cos\theta \leq 0$,

$$\begin{aligned}
&|R_0\left(t\right)|^2 \\
&\leq \quad 1 + r^2\left\{1 + 4\cos 2\theta + 4r^2\right\} \\
&\leq \quad 1 + r^2\left\{1 - 2\sqrt{2} + \frac{4}{169}\right\} \\
&\leq \quad 1 - r^2\left\{\sqrt{2} - 1\right\}.
\end{aligned}$$

If instead, $\theta \in \left(\frac{5\pi}{8}, \pi\right]$, then

$$\begin{aligned}
&|R_0\left(t\right)|^2 \\
&\leq \quad 1 + r\left\{-2\sin\frac{\pi}{8} + 5r + 4r^2 + 4r^3\right\} \\
&\leq \quad 1 - r\sin\frac{\pi}{8},
\end{aligned}$$

as $5r + 4r^2 + 4r^3 < 13r \leq \sin\frac{\pi}{8}$. Thus for all $t \in \bar{G}$,

$$|R_0\left(t\right)|^2 \leq \max\left\{1 - r^2\left\{\sqrt{2} - 1\right\}, 1 - r\sin\frac{\pi}{8}\right\}.$$

In particular, if $t \in G\backslash B\left(0, \frac{\delta}{2}\right)$, so that $|t| \geq \frac{\delta}{2}$, we have

$$|R_0\left(t\right)|^2 \leq \max\left\{1 - \frac{\delta^2}{4}\left\{\sqrt{2} - 1\right\}, 1 - \frac{\delta}{2}\sin\frac{\pi}{8}\right\}.$$

This latter bound can be taken as our η^2, since it will clearly work uniformly
for $z \in J_1$. ∎

Remark

Uniform peak polynomial sets were discussed in [1] and [31], though with

different terminology. It is easy to see that some geometric restriction is needed. Indeed, let

$$G = \{z : |z| \leq 1 \text{ with } \arg(z) \in [\beta\pi, \pi] \cup [-\pi, -\beta\pi]\},$$

where $0 < \beta < \frac{\pi}{2}$. Thus G is the ball center 0, radius 1, with a sector of width $2\beta\pi < \pi$ removed about the positive real axis. Then there does not exist a polynomial R_0 with the properties listed above. Indeed, if

$$R_0(z) = 1 + \gamma z + O(z^2),$$

then in order that $|R_0| < 1$ in that part of G in the left-half plane close to 0, we need $\gamma > 0$. But then for z in G close to 0 in the right-half plane, we obtain $|R_z| > 1$.

Proof of Theorem 3.1 under the hypotheses of Theorem 1.1

(a) Let $\varepsilon > 0$ and choose $\delta > 0$ such that μ is absolutely continuous with respect to planar measure in $J(\delta)$, and such that

$$(3.3) \qquad (1+\varepsilon)^{-1} \leq \frac{w(z)}{w(u)} \leq 1+\varepsilon, \ z, u \in J(\delta) \cap \bar{G} \text{ with } |z - u| \leq \delta.$$

(This is possible because of compactness of J and continuity and positivity of w at every point of J). We may assume that δ is so small that $J(\delta) \cap \partial G$ is a uniform peak polynomial set for $\bar{G}$. Let us fix $z_0 \in J$, $z \in B(z_0, \delta/2) \cap \partial G$ and let R_z be the quadratic polynomial of Lemma 3.2. Thus $R_z(z) = 1$ and there exists $r < 1$ such that

$$|R_z| \leq r \text{ in } G \backslash B\left(z, \frac{\delta}{2}\right).$$

Here r is independent of z and z_0. Let $\eta \in \left(0, \frac{1}{2}\right)$, $n > 2[\eta n]$, and consider the polynomial

$$P(t) = \frac{K^C_{n-2[n\eta]}(t, z)}{K^C_{n-2[n\eta]}(z, z)} R_z(t)^{[n\eta]}.$$

We see that $P(z) = 1$, and P has degree $\leq n - 1$, so, using the properties of R_z, and (3.3), and recalling $|z - z_0| < \delta/2$,

$$\lambda_n(z)$$
$$\leq \int_G |P|^2 \, d\mu$$
$$\leq \frac{w(z_0)(1+\varepsilon)}{K^C_{n-2[n\eta]}(z, z)^2} \int_{G \cap B\left(z, \frac{\delta}{2}\right)} \left|K^C_{n-2[n\eta]}(t, z)\right|^2 dA(t)$$
$$(3.4) \qquad + \frac{\left\|K^C_{n-2[n\eta]}(\cdot, z)\right\|^2_{L_\infty\left(G \backslash B\left(z, \frac{\delta}{2}\right)\right)}}{K^C_{n-2[n\eta]}(z, z)^2} r^{2[n\eta]} \int_{G \backslash B\left(z, \frac{\delta}{2}\right)} d\mu.$$

Here

$$\int_{G \cap B\left(z, \frac{\delta}{2}\right)} \left| K_{n-2[n\eta]}^{C}\left(t, z\right) \right|^{2} dA\left(t\right)$$

$$\leq \int_{G} \left| K_{n-2[n\eta]}^{C}\left(t, z\right) \right|^{2} dA\left(t\right) = K_{n-2[n\eta]}^{C}\left(z, z\right).$$

Moreover, by Cauchy-Schwarz, the maximum-modulus principle, and Theorem 2.1(b),

$$\left\| K_{n-2[n\eta]}^{C}\left(\cdot, z\right) \right\|_{L_{\infty}(G)} \leq C n^{2}.$$

Hence, using $\lambda_{n-2[n\eta]}^{C}\left(z\right) = 1/K_{n-2[n\eta]}^{C}\left(z, z\right)$, and Theorem 2.1(b) again, (3.4) gives

$$\lambda_{n}\left(z\right) / \lambda_{n-2[\eta n]}^{C}\left(z\right)$$

$$\leq w\left(z_{0}\right)\left(1 + \varepsilon\right) + C n^{2} r^{2[\eta n]}.$$

This bound holds uniformly for $z \in B\left(z_{0}, \frac{\delta}{2}\right) \cap \partial G$ and $z_{0} \in J$. Now let $M > 0$ and $\{z_{n}\} \subset \partial G$ with $|z_{n} - z_{0}| \leq \frac{M}{n}$, $n \geq 1$. We then obtain for $n \geq \frac{2M}{\delta}$,

$$\lambda_{n}\left(z_{n}\right) / \lambda_{n-2[n\eta]}^{C}\left(z_{n}\right)$$

$$\leq w\left(z_{0}\right)\left(1 + \varepsilon\right) + C n^{2} r^{[\eta n]}.$$

Using the asymptotic (2.2) in Theorem 2.1(b), we deduce that

$$\limsup_{n \to \infty} n^{2} \lambda_{n}\left(z_{n}\right) \leq \frac{2\pi}{\left| \phi'\left(z_{0}\right) \right|^{2}} w\left(z_{0}\right)\left(1 + \varepsilon\right)\left(1 - 2\eta\right)^{-2}.$$

As ε and $\eta > 0$ are arbitrary, we obtain, uniformly for $z_{0} \in J$ and such sequences $\{z_{n}\}$,

$$(3.5) \qquad \limsup_{n \to \infty} n^{2} \lambda_{n}\left(z_{n}\right) \leq \frac{2\pi}{\left| \phi'\left(z_{0}\right) \right|^{2}} w\left(z_{0}\right).$$

The converse inequality is a little more difficult, and *does* use regularity of the measure μ. Much as before, we let R_{z} and η be as above, and for $n > 2[\eta n]$, we set

$$P\left(t\right) = \frac{K_{n-2[n\eta]}\left(t, z\right)}{K_{n-2[n\eta]}\left(z, z\right)} R_{z}\left(t\right)^{[n\eta]}.$$

Much as before, we obtain

$$\lambda_n^C (z)$$

$$\leq \int_G |P|^2 \, dA$$

$$\leq \frac{w(z_0)^{-1}(1+\varepsilon)}{K_{n-2[n\eta]}(z,z)^2} \int_{G\cap B(z,\frac{\delta}{2})} \left|K_{n-2[n\eta]}(t,z)\right|^2 w(t) \, dA(t)$$

(3.6)
$$+\frac{\left\|K_{n-2[n\eta]}(\cdot,z)\right\|^2_{L_\infty(G\setminus B(z,\delta/2))}}{K_{n-2[n\eta]}(z,z)^2} r^{2[\eta n]} \int_{G\setminus B(z,\delta/2)} dA.$$

Here

$$\int_{G\cap B\left(z,\frac{\delta}{2}\right)} \left|K_{n-2[n\eta]}(t,z)\right|^2 w(t) \, dA(t)$$

$$\leq \int_G \left|K_{n-2[n\eta]}(t,z)\right|^2 d\mu(t) = K_{n-2[n\eta]}(z,z).$$

Next, by the regularity of μ, for any sequence $\{S_n\}$ of polynomials, with each S_n of degree $\leq n$, we have [27, Thm. 3.2.1(iii), p. 66],

$$\limsup_{n\to\infty} \left[\frac{\|S_n\|_{L_\infty(G)}}{\left(\int_G |S_n|^2 \, d\mu\right)^{1/2}}\right]^{1/n} \leq 1.$$

In particular,

$$\left\|K_{n-2[n\eta]}(\cdot,z)\right\|^2_{L_\infty(G)}$$

$$\leq (1+o(1))^n \int_G \left|K_{n-2[n\eta]}(t,z)\right|^2 d\mu(z) = (1+o(1))^n K_{n-2[n\eta]}(z,z).$$

Substituting these estimates in (3.6) gives for such z, z_0, and n,

$$\lambda_n^C (z) / \lambda_{n-2[n\eta]}(z)$$

$$\leq w(z_0)^{-1}(1+\varepsilon) + (1+o(1))^n r^{2[\eta n]} \int_G dA.$$

Now recall that $\{z_n\}$ is a sequence in ∂G with $|z_n - z_0| \leq M/n$ for $n \geq 1$. Setting $z = z_{n-2[\eta n]}$ and using our asymptotic Theorem 2.1(b) yields

$$\limsup_{n\to\infty} n^{-2}/\lambda_{n-2[n\eta]}\left(z_{n-2[n\eta]}\right)$$

$$\leq \left(\frac{2\pi}{|\phi'(z_0)|^2} w(z_0)\right)^{-1}(1+\varepsilon).$$

Now for any m large enough, we can write $m = n - 2[n\eta]$ for some n: indeed, the difference between successive elements, namely $(n+1-2[(n+1)\eta]) -$

$(n - 2[n\eta]) = 1 + 2([n\eta] - [(n+1)\eta])$ is no larger than 1. We deduce that

$$\limsup_{m\to\infty} m^{-2}/\lambda_m(z_m) \leq \left(\frac{2\pi}{|\phi'(z_0)|^2} w(z_0)\right)^{-1} (1+\varepsilon)(1-2\eta)^{-2}.$$

Here ε and η may be taken arbitrarily small. Together with (3.5), this gives the result.

(b) This was essentially proved in (a) - use the estimates established uniformly for $z \in B(z_0, \delta/2)$ and $z_0 \in J$. ∎

Remark
Note that we did not use regularity of μ in the upper bound. Thus for any measure μ on G, (3.5) holds. Of course, this is a familiar result in the context of measures on the real line, and holds even when w is not continuous at z_0 [16], [29].

Proof of Theorem 3.1 under the hypotheses of Theorem 1.3
The only difference is that we let

$$P(t) = \frac{K_{n-k[n\eta]}^C(t,z)}{K_{n-k[n\eta]}^C(z,z)} R_z(t)^{[n\eta]}$$

for the asymptotic upper bound, and

$$P(t) = \frac{K_{n-k[n\eta]}(t,z)}{K_{n-k[n\eta]}(z,z)} R_z(t)^{[n\eta]}$$

for the asymptotic lower bound, where R_z is the polynomial of degree, say, $\leq k$ of Definition 1.2. Here k is by hypothesis, independent of z. Moreover, $\eta \in \left(0, \frac{1}{k}\right)$. The rest of the details are as before. ∎

4. Proof of Theorems 1.1, 1.3 and Corollary 1.2

We begin with an inequality:

Lemma 4.1
Suppose that μ, μ^* are measures on G with $\mu \leq \mu^*$ in G. Then for $z, t \in \mathbb{C}$,

$$(4.1) \qquad \begin{aligned} &|K_n(z,t) - K_n^*(z,t)|/K_n(z,z) \\ &\leq \left(\frac{K_n(t,t)}{K_n(z,z)}\right)^{1/2} \left[1 - \frac{K_n^*(z,z)}{K_n(z,z)}\right]^{1/2}. \end{aligned}$$

Proof
The idea is to estimate the L_2 norm of $K_n - K_n^*$ over G, and then to use

Christoffel function estimates. Now

$$\int_G |(K_n - K_n^*)(z,t)|^2 \, d\mu(t)$$

$$= \int_G |K_n(z,t)|^2 \, d\mu(t) - 2\operatorname{Re} \int_G \left(\overline{K_n} K_n^*\right)(z,t) \, d\mu(t)$$

$$+ \int_G |K_n^*(z,t)|^2 \, d\mu(t)$$

$$= K_n(z,z) - 2K_n^*(z,z) + \int_G |K_n^*(z,t)|^2 \, d\mu(t),$$

by the reproducing kernel property. As $\mu \leq \mu^*$, we also have

$$\int_G |K_n^*(z,t)|^2 \, d\mu(t) \leq \int_G |K_n^*(z,t)|^2 \, d\mu^*(t) = K_n^*(z,z).$$

So

$$\int_G |(K_n - K_n^*)(z,t)|^2 \, d\mu(t)$$

(4.2) $$\leq (K_n - K_n^*)(z,z).$$

Next for any polynomial P of degree $\leq n-1$, we have the Christoffel function estimate

$$|P(t)| \leq K_n^{1/2}(t,t) \left(\int_G |P|^2 \, d\mu\right)^{1/2}.$$

Applying this to the polynomial $P(t) = (K_n - K_n^*)(z,t)$, and using (4.2) gives the result. ∎

From this we readily deduce:

Lemma 4.2
Assume that μ, μ^* are regular measures on G, and that G, J, μ are as in Theorem 1.1 or 1.3. Assume μ^* is also absolutely continuous with respect to planar measure in a relatively open subset of $\bar{G}$ set containing J. Assume that

$$d\mu = d\mu^* \text{ in } J.$$

Let $A > 0$. Then as $n \to \infty$,

(4.3) $$|(K_n - K_n^*)(z_n, t_n)|/n^2 = o(1),$$

uniformly for all sequences $\{z_n\}, \{t_n\} \subset \partial G$ with the following properties: for some $z_0 \in J$ we have for all $n \geq 1$, $|z_n - z_0| \leq A/n$, $|t_n - z_0| \leq A/n$.
Proof
We initially assume that

(4.4) $$d\mu \leq d\mu^* \text{ in } G.$$

Then the inequality (4.1) of Lemma 4.1 holds. Now we set $z = z_n$ and $t = t_n$ where $\{z_n\}, \{t_n\}$ are as above. By Theorem 3.1, uniformly for such

sequences, $\frac{K_n^*(z_n,z_n)}{K_n(z_n,z_n)} = 1 + o(1)$, for $(\mu^*)' = \mu' = w$ in J. Moreover,

$$K_n(z_n, z_n) \sim K_n(t_n, t_n) \sim n^2,$$

so uniformly for such $\{z_n\}, \{t_n\}$, (4.1) gives

$$|(K_n - K_n^*)(z_n, t_n)| / n^2 = o(1).$$

Now we drop the extra hypothesis (4.4). Define a measure ν by $\nu = \mu = \mu^*$ in J; and in $G\backslash J$, let

$$d\nu(t) = \max\{dist(t, J), w(t), w^*(t)\} dA(t) + d\mu_s(t) + d\mu_s^*(t),$$

where w, w^* and μ_s, μ_s^* are respectively the absolutely continuous and singular components of μ, μ^*. Then $\mu \leq \nu$ and $\mu^* \leq \nu$, and ν is regular as its absolutely continuous component is positive in G. Moreover, ν is absolutely continuous in an open subset of G whose boundary contains J, and $w^* = w$ in J. The case above shows that the reproducing kernels for μ and μ^* have the same asymptotics as that for ν, in the sense of (4.3), and hence the same asymptotics as each other. ■

We next approximate μ of Theorem 1.1 by a multiple $\mu^\#$ of Lebesgue measure on G, and then prove Theorem 1.1. Recall that $\tilde{K}_n$ is the normalized kernel, given by (1.4).

Lemma 4.3
Let G, J, μ be as in Theorem 1.1 or 1.3. Let $A > 0$, $\varepsilon \in \left(0, \frac{1}{2}\right)$ and choose $\delta > 0$ such that (3.3) holds. Then there exists n_0 such that for $n \geq n_0$, $z_0 \in J$ and $z, t \in B\left(z_0, \frac{\delta}{2}\right) \cap \partial G$,

$$(4.5) \qquad \left|\left(\tilde{K}_n - \tilde{K}_n^C\right)(z, t)\right| / n^2 \leq C\sqrt{\varepsilon},$$

Here C is independent of $n, z, t, z_0, \varepsilon, \delta$.
Proof
Fix $z_0 \in J$ and let $w^\#$ be the scaled Legendre weight

$$w^\# = w(z_0) \text{ in } G.$$

Note that

$$(4.6) \qquad K_n^\#(\cdot, \cdot) = \frac{1}{w(z_0)} K_n^C(\cdot, \cdot).$$

Because of our localization result Lemma 4.2, we may replace $d\mu$ by $w^*(t) dA(t)$, where

$$w^* = w \text{ in } B(z_0, \delta)$$

and

$$w^* = w(z_0) \text{ in } G\backslash B(z_0, \delta),$$

without affecting the asymptotics for $K_n(z, t) / n^2$ in the ball $B\left(z_0, \frac{\delta}{2}\right)$. (Note that ε and δ play no role in Lemma 4.2). So in the sequel, we assume

that $w = w(z_0) = w^{\#}$ in $G \backslash B(z_0, \delta)$, while not changing w in $B(z_0, \delta)$. Observe that (3.3) implies that

(4.7) $$(1 + \varepsilon)^{-1} \leq \frac{w}{w^{\#}} \leq 1 + \varepsilon, \text{ in } G.$$

Let

$$w_1 = (1 + \varepsilon)w.$$

Then $w_1 \geq w^{\#}$ in G, and if K_n^1 denotes the kernel function for w_1, we have

$$K_n^1(\cdot, \cdot) = \frac{1}{1 + \varepsilon} K_n(\cdot, \cdot).$$

By Lemma 4.1, for all $u, v \in \mathbb{C}$,

$$\left| K_n^{\#}(u, v) - K_n^1(u, v) \right| / K_n^{\#}(u, u)$$

(4.8) $$\leq \left(\frac{K_n^{\#}(v, v)}{K_n^{\#}(u, u)} \right)^{1/2} \left[1 - \frac{K_n^1(u, u)}{K_n^{\#}(u, u)} \right]^{1/2}.$$

Here

$$\frac{K_n^1(u, u)}{K_n^{\#}(u, u)} = \frac{1}{1 + \varepsilon} \frac{K_n(u, u)}{K_n^{\#}(u, u)} \geq \frac{1}{(1 + \varepsilon)^2},$$

in view of (4.7). Then (4.6) and (4.8) give for all $z, t \in B\left(z_0, \frac{\delta}{2}\right) \cap \partial G$,

$$\left| K_n^{\#}(z, t) - \frac{1}{1 + \varepsilon} K_n(z, t) \right| / K_n^{\#}(z, z)$$

$$\leq \left(\frac{K_n^C(t, t)}{K_n^C(z, z)} \right)^{1/2} \left[1 - \frac{1}{(1+\varepsilon)^2} \right]^{1/2}.$$

Using that $K_n^{\#}(z, t) = O(n^2)$ and $K_n^{\#}(z, z) \geq Cn^2$, we obtain for all $z, t \in B\left(z_0, \frac{\delta}{2}\right) \cap \partial G$,

$$\left| \left(K_n - K_n^{\#} \right)(z, t) \right| / n^2 \leq C\sqrt{\varepsilon},$$

so (4.6) gives

(4.9) $$\left| w(z_0) K_n(z, t) - K_n^C(z, t) \right| / n^2 \leq C\sqrt{\varepsilon}.$$

Next, by (3.3), we have for $z, t \in B\left(z_0, \frac{\delta}{2}\right) \cap G$,

$$(1 + \varepsilon)^{-1} \leq \frac{w(z)^{1/2} w(t)^{1/2}}{w(z_0)} \leq 1 + \varepsilon.$$

Then for such z, t and z_0,

$$\left|(\tilde{K}_n - \tilde{K}_n^C)(z,t)\right|/n^2$$

$$= \left|\frac{w(z)^{1/2}w(t)^{1/2}}{w(z_0)}w(z_0)K_n(z,t) - K_n^C(z,t)\right|/n^2$$

$$\leq \left|\frac{w(z)^{1/2}w(t)^{1/2}}{w(z_0)} - 1\right|w(z_0)|K_n(z,t)|/n^2 + \left|w(z_0)K_n(z,t) - K_n^C(z,t)\right|/n^2$$

$$\leq C\sqrt{\varepsilon},$$

by (4.9). ∎

Our last lemma is a growth estimate for polynomials:

Lemma 4.4
Let $\eta > 0, M > 0$. There exists $C > 0$ such that for $n \geq 1$ and polynomials P of degree $\leq n$,

$$(4.10) \qquad\qquad |P(z)| \leq C \sup_{J(\eta) \cap \partial G} |P|$$

for all z with $dist(z, J \cap \partial G) \leq \frac{M}{n}$.
Proof
We use the fact that $J(\eta) \cap \partial G$ consists of finitely many smooth arcs of length at least 2η, so that the Green's function for the complement of the set is well behaved. More precisely, let $g(z)$ denote the Green's function for $\mathbb{C} \setminus (J(\eta) \cap \partial G)$ with pole at ∞. Since $\Gamma = \partial G \in C(1, \alpha)$ with $\alpha > \frac{1}{2}$, each arc of $J(\eta) \cap \partial G$ is smooth and of length $\geq 2\eta$. Then for $z \in \mathbb{C} \setminus (J(\eta) \cap \partial G)$ with $dist(z, J) \leq \eta/2$,

$$g(z) \leq C dist(z, J(\eta) \cap \partial G).$$

To see this, we use the classic representation of the Greens function in terms of the equilibrium potential for $J(\eta) \cap \partial G$, and the fact that the equilibrium density is continuous, and so bounded, in $J(\eta/2) \cap \partial G$. See, for example, [24, p. 216]. We now use the Bernstein-Walsh inequality

$$|P(z)| \leq e^{ng(z)} \sup_{J(\eta) \cap \partial G} |P|$$

and the estimate above for the Green's function, which gives

$$g(z) \leq C dist(z, J \cap \partial G) \leq C\frac{M}{n}.$$

∎

Proof of Theorems 1.1 and 1.3
Let $A, \varepsilon > 0$. Choose $\delta > 0$ such that (3.3) holds. Now cover J by, say, L balls $B\left(\hat{z}_j, \frac{\delta}{2}\right)$, $1 \leq j \leq L$, each of diameter δ. For each j, there

exists a threshhold $n_0 = n_0(j)$ for which (4.5) holds for $n \geq n_0(j)$ and $z, t \in B\left(\hat{z}_j, \frac{\delta}{2}\right) \cap \partial G$. Let n_1 denote the largest of these. Then we obtain, for $n \geq n_1$, $z_0 \in J$, $z, t \in B\left(z_0, \frac{\delta}{2}\right) \cap \partial G$,

$$\left|\left(\tilde{K}_n - K_n^C\right)(z, t)\right|/n^2 \leq C\sqrt{\varepsilon}.$$

It follows that if $\{z_n\}, \{t_n\}$ are sequences in ∂G such that for some $M > 0$ and $z_0 \in J$, we have $|z_n - z_0| \leq \frac{M}{n}$ and $|t_n - z_0| \leq \frac{M}{n}$ for all n, then

(4.11) $$\lim_{n \to \infty} \left|\left(\tilde{K}_n - K_n^C\right)(z_n, t_n)\right|/n^2 = 0.$$

In particular, suppose that for some given a, b,

(4.12) $z_n = z_0 + \dfrac{a_n}{n}$ and $t_n = z_0 + \dfrac{b_n}{n}$, with $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$.

From (4.11) and the uniform convergence in Theorem 2.1(c) for Carleman polynomials, we have

$$\lim_{n \to \infty} \frac{K_n\left(z_0 + \frac{a_n}{n}, z_0 + \frac{b_n}{n}\right)}{K_n(z_0, z_0)} = \lim_{n \to \infty} \frac{\tilde{K}_n\left(z_0 + \frac{a_n}{n}, z_0 + \frac{b_n}{n}\right)}{\tilde{K}_n(z_0, z_0)}$$

$$= \lim_{n \to \infty} \frac{K_n^C\left(z_0 + \frac{a_n}{n}, z_0 + \frac{b_n}{n}\right)}{K_n^C(z_0, z_0)}$$

$$= H\left(a\phi'(z_0)\overline{\phi(z_0)} + \overline{b\phi'(z_0)}\phi(z_0)\right).$$

We claim that then this implies for the given a, b,

(4.13) $$\lim_{n \to \infty} \frac{K_n\left(z_0 + \frac{a}{n}, z_0 + \frac{b}{n}\right)}{K_n(z_0, z_0)} = H\left(a\phi'(z_0)\overline{\phi(z_0)} + \overline{b\phi'(z_0)}\phi(z_0)\right).$$

This will follow if we can show that $\left\{\frac{K_n\left(z_0 + \frac{s}{n}, z_0 + \frac{t}{n}\right)}{K_n(z_0, z_0)}\right\}_n$ is a normal family for s, t in compact subsets of the plane. To show the latter, recall from Theorem 3.1(b) and Cauchy-Schwarz,

$$\sup_{u, v \in J(\eta) \cap \partial G} |K_n(u, v)| \leq Cn^2.$$

Lemma 4.4 gives that for all u, v with $\mathrm{dist}(u, J \cap \partial G) \leq \frac{M}{n}$ and $\mathrm{dist}(v, J \cap \partial G) \leq \frac{M}{n}$,

$$|K_n(u, v)| \leq Cn^2 \left(\leq CK_n(z_0, z_0)\right).$$

The desired normality then follows. Finally, we can choose sequences of infinitely many distinct a, b with a finite limit point for which there exist sequences $\{z_n\}, \{t_n\}$ in $J(\eta) \cap \partial G$ with the properties (4.11) and (4.12). (In fact, there is a continuum of such a, b). The normality of $\left\{\frac{K_n\left(z_0 + \frac{a}{n}, z_0 + \frac{b}{n}\right)}{K_n(z_0, z_0)}\right\}_n$ and convergence continuation theorems then show that (4.13) holds uniformly for a, b in compact subsets of the plane. $\blacksquare$

Proof of Corollary 1.4

Exactly as in the proof of Corollary 2.2, we have the Taylor series identity
(2.8) for K_n instead of K_n^C. Moreover, as at (4.13), we have the uniform
convergence of $\frac{1}{K_n(z_0,z_0)} K_n \left(z_0 + \frac{a}{n}, z_0 + \frac{b}{n} \right)$ for a, b in compact subsets of
the complex plane. Recalling the identity (2.9) from Theorem 2.1, the re-
sult now follows, for each fixed $z \in J$. However, the uniformity in z still
must be proved separately. For this, we just use the uniform boundedness
in z also. $\blacksquare$

Acknowledgement

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