SOME RECENT METHODS FOR ESTABLISHING UNIVERSALITY LIMITS

D. S. LUBINSKY

Abstract. We survey some recent methods for establishing universality limits for random matrices in the unitary case. These include Levin's method using a Markov-Bernstein inequality, a comparison inequality of the author, and a method based on complex analysis and reproducing kernels. We focus on the bulk of the spectrum for measures with compact support, but the methods may also be used at the soft or hard edge, and for measures with unbounded support.

1. Introduction[1]

Let $\mathcal{M}(n)$ denote the space of n by n Hermitian matrices $M = (m_{ij})_{1 \leq i,j \leq n}$. Consider a probability distribution on $\mathcal{M}(n)$,

$$\begin{aligned} P^{(n)}(M) &= cw(M)\,dM \\ &= cw(M)\left(\prod\nolimits_{j=1}^{n} dm_{jj}\right)\left(\prod\nolimits_{j<k} d(\operatorname{Re} m_{jk})\,d(\operatorname{Im} m_{jk})\right). \end{aligned}$$

Here $w(M)$ is a function defined on $\mathcal{M}(n)$, and c is a normalizing constant. The most important case is

$$w(M) = \exp(-2n\ Q(M)),$$

for appropriate functions Q defined on $\mathcal{M}(n)$. In particular, the choice

$$Q(M) = \operatorname{tr}\left(M^2\right),$$

leads to the Gaussian unitary ensemble (apart from scaling) that was considered by Wigner, in the context of scattering theory for heavy nuclei. One may identify $P^{(n)}$ above with a probability density on the eigenvalues $x_1 \leq x_2 \leq ... \leq x_n$ of M,

$$P^{(n)}(x_1, x_{2,}..., x_n) = c\left(\prod_{j=1}^{m} w(x_j)\right)\left(\prod\nolimits_{i<j}(x_i - x_j)^2\right).$$

See [15, p. 102 ff.]. Again, c is a normalizing constant.

Date: May 12, 2009.

[1]Research supported by NSF grant DMS0400446 and US-Israel BSF grant 2004353

It is at this stage that orthogonal polynomials arise [15], [43]. Let μ be a finite positive Borel measure with compact support and infinitely many points in the support. Define orthonormal polynomials

$$p_n(x) = \gamma_n x^n + ..., \gamma_n > 0,$$

$n = 0, 1, 2, ...$, satisfying the orthonormality conditions

$$\int p_j p_k d\mu = \delta_{jk}.$$

Throughout we use w to denote the Radon-Nikodym derivative of μ. The nth reproducing kernel for μ is

$$K_n(x, y) = \sum_{k=0}^{n-1} p_k(x) p_k(y), \tag{1.1}$$

and the normalized kernel is

$$\widetilde{K}_n(x, y) = w(x)^{1/2} w(y)^{1/2} K_n(x, y). \tag{1.2}$$

When

$$w(x) = e^{-2nQ(x)} dx,$$

there is the basic formula for the probability distribution $P^{(n)}$ [15, p.112]:

$$P^{(n)}(x_1, x_2, ..., x_n) = \frac{1}{n!} \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le n}.$$

One may use this to compute a host of statistical quantities - for example the probability that a fixed number of eigenvalues of a random matrix lie in a given interval. One particularly important quantity is the $m-$point correlation function for $M(n)$ [15, p. 112]:

$$\begin{aligned} R_m(x_1, x_{2,}..., x_m) &= \frac{n!}{(n-m)!} \int ... \int P^{(n)}(x_1, x_2 ..., x_n)\, dx_{m+1}\ dx_{m+2}\ ...dx_n \\ &= \det\left(\tilde{K}_n(x_i, x_j)\right)_{1 \le i,j \le m}. \end{aligned}$$

The *universality limit in the bulk* asserts that for fixed $m \ge 2$, and ξ in the interior of the support of $\{\mu\}$, and real $a_1, a_2, ..., a_m$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\xi,\xi)^m} R_m\left(\xi + \frac{a_1}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{a_2}{\tilde{K}_n(\xi,\xi)}, ..., \xi + \frac{a_m}{\tilde{K}_n(\xi,\xi)}\right) \\ = \ &\det\left(\frac{\sin\pi(a_i - a_j)}{\pi(a_i - a_j)}\right)_{1 \le i,j \le m}. \end{aligned}$$

Of course, when $a_i = a_j$, we interpret $\frac{\sin\pi(a_i - a_j)}{\pi(a_i - a_j)}$ as 1. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$\lim_{n\to\infty} \frac{\tilde{K}_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{\tilde{K}_n(\xi,\xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{1.3}$$

Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. The term universal is quite justified: the limit on the right-hand side of (1.3) is independent of ξ, but more importantly is independent of the underlying measure.

Typically, the limit (1.3) is established uniformly for a, b in compact subsets of the real line, but if we remove the normalization from the outer K_n, we can also establish its validity for complex a, b, that is,

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin \pi (a-b)}{\pi (a-b)}. \tag{1.4}$$

The most obvious approach is to use the Christoffel-Darboux formula,

$$K_n(u,v) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(u)\, p_{n-1}(v) - p_{n-1}(u)\, p_n(v)}{u-v}, \ u \neq v; \tag{1.5}$$

$$K_n(u,u) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n'(u)\, p_{n-1}(u) - p_n(u)\, p_{n-1}'(u)\right), \tag{1.6}$$

leading to (for $b \neq a$),

$$\begin{aligned} & \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} \\ = \ & w(\xi)\, \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}\right) p_{n-1}\left(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right) - p_{n-1}\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}\right) p_n\left(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{a-b} \\ =\ & : w(\xi)\, \frac{\Gamma_n}{a-b}. \end{aligned} \tag{1.7}$$

It is clear from this that if we have sufficient knowledge of the asymptotic behavior of p_n as $n \to \infty$, then we can substitute in these asymptotics, and deduce universality. Of course, the question is: what is sufficient? For classical weights, such as Jacobi weights, complete asymptotic expansions (such as Plancherel-Rotach asymptotics) are available, and these yield far more than universality.

In recent years, the deep and powerful Riemann-Hilbert methods have also yielded far more than is required for universality. Originally, they were applied to $w = e^{-Q}$ or varying weights $w = e^{-nQ}$, for analytic Q [4], [5], [8], [9], [15], [17]. The ∂-bar method has permitted their application to non analytic Q, for example, when Q'' satisfies a Lipschitz condition [41], [42]. The Riemann-Hilbert literature is extensive; some recent references include [3], [10], [11], [12], [13], [14], [16], [24], [25], [26], [27], [40], [59]. Other useful methods arise from techniques in mathematical physics, probability theory, and operator theory [1], [6], [7], [18], [20], [21], [23], [46], [47], [51], [56], [57], [58], [60]. We shall not survey these methods here. We shall simply survey three recent methods, giving an outline of proofs, and some relevant references.

In the sequel, $C, C_1, C_2, ...$ denote positive constants independent of n, x, t, z and polynomials of degree $\leq n$. The same symbol does not necessarily denote the same constant in different occurrences. For $x \geq 0$, we let $[x]$ denote the greatest integer $\leq x$. For sequences $\{c_n\}$ and $\{d_n\}$, we write

$$c_n \sim d_n$$

if there exist positive constants C_1 and C_2 such that for all n,

$$C_1 \leq c_n/d_n \leq C_2.$$

Similar notation is used for functions, and sequences of functions. While K_n is associated with μ, we shall use K_n^* for the kernel associated with a measure μ^*. For other measures such as ν, we shall use K_n^ν. Similar superscripts are used for other orthogonal polynomial quantities.

2. The Chebyshev weight

Let us start with the Chebyshev weight

$$w(x) = \frac{1}{\sqrt{1-x^2}}, x \in (-1,1),$$

for this gives substantial insight about the general case. Then $p_0 = \frac{1}{\sqrt{\pi}}$, while for $n \geq 1$,

$$\frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2} \text{ and } p_n(\cos\theta) = \sqrt{\frac{2}{\pi}} \cos(n\theta).$$

Moreover, if

$$\xi = \cos\theta \in (-1,1),$$

straightforward manipulations give

$$\begin{aligned} \tilde{K}_n(\xi,\xi) &= \frac{1}{\pi\sqrt{1-\xi^2}} \left(1 + 2\sum_{j=1}^{n-1} (\cos j\theta)^2\right) \\ &= \frac{1}{\pi \sin\theta} \left(n - \frac{1}{2} + \frac{\sin(2n-1)\theta}{2\sin\theta}\right) = \frac{n}{\pi\sin\theta} + O(1). \end{aligned} \tag{2.1}$$

Let us set, for a given ξ, a and b, with $b \neq a$,

$$\begin{aligned} \xi + \frac{a}{\tilde{K}_n(\xi,\xi)} &= \cos\theta_n; \\ \xi + \frac{b}{\tilde{K}_n(\xi,\xi)} &= \cos(\theta_n + \delta_n). \end{aligned}$$

Here, expanding to a Taylor series of first order,

$$\begin{aligned} \frac{a-b}{\tilde{K}_n(\xi,\xi)} &= \cos\theta_n - \cos(\theta_n+\delta_n) \\ &= \delta_n \sin\theta_n + O\left(\delta_n^2\right), \end{aligned}$$

so $\delta_n = O\left(\frac{1}{n}\right)$ and

$$\begin{aligned}
\delta_n &= \frac{a-b}{\sin\theta_n \tilde{K}_n(\xi,\xi)}\left(1+O(\delta_n)\right) \\
&= \frac{a-b}{\sin\theta_n \tilde{K}_n(\xi,\xi)} + O\left(n^{-2}\right). \qquad (2.2)
\end{aligned}$$

It is also easy to see that

$$\cos\theta_n = \cos\theta + O\left(\frac{1}{n}\right); \;\; \sin\theta_n = \sin\theta + O\left(\frac{1}{n}\right). \qquad (2.3)$$

Then from (2.1),

$$\delta_n = \frac{\pi(a-b)}{n} + O\left(\frac{1}{n^2}\right). \qquad (2.4)$$

Then we can express the numerator Γ_n in the right-hand side of (1.7) as

$$\begin{aligned}
\Gamma_n &= \frac{1}{\pi}\left[(\cos n\theta_n)(\cos(n-1)(\theta_n+\delta_n)) - (\cos(n-1)\theta_n)\cos n(\theta_n+\delta_n)\right] \\
&= \frac{1}{2\pi}\left[\begin{array}{c} \cos((2n-1)\theta_n + (n-1)\delta_n) + \cos(\theta_n - (n-1)\delta_n) \\ -\cos((2n-1)\theta_n + n\delta_n) - \cos(\theta_n + n\delta_n) \end{array}\right] \\
&= \frac{1}{\pi}\left[\sin\left((2n-1)\theta_n + \left(n-\frac{1}{2}\right)\delta_n\right)\sin\left(\frac{\delta_n}{2}\right) + \sin\left(\theta_n + \frac{\delta_n}{2}\right)\sin\left(\left(n-\frac{1}{2}\right)\delta_n\right)\right].
\end{aligned}$$

(2.5)

Here in the first step, we used the identity

$$\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$$

and in the second step, we used

$$\cos A - \cos B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{B-A}{2}\right).$$

Using (2.3) and (2.4), we continue (2.5) as

$$\begin{aligned}
\Gamma_n &= \frac{1}{\pi}\left[O\left(\frac{1}{n}\right) + \left(\sin\theta + O\left(\frac{1}{n}\right)\right)\left(\sin(\pi(a-b)) + O\left(\frac{1}{n}\right)\right)\right] \\
&= \frac{1}{\pi}(\sin\theta)\sin(\pi(a-b)) + O\left(\frac{1}{n}\right).
\end{aligned}$$

Substituting into (1.7), we obtain

$$\frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} = \frac{\sin(\pi(a-b))}{\pi(a-b)} + O\left(\frac{1}{n}\right).$$

Thus we have the universality limit (1.3) for a fixed $\xi \in (-1,1)$ and for fixed a, b with $a \neq b$.

3. Levin's Method

It is clear that the proof in the previous section requires modification if $b = a$. Indeed, one of the problems of substituting asymptotics for p_n and p_{n-1} into (1.7) is treating the case where b is close to a. In the past this has been circumvented by using higher order terms in asymptotics for p_n. It was Eli Levin who first observed that first order asymptotics suffice, and that remainders can be estimated using a Markov-Bernstein inequality. This was applied in [31] to exponential weights. We illustrate the method in this section for measures with compact support.

Assume that our measure μ has support $[-1, 1]$. Fix $\xi = \cos\theta \in (-1, 1)$, and assume that uniformly for x in a neighborhood of ξ, we have

$$\tilde{K}_n(x, x) = \frac{n}{\pi\sqrt{1-x^2}}(1 + o(1)). \tag{3.1}$$

Assume, moreover, uniformly for such $x = \cos s$,

$$p_n(x)\, w(x)^{1/2}\left(1 - x^2\right)^{1/4} = \sqrt{\frac{2}{\pi}}\cos(ns + h(s)) + o(1), \tag{3.2}$$

where h is a continuously differentiable function with bounded derivative. We also assume that w is positive and continuous near ξ. For such measures, we also have

$$\frac{\gamma_{n-1}}{\gamma_n} = \frac{1}{2} + o(1). \tag{3.3}$$

We note that if μ satisfies Szegő's condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty,$$

while μ is absolutely continuous, and μ' is continuous in a neighborhood of ξ, then (3.1) is true [39], [44], [45]. If in addition, μ' satisfies a Lipschitz condition of order greater than $\frac{1}{2}$ near ξ, then (3.2) is true [22, p. 246, Table II]. Of course, more general results are available, but these are easy to formulate.

We shall use the Christoffel-Darboux formula (1.5) and its confluent form (1.6). We assume $b \neq 0$, set $a = 0$ (effectively, this is a change of variable

$$\xi \to \xi + \frac{a}{\tilde{K}_n(\xi, \xi)}),$$

and establish the limit

$$\lim_{n\to\infty} \tilde{K}_n\left(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)}, \xi\right) / \tilde{K}_n(\xi, \xi) = \frac{\sin \pi b}{\pi b}, \tag{3.4}$$

uniformly for b in compact subsets of the real line. Let us set

$$\xi_{n,b} = \xi + \frac{b}{\tilde{K}_n(\xi, \xi)} = \xi + O\left(\frac{1}{n}\right), \tag{3.5}$$

recall (3.1). Also write

$$\xi_{n,b} = \cos \theta_{n,b};\ \xi = \cos \theta,$$

so that

$$\begin{aligned} \frac{b}{\tilde{K}_n(\xi,\xi)} &= \cos\theta_{n,b} - \cos\theta \\ &= -(\sin\theta)(\theta_{n,b}-\theta) + O\left((\theta_{n,b}-\theta)^2\right). \end{aligned}$$

Then

$$\theta_{n,b} - \theta = -\frac{b}{(\sin\theta)\,\tilde{K}_n(\xi,\xi)} + O\left(n^{-2}\right) = -\frac{\pi b}{n} + o\left(n^{-1}\right) \tag{3.6}$$

so the asymptotics for p_n and p_{n-1} at $\xi_{n,b}$ take the form

$$\begin{aligned} p_n\left(\xi_{n,b}\right) &= w(\xi)^{-1/2}\left(1-\xi^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\left(n\theta_{n,b} + h\left(\theta_{n,b}\right)\right) + o(1) \\ &= w(\xi)^{-1/2}\left(1-\xi^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\left(n\theta + h(\theta) - \pi b\right) + o(1); \end{aligned} \tag{3.7}$$

(3.8)

$$p_{n-1}\left(\xi_{n,b}\right) = w(\xi)^{-1/2}\left(1-\xi^2\right)^{-1/4}\sqrt{\frac{2}{\pi}}\cos\left((n-1)\theta + h(\theta) - \pi b\right) + o(1).$$

The Christoffel-Darboux formula in the form (1.7) gives

$$\begin{aligned} &K_n\left(\xi_{n,b},\xi\right)/K_n(\xi,\xi) \\ = \;& w(\xi)\frac{\gamma_{n-1}}{\gamma_n}\left(\frac{p_n\left(\xi_{n,b}\right)p_{n-1}(\xi) - p_{n-1}\left(\xi_{n,b}\right)p_n(\xi)}{b}\right). \end{aligned}$$

Inserting here the asymptotics (3.2), (3.3), (3.7), (3.8), we obtain uniformly for b in a compact subset of $\mathbb{R}\backslash\{0\}$,

$$\begin{aligned} &K_n\left(\xi_{n,b},\xi\right)/K_n(\xi,\xi) \\ = \;& \frac{1}{\pi b}\left(1-\xi^2\right)^{-1/2}\left\{\begin{array}{c} \cos\left(n\theta + h(\theta) - \pi b\right)\cos\left((n-1)\theta + h(\theta)\right) \\ -\cos\left((n-1)\theta + h(\theta) - \pi b\right)\cos\left(n\theta + h(\theta)\right) \end{array}\right\} + o(1). \end{aligned}$$

Using some elementary trigonometry, as in the previous section, the cosine terms in $\{\}$ are reduced to

$$(\sin \pi b)\sin\theta,$$

and we finally obtain

$$K_n\left(\xi + \frac{b}{\tilde{K}_n(\xi,\xi)},\xi\right)/K_n(\xi,\xi) = \frac{\sin \pi b}{\pi b} + o(1).$$

This gives the result, but the uniformity in b follows only for b in compact subsets of $\mathbb{R}\backslash\{0\}$. For $b = 0$, the result is immediate.

Now comes Levin's main idea on bounding the tail. His method shows that given a sequence $\{b_n\}$ of non-zero numbers with limit 0, we have

$$\lim_{n\to\infty} K_n\left(\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)},\xi\right)/K_n(\xi,\xi)=1. \tag{3.9}$$

This will give the uniformity in b. We again use the Christoffel-Darboux formula, and expand $p_n\left(\xi+b/\tilde{K}_n(\xi,\xi)\right)$ and $p_{n-1}\left(\xi+b/\tilde{K}_n(\xi,\xi)\right)$ about ξ_n to the second order:

$$\begin{aligned}
&K_n\left(\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)},\xi\right)/K_n(\xi,\xi)\\
=\;&\frac{1}{b_n}w(\xi)\frac{\gamma_{n-1}}{\gamma_n}\left\{p_n\left(\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)}\right)p_{n-1}(\xi)-p_{n-1}\left(\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)}\right)p_n(\xi)\right\}\\
=\;&\frac{1}{b_n}w(\xi)\frac{\gamma_{n-1}}{\gamma_n}\left\{\begin{array}{c}[p_n(\xi)p_{n-1}(\xi)-p_{n-1}(\xi)p_n(\xi)]\\ +\frac{b_n}{\tilde{K}_n(\xi,\xi)}\left[p_n'(\xi)p_{n-1}(\xi)-p_{n-1}'(\xi)p_n(\xi)\right]\\ +\frac{1}{2}\left(\frac{b_n}{\tilde{K}_n(\xi,\xi)}\right)^2\left[p_n''(r)p_{n-1}(\xi)-p_{n-1}''(s)p_n(\xi)\right]\end{array}\right\}+o(1),
\end{aligned}$$

where r and s are between ξ and $\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)}$. Using the confluent form (1.6) of the Christoffel-Darboux formula, we continue this as

$$0+1+O\left(\frac{|b_n|}{n^2}\right)\left(\max_J|p_n''|+\max_J|p_{n-1}''|\right)\left(\max_J|p_n|+\max_J|p_{n-1}|\right). \tag{3.10}$$

Here J is some interval containing ξ in its interior. Now the asymptotic (3.2) ensures that p_n is uniformly bounded in some open interval containing ξ. To bound p_n'', we use the Bernstein inequality

$$|P'(t)|\le\frac{n}{\sqrt{1-t^2}}\|P\|_{L_\infty[-1,1]},\ t\in(-1,1).$$

This is valid for polynomials P of degree $\le n$. In particular, $|P'|$ grows no faster than $n\|P\|_{L_\infty[-1,1]}$ in any compact subset of $(-1,1)$. Applying this twice, with appropriate intervals, we obtain

$$\max_J|p_n''|+\max_J|p_{n-1}''|=O\left(n^2\right),$$

so we can continue (3.10) as

$$1+O(|b_n|)=1+o(1).$$

Thus for any sequence $\{b_n\}$ of non-zero numbers with limit 0,

$$\lim_{n\to\infty} K_n\left(\xi+\frac{b_n}{\tilde{K}_n(\xi,\xi)},\xi\right)/K_n(\xi,\xi)=1.$$

We have thus proven that

$$\lim_{n\to\infty}\frac{K_n\left(\xi+\frac{b}{\tilde{K}_n(\xi,\xi)},\xi\right)}{K_n(\xi,\xi)}=\frac{\sin(\pi b)}{\pi b},$$

uniformly for b in compact subsets of the real line. Because of the uniformity in b, and local uniformity in ξ, we can, as noted above, make the substitution $\xi \to \xi + \frac{a}{\tilde{K}_n(\xi,\xi)}$ and deduce

Proposition 3.1
Under the assumptions (3.1), (3.2), and (3.3), we have

$$\lim_{n\to\infty} \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n\left(\xi,\xi\right)} = \frac{\sin \pi\left(a-b\right)}{\pi\left(a-b\right)},$$

uniformly for a, b in compact subsets of the real line.

Levin's method should be useful whenever we have both (i) first order asymptotics for orthogonal polynomials, and (ii) a suitable Markov-Bernstein inequality.

4. A Comparison Inequality

The method of the previous section, like all of its predecessors, requires asymptotics for the orthonormal polynomials themselves. Inspired by Percy Deift's 60th birthday conference, the author came up with an inequality that allows one to establish universality when asymptotics for p_n are not available, but asymptotics are available for the reproducing kernel

$$K_n\left(x,x\right) = \sum_{j=0}^{n-1} p_j^2\left(x\right)$$

along the diagonal. Since $\frac{1}{n}K_n\left(x,x\right)$ is an average of squares of orthonormal polynomials, it is likely to have more regular behavior. Moreover, it satisfies an extremal property,

$$K_n\left(x,x\right) = \sup_{\deg(P)\leq n-1} P^2\left(x\right) / \int P^2 d\mu.$$

This is more commonly formulated for the *Christoffel function*

$$\lambda_n\left(x\right) = \frac{1}{K_n\left(x,x\right)},$$

as

$$\lambda_n\left(x\right) = \inf_{\deg(P)\leq n-1} \int P^2 d\mu / P^2\left(x\right). \tag{4.1}$$

Christoffel functions have been studied for many decades, and serve as a cornerstone of what one might call the *Hungarian approach* to orthogonal polynomials - a theme studied by Erdős, Turán, Freud, Nevai, Máté, Totik, and others [22], [39], [45], [54].

The key inequality is:

Lemma 4.1
Assume that μ *and* μ^* *are measures on the real line such that* $\int x^j d\mu^*(x)$ *is finite for each* $j \geq 0$*, and such that*

$$\mu \leq \mu^*.$$

Let K_n^* *denote the nth reproducing kernel for* μ^**. Then for all real* x, y*,*

$$\begin{aligned} & \left|K_n(x,y) - K_n^*(x,y)\right| / K_n(x,x) \\ \leq \; & \left(\frac{K_n(y,y)}{K_n(x,x)}\right)^{1/2} \left[1 - \frac{K_n^*(x,x)}{K_n(x,x)}\right]^{1/2}. \end{aligned} \tag{4.2}$$

Proof
The idea is to estimate the L_2 norm of $K_n(x,t) - K_n^*(x,t)$ and then to use Christoffel function estimates. Now

$$\begin{aligned} & \int \left(K_n(x,t) - K_n^*(x,t)\right)^2 d\mu(t) \\ = \; & \int K_n^2(x,t)\, d\mu(t) - 2\int K_n(x,t)\, K_n^*(x,t)\, d\mu(t) + \int K_n^{*2}(x,t)\, d\mu(t) \\ = \; & K_n(x,x) - 2K_n^*(x,x) + \int K_n^{*2}(x,t)\, d\mu(t), \end{aligned}$$

by the reproducing kernel property. As $\mu \leq \mu^*$, we also have

$$\int K_n^{*2}(x,t)\, d\mu(t) \leq \int K_n^{*2}(x,t)\, d\mu^*(t) = K_n^*(x,x).$$

So

$$\begin{aligned} & \int \left(K_n(x,t) - K_n^*(x,t)\right)^2 d\mu(t) \\ \leq \; & K_n(x,x) - K_n^*(x,x). \end{aligned} \tag{4.3}$$

Next for any polynomial P of degree $\leq n-1$, (4.1) yields the Christoffel function estimate

$$|P(y)| \leq K_n(y,y)^{1/2} \left(\int P^2 d\mu\right)^{1/2}. \tag{4.4}$$

Applying this to $P(t) = K_n(x,t) - K_n^*(x,t)$ and using (4.3) gives, for all $x, y \in [-1,1]$,

$$\begin{aligned} & \left|K_n(x,y) - K_n^*(x,y)\right| \\ \leq \; & K_n(y,y)^{1/2} \left[K_n(x,x) - K_n^*(x,x)\right]^{1/2}. \end{aligned}$$

■

The essential feature is that in the left-hand side of (4.2), we have $K_n(x,y)$ with x and y different, while the right-hand side involves values of K_n "along the diagonal". If $K_n^*(x,x)$ is close to $K_n(x,x)$, and $K_n(y,y) = O(K_n(x,x))$, then the left-hand side is small. Thus, assume that for x in a neighborhood $\mathcal{N}$ of ξ,

$$\tilde{K}_n(x,x) \sim K_n(x,x) \sim n$$

while for some $\varepsilon \in \left(0, \frac{1}{2}\right)$, large enough n, and $x \in \mathcal{N}$,

$$K_n^*(x,x) \geq K_n(x,x) / (1+\varepsilon).$$

As $\mu \leq \mu^*$, we also automatically have

$$K_n^*(x,x) \leq K_n(x,x).$$

Then given $A > 0$, (4.2) yields for $n \geq n_0(A)$ and $|a|, |b| \leq A$,

(4.5)

$$\left| \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} - \frac{K_n^*\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n^*(\xi,\xi)} \right| \leq C\varepsilon^{1/2},$$

where C is independent of ε. To replace $\tilde{K}_n(\xi,\xi)$ by $\tilde{K}_n^*(\xi,\xi)$, in $\frac{a}{\tilde{K}_n(\xi,\xi)}$ and $\frac{b}{\tilde{K}_n(\xi,\xi)}$ inside K_n^*, one would use uniform convergence in a, b as $n \to \infty$.

This approach is powerful because the Christoffel function $\lambda_n(x) = 1/K_n(x,x)$ depends primarily on the structure of the support supp$[\mu]$ of μ, and the value of $\mu'(x)$. In particular, if in some open neigborhood $\mathcal{N}$ of x

$$\mu_{|\mathcal{N}} = \mu^*_{|\mathcal{N}},$$

and supp$[\mu]$=supp$[\mu^*]$, then we expect that

$$\lim_{n\to\infty} K_n^*(x,x) / K_n(x,x) = 1,$$

and once we know universality for μ^*, it follows for μ.

To establish this rigorously, one needs the concept of regularity in the sense of Ullman or, Stahl and Totik [52]. We say the measure μ is *regular* if

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{1}{\text{cap}(\text{supp}[\mu])},$$

where cap denotes logarithmic capacity. For example, if supp$[\mu] = [a,b]$, the capacity is $(b-a)/4$ and the requirement is that

$$\lim_{n\to\infty} \gamma_n^{1/n} = \frac{4}{b-a}.$$

For those new to the concept, this definition may seem implicit. It is used primarily because it is easy to state. One transparent sufficent condition for regularity is that $\mu' > 0$ a.e. in supp$[\mu]$. In applying it to universality, the crucial feature of a regular measure is the following: if J is a compact subset of supp$[\mu]$, then

$$\left\{ \sup_{\deg(P)\leq n} \|P\|_{L_\infty(J)} / \left(\int |P|^2 \, d\mu \right)^{1/2} \right\}^{1/n} \to 1 \text{ as } n \to \infty. \tag{4.6}$$

That is, sup norms of polynomials of deg $\leq n$ over J, are dominated by their $L^2(\mu)$ norms, up to a factor that is $e^{o(n)}$. It is the latter property that enables one to prove, for example:

Proposition 4.2

Suppose that μ and μ^* have the same support and both are regular measures. Let $x \in$supp$[\mu]$, and assume that $\mu = \mu^*$ in a neighborhood of x. Assume, moreover, that given $\delta > 0$, there exists $\eta > 0$ such that for large enough n,

$$K_n^*(x,x)/K_{[n(1-\eta)]}^*(x,x) \le 1+\delta. \tag{4.7}$$

Then

$$\lim_{n\to\infty} K_n(x,x)/K_n^*(x,x) = 1.$$

Proof when supp$[\mu] \subset [-1,1]$

Let $\delta \in \left(0, \frac{1}{2}\right)$ be such that $\mu^* = \mu$ in $(x-\delta, x+\delta)$ and let $n \ge m \ge 1$. Choose a polynomial P_m of degree $\le m-1$, such that $P_m(x) = 1$ and

$$\lambda_m^*(x) = 1/K_m^*(x,x) = \int P_m^2 d\mu^*.$$

We use this to estimate $\lambda_n(x)$ above. There exists $r \in (0,1)$ depending only on δ such that

$$0 \le 1 - \left(\frac{t-x}{2}\right)^2 \le r \text{ for } t \in [-1,1] \setminus (x-\delta, x+\delta). \tag{4.8}$$

(We may take $r = 1 - \left(\frac{\delta}{2}\right)^2$). Let

$$S_n(t) = P_m(t)\left(1 - \left(\frac{t-x}{2}\right)^2\right)^{\left[\frac{n-m}{2}\right]},$$

a polynomial of degree $\le m-1+2\left[\frac{n-m}{2}\right] \le n-1$ with $S_n(x) = 1$. Then using the extremal property of Christoffel functions,

$$\begin{aligned}
\lambda_n(x) &\le \int S_n^2 d\mu \\
&\le \int_{x-\delta}^{x+\delta} P_m^2 d\mu + \|P_m\|^2_{L_\infty(\mathrm{supp}[\mu]\setminus(x-\delta,x+\delta))} r^{2\left[\frac{n-m}{2}\right]} \int_{\mathrm{supp}[\mu]\setminus(x-\delta,x+\delta)} d\mu \\
&= \int_{x-\delta}^{x+\delta} P_m^2 d\mu^* + \|P_m\|^2_{L_\infty(\mathrm{supp}[\mu^*]\setminus(x-\delta,x+\delta))} r^{2\left[\frac{n-m}{2}\right]} \int_{\mathrm{supp}[\mu]\setminus(x-\delta,x+\delta)} d\mu \\
&\le \lambda_m^*(x) + \|P_m\|^2_{L_\infty(\mathrm{supp}[\mu^*]\setminus(x-\delta,x+\delta))} r^{2\left[\frac{n-m}{2}\right]} \int d\mu.
\end{aligned} \tag{4.9}$$

Now we use the regularity of μ^*. Assume that $m = m(n)$ is chosen so that for some fixed $\eta > 0$, we have $n - m \ge \eta n$. For example, choosing $m = n - [\eta n] - 2$ suffices. By the regularity of μ^*,

$$\begin{aligned}
\|P_m\|^2_{L_\infty(\mathrm{supp}[\mu^*]\setminus(x-\delta,x+\delta))} &\le (1+o(1))^n \int P_m^2 d\mu^* \\
&= (1+o(1))^n \lambda_m^*(x).
\end{aligned}$$

Since

$$r^{2\left[\frac{n-m}{2}\right]}(1+o(1))^n \le r^{\eta n}(1+o(1))^n = o(1),$$

putting this in (4.9) gives

$$\lambda_n(x) \leq \lambda_m^*(x)(1+o(1)).$$

By hypothesis (4.7), we can choose η in the definition of $m = m(n)$ so small, that for a given δ and large enough n,

$$\lambda_m^*(x) \leq \lambda_n^*(x)(1+\delta). \tag{4.10}$$

Assuming this, we obtain

$$\limsup_{n\to\infty} \lambda_n(x)/\lambda_n^*(x) \leq 1+\delta,$$

and since the left-hand side is independent of δ, we obtain

$$\limsup_{n\to\infty} \lambda_n(x)/\lambda_n^*(x) \leq 1.$$

In a similar fashion, we can establish

$$\limsup_{n\to\infty} \lambda_n^*(x)/\lambda_n(x) \leq 1,$$

and then have the result. The use of (4.7) for this converse direction is more tricky (we did not assume it for μ) but still doable. ■

Corollary 4.3
Suppose that μ is a regular measure on $[-1,1]$. Let $x \in$supp$[\mu]$, and assume that μ is absolutely continuous in a neighborhood of x, and that μ' is positive and continuous at x. Then

$$\lim_{n\to\infty} \frac{1}{n} K_n\left(x+\frac{a}{n}, x+\frac{a}{n}\right) = \frac{1}{\pi}\left(\sqrt{1-x^2}\mu'(x)\right)^{-1}.$$

Sketch of Proof for $a = 0$
See [33] or [55] for more comprehensive results. The original ideas go back to Nevai [44], [45], and Máté, Nevai and Totik [39]. Let μ^* be the Legendre weight on $[-1,1]$,

$$d\mu^*(x) = dx \text{ on } \operatorname{supp}[\mu].$$

It is easy to establish (4.7) using bounds on Legendre polynomials, for

$$K_n^*(x,x) - K_m^*(x,x) = \sum_{k=m}^{n-1} p_k^{*2}(x) \leq C(n-m)$$

uniformly in n, m and for x in compact subsets of $(-1,1)$. Moreover, it is classical that uniformly for such x,

$$\lim_{n\to\infty} \frac{1}{n} K_n^*(x,x) = \frac{1}{\pi}\left(\sqrt{1-x^2}\right)^{-1}.$$

From Proposition 4.2, it follows that for any regular measure ν on $[-1,1]$, with ν absolutely continuous near x, and $\nu' = 1$ near x, we also have

$$\lim_{n\to\infty} \frac{1}{n} K_n^{\nu}(x,x) = \frac{1}{\pi}\left(\sqrt{1-x^2}\right)^{-1}.$$

To convert to the general case above, one considers a measure ω such that $\omega = \mu$ outside a neighborhood $\mathcal{N}$ of x, while in $\mathcal{N}$, $\omega' = \mu'(x)(1+\varepsilon)$. The last limit, suitably scaled, shows that

$$\lim_{n\to\infty} \frac{1}{n} K_n^{\omega}(x,x) = \frac{1}{\pi}\left(\sqrt{1-x^2}\mu'(x)\right)^{-1}(1+\varepsilon)^{-1}.$$

If the neighborhood $\mathcal{N}$ of x is small enough, the assumed continuity of μ' at x shows that throughout $[-1,1]$,

$$\omega \geq \mu.$$

We deduce that $\lambda_n^{\omega} \geq \lambda_n$ globally, and hence,

$$\limsup_{n\to\infty} \frac{1}{n} K_n(x,x) \leq \frac{1}{\pi}\left(\sqrt{1-x^2}\mu'(x)\right)^{-1}(1+\varepsilon)^{-1}.$$

Of course ε here is arbitrary, and we can similarly establish an asymptotic lower bound. ■

This leads to:

Theorem 4.4
Let μ be a finite positive Borel measure on $(-1,1)$ that is regular. Let $\xi \in (-1,1)$ and such that μ is absolutely continuous in an open set containing ξ. Assume moreover, that μ' is positive and continuous at ξ. Then uniformly for a, b in compact subsets of the real line, we have

$$\lim_{n\to\infty} \frac{\widetilde{K}_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{\widetilde{K}_n(\xi,\xi)} = \frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{4.11}$$

Sketch of proof
Step 1. Change μ outside a neighborhood of ξ
Consider the measure μ^* that is absolutely continuous in $[-1,1]$, and which is equal to the Legendre weight multiplied by $\mu'(\xi)$ outside a neighborhood of ξ, while $\mu^* = \mu$ in that neighborhood of ξ. Let

$$\nu = \max\{\mu^*, \mu\}.$$

Then $\nu = \mu = \mu^*$ near ξ, while $\nu \geq \mu$ and $\nu \geq \mu^*$. Moreover, by Corollary 4.3, K_n^{ν}, K_n^*, and K_n all have the same asymptotic behavior along the diagonal at x, so

$$\begin{aligned} &\lim_{n\to\infty} K_n^{\nu}\left(\xi + \frac{a}{n}, \xi + \frac{a}{n}\right) / K_n^*\left(\xi + \frac{a}{n}, \xi + \frac{a}{n}\right) \\ = \;& 1 = \lim_{n\to\infty} K_n^{\nu}\left(\xi + \frac{a}{n}, \xi + \frac{a}{n}\right) / K_n\left(\xi + \frac{a}{n}, \xi + \frac{a}{n}\right). \end{aligned}$$

We can then apply Lemma 4.1, as in the discussion after there, to deduce that ν and μ^* have the same universality behavior at ξ, and so do ν and μ. Then μ and μ^* also have the same universality behavior at ξ.
Step 2. Use the continuity of μ' at ξ
Because of Step 1, we can assume that μ is a constant multiple of the Legendre weight outside a neighborhood $\mathcal{N}$ of x, say $\mu' = \mu'(\xi)$ outside $\mathcal{N}$.

Inside $\mathcal{N}$, we leave μ unchanged. We emphasize that this does not change the universality behavior at ξ, as shown by Step 1. Next, we let μ^* be a constant multiple of the Legendre weight in $(-1,1)$, namely $(\mu^*)' = \mu'(\xi)$ throughout $(-1,1)$. If $\mathcal{N}$ is small enough, we will have

$$\frac{1}{1+\varepsilon} \leq \frac{(\mu^*)'}{\mu'} \leq 1+\varepsilon$$

in $\mathcal{N}$ and hence globally. It follows that also globally,

$$\frac{1}{1+\varepsilon} \leq \frac{\lambda_n^*}{\lambda_n} \leq 1+\varepsilon.$$

We can then apply Lemma 4.1 to the measures μ and $(1+\varepsilon)\mu^*$ to show that the universality behaviors of μ and μ^* at ξ are $O(\varepsilon)$ apart. Finally, let ε approach 0 carefully. ■

We note that this approach has been taken far beyond the confines of the above results, especially by Findley, Simon, and Totik [19], [49], [50], [55]. In particular, Findley and Totik have shown that if μ is a regular measure on a compact set, then universality holds a.e. in a neighborhood of any point where $\log \mu'$ is integrable. The problem with the extension to this case, is that there is no nice measure, such as the Legendre weight on $(-1,1)$, for which universality is known. So Totik manufactured one. He first consider supports of the form $P^{[-1]}[-1,1]$, where P is a suitable polynomial - this handles the case of supports that consist of several intervals. He then approximates arbitrary compact sets by such "polynomial pullbacks". The proof also shows that if μ' is positive and continuous at a given point x, or more generally, the local Szegő function satisfies a Lebesgue point type condition at x, then universality is true. A different approach to extension was taken by Barry Simon, who used Jost functions, and obtained results that are closely related to those of Findley and Totik.

The comparison approach has also been applied to universality on the unit circle [29], to exponential weights [31], at the hard edge of the spectrum [34], and in a generalized setting [35].

5. A normal families approach

The main drawback of the comparison inequality, is that it requires a comparison measure for which universality is known. This leads to the (weak) global restriction of regularity, which is used in much the same way as outlined in Proposition 4.2. In [36], a method for establishing universality was introduced, based on ideas from complex analysis, that avoids this pitfall. It uses basic tools of complex analysis, such as normal families, together with some of the theory of entire functions, and reproducing kernels.

Suppose that μ is a measure with compact support and that $w = \mu'$ is bounded above and below in some open interval O containing the closed interval J. Then it is well known that for some $C_1, C_2 > 0$,

$$C_1 \leq \frac{1}{n} K_n(x, x) \leq C_2, \tag{5.1}$$

in any proper open subset O_1 of O. Indeed, this follows by comparing λ_n below to the Christoffel function of the weight 1 on a suitable subinterval of O, and comparing it above to a suitable dominating measure. Cauchy-Schwarz inequality's then gives

$$\frac{1}{n} |K_n(\xi, t)| \leq C \tag{5.2}$$

for $\xi, t \in O_1$. We can extend this estimate into the complex plane, as follows:

Lemma 5.1
Let $[c, d]$ be a real interval and $\mathcal{J}$ be a compact subset of (c, d). Let $A > 0$. There exists n_0 and C such that for $n \geq n_0$, polynomials P of degree $\leq n$, $x \in \mathcal{J}$ and $|a| \leq A$,

$$\left| P\left(x + i\frac{a}{n}\right) \right| \leq e^{C|a|} \|P\|_{L_\infty[c,d]}. \tag{5.3}$$

Proof for $[c, d] = [-1, 1]$.
Let $x \in \mathcal{J}$ and $z = x + i\frac{a}{n}$. By Bernstein's growth inequality,

$$|P(z)| \leq \left| z + \sqrt{z^2 - 1} \right|^n \|P\|_{L_\infty[-1,1]}. \tag{5.4}$$

As $\left| x + \sqrt{x^2 - 1} \right| = 1$, straightforward estimation gives

$$\log \left| z + \sqrt{z^2 - 1} \right| \leq C\frac{|a|}{n} + O\left(n^{-2}\right),$$

where C is independent of a and n. On substituting this into (5.4), we obtain (5.3) in the special case $[c, d] = [-1, 1]$. ■

Let $A > 0$ and fix $\xi \in J$. We apply Lemma 5.1 to $\frac{1}{n} K_n\left(\xi + \frac{a}{n}, \xi + \frac{b}{n}\right)$, separately in each variable a and b. This yields an n_0 such that for $n \geq n_0$ and $|a|, |b| \leq A$,

$$\left| \frac{1}{n} K_n\left(\xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right| \leq C_1 e^{C_2(|\mathrm{Im}\, a| + |\mathrm{Im}\, b|)}.$$

Here C_1 and C_2 are independent of n, A, a and b. Thus

$$\left\{ \frac{1}{n} K_n\left(\xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right\}_{n=1}^{\infty}$$

is uniformly bounded in compact sets, and so is a normal family. In view of (5.1), the same is true of $\{f_n(a, b)\}_{n=1}^{\infty}$, where

$$f_n(a, b) = \frac{K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi, \xi)}. \tag{5.5}$$

Thus, given $A > 0$, we have for $n \geq n_0$ and $|a|, |b| \leq A$, that

$$|f_n(a,b)| \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}. \tag{5.6}$$

We emphasize that C_1 and C_2 are independent of n, A, a and b.

Let $f(a,b)$ be the limit of some subsequence $\{f_n(\cdot,\cdot)\}_{n\in\mathcal{S}}$ of $\{f_n(\cdot,\cdot)\}_{n=1}^{\infty}$. It is an entire function in a, b, but (5.6) shows even more: namely that for all complex a, b,

$$|f(a,b)| \leq C_1 e^{C_2(|\operatorname{Im} a| + |\operatorname{Im} b|)}. \tag{5.7}$$

So f is bounded for $a, b \in \mathbb{R}$, and is an entire function of exponential type in each variable. We can then apply the very rich theory of entire functions of exponential type [28]. Recall, here, that the exponential type of an entire function g of order 1, is the smallest number τ such that for any given $\varepsilon > 0$, we have

$$|g(z)| \leq e^{(\tau+\varepsilon)|z|},$$

for large enough $|z|$.

Our goal is to show that

$$f(a,b) = \frac{\sin \pi (a-b)}{\pi (a-b)}. \tag{5.8}$$

So we study the properties of f. Our main tool is to take elementary properties of the reproducing kernel K_n, and then after scaling and taking limits, to deduce that an analogous property is true for f. Let us list some of these:

(I) Real Zeros

Let us fix a. Since for each real ξ, $K_n(\xi, t)$ has only real zeros, the same is true of $f(a, \cdot)$. Moreover, $f(a,\cdot)$ has countably many such zeros.

(II) Squares Inequality

For all $a \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} |f(a,s)|^2 \, ds \leq f(a, \bar{a}). \tag{5.9}$$

To prove this, we start with the identity

$$\int |K_n(b,t)|^2 \, d\mu(t) = K_n\left(b, \bar{b}\right).$$

We let $b = \xi + \frac{a}{\tilde{K}_n(\xi,\xi)}$, make the substitution $t = \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}$, and drop most of the range of integration, leading to

$$\begin{aligned} &\int_{-r}^{r} \left| K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right)\right|^2 \frac{\mu'\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi) K_n(\xi,\xi)} ds \\ \leq\ & K_n\left(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{\bar{a}}{\tilde{K}_n(\xi,\xi)}\right). \end{aligned}$$

Recalling (5.5), we can reformulate this as

$$\int_{-r}^{r} |f_n(a,s)|^2 \frac{\mu'\left(\xi + \frac{s}{\tilde{K}_n(\xi,\xi)}\right)}{\mu'(\xi)} ds \leq f_n(a,\bar{a})$$

If we now assume that μ' is positive and continuous at ξ, then we can let $n \to \infty$ through S, and deduce (5.9), but with $(-\infty,\infty)$ replaced by $(-r,r)$. One simply lets $r \to \infty$. With a little more work, this argument extends to the case where ξ is a Lebesgue point of μ.

(III) f is bounded above and below on real diagonal

Uniformly for $u \in \mathbb{R}$,

$$f(u,u) \sim 1. \tag{5.10}$$

Indeed, this follows from (5.1): for some $C_1, C_2, \eta > 0$, for $|t| \leq \eta$, and for large enough n,

$$C_1 \leq \frac{K_n(\xi+t,\xi+t)}{K_n(\xi,\xi)} \leq C_2,$$

while

$$f(u,u) = \lim_{n\to\infty, n\in\mathcal{S}} \frac{K_n\left(\xi + \frac{u}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{u}{\tilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}.$$

(IV) If σ_a is the exponential type of $f(a,\cdot)$, then $\sigma_a = \sigma$, independent of a

For this one uses interlacing properties of zeros of K_n. Namely, that given any real ξ, the zeros of $K_n(\xi,t)$ interlace those of p_n. This implies that in any real interval $[c,d]$, the difference between the number of zeros of $f_n(a,\cdot)$ and of $f_n(b,\cdot)$ is at most 2. By taking limits as $n \to \infty$ through S, one can deduce that in any interval, the difference between the number of zeros of $f(a,\cdot)$ and $f(b,\cdot)$ is bounded independent of the interval. By classical results on entire functions of exponential type,

$$\frac{1}{2r} \times \text{Number of zeros of } f(a,\cdot) \text{ in } [-r,r] \to \frac{\sigma_a}{\pi}$$

as $r \to \infty$, and this yields that σ_a is independent of a.

(V) Key least squares inequality

For real a,

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}} \left(\frac{f(a,s)}{f(a,a)} - \frac{\sin\sigma(s-a)}{\sigma(s-a)}\right)^2 ds \\ &\leq \frac{1}{f(a,a)} - \frac{\pi}{\sigma}. \end{aligned} \tag{5.11}$$

To prove this, observe that the left-hand side equals

$$\frac{1}{f(a,a)^2}\int_{-\infty}^{\infty} f(a,s)^2 ds - \frac{2}{f(a,a)}\int_{-\infty}^{\infty} f(a,s)\frac{\sin\sigma(a-s)}{\sigma(a-s)} ds + \int_{-\infty}^{\infty}\left(\frac{\sin\sigma(a-s)}{\sigma(a-s)}\right)^2 ds. \tag{5.12}$$

We now apply (5.9) to the first term, obtaining the upper bound $\frac{1}{f(a,a)}$. For the second term, we use the reproducing kernel identity [53, p. 95]

$$g(x) = \int_{-\infty}^{\infty} g(t) \frac{\sin \sigma (x-t)}{\pi (x-t)} dt, \; x \in \mathbb{R}, \tag{5.13}$$

which holds for g that is entire of exponential type at most σ and $g \in L_2(\mathbb{R})$. The second term in (5.12) then becomes $-\frac{2\pi}{\sigma}$. Choosing $g(t) = \frac{\sin \sigma t}{\pi t}$ also leads to

$$\int_{-\infty}^{\infty} \left(\frac{\sin \sigma (a-s)}{\sigma (a-s)} \right)^2 ds = \frac{\pi}{\sigma},$$

which can be substituted for the third term in (5.12). Then (5.11) follows.

(VI) A formula for the type

$$\sigma = \pi \sup_{x \in \mathbb{R}} f(x,x). \tag{5.14}$$

Since the left-hand side of (5.11) is nonnegative, we obtain

$$\sigma \geq \pi \sup_{a \in \mathbb{R}} f(a,a).$$

The converse direction is more difficult. One uses classical inequalities, the so-called Markov-Stieltjes inequalities. If $\{t_{jn}\}$ are the zeros of $(t - \xi_n) K_n(\xi, t)$ in increasing order, these assert that for $k > \ell$,

$$\sum_{j=k+1}^{\ell-1} \lambda_n(t_{jn}) \leq \int_{t_{kn}}^{t_{\ell n}} d\mu(t) \leq \sum_{j=k}^{\ell} \lambda_n(t_{jn}).$$

Suitably scaled, these zeros t_{jn} correspond to zeros of $f_n(0,z)$. In the limit as $n \to \infty$ through the subsequence $\mathcal{S}$, the zeros of $f_n(0,z)$ converge to zeros of $f(0,z)$. If we denote the zeros of $zf(0,z)$ by $\{\rho_j\}$ in increasing order, we obtain

$$\sum_{j=k+1}^{\ell-1} \frac{1}{f(\rho_j, \rho_j)} \leq \rho_\ell - \rho_k \leq \sum_{j=k}^{\ell} \frac{1}{f(\rho_j, \rho_j)}.$$

The left-hand inequality leads to

$$\frac{\ell - k - 1}{\sup_{x \in \mathbb{R}} f(x,x)} \leq \rho_\ell - \rho_k.$$

This enables us to estimate above the number of zeros of $f(0,z)$ in any interval $[-r,r]$, aka $[\rho_k, \rho_\ell]$. Dividing by $2r$, and using classic results on zero distribution of entire functions of exponential type, leads to

$$\frac{\sigma}{\sup_{x \in \mathbb{R}} f(x,x)} \leq \pi.$$

Then (5.14) follows.

Plugging into (5.11) gives

$$\int_{-\infty}^{\infty}\left(\frac{f(a,s)}{f(a,a)}-\frac{\sin\sigma(a-s)}{\sigma(a-s)}\right)^2 ds$$
$$\leq \frac{1}{f(a,a)}-\frac{1}{\sup_{x\in\mathbb{R}} f(x,x)}. \tag{5.15}$$

Finally, if we assume universality along the diagonal, which we formulate as

$$\lim_{n\to\infty} f_n(a,a)=1$$

for all a, then $f(a,a)=1$ for all a. The right-hand side of (5.15) vanishes, while $\sigma=\pi$, and we obtain

$$f(a,s)=\frac{\sin\pi(a-s)}{\pi(a-s)}.$$

Since the limit is independent of the subsequence, we have sketched the proof of [36]:

Theorem 5.2
Let μ be a finite positive Borel measure on the real line with compact support. Let $J\subset \operatorname{supp}[\mu]$ be compact, and such that μ is absolutely continuous in an open set containing J. Assume that w is positive and continuous at each point of J. The following are equivalent:
(I) Uniformly for $\xi\in J$ and a in compact subsets of the real line,

$$\lim_{n\to\infty}\frac{K_n\left(\xi+\frac{a}{n},\xi+\frac{a}{n}\right)}{K_n(\xi,\xi)}=1. \tag{5.16}$$

(II) Uniformly for $\xi\in J$ and a,b in compact subsets of the complex plane, we have

$$\lim_{n\to\infty}\frac{K_n\left(\xi+\frac{a}{\widetilde{K}_n(\xi,\xi)},\xi+\frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)}=\frac{\sin\pi(a-b)}{\pi(a-b)}. \tag{5.17}$$

The clear advantage of the theorem is that there is no global restriction on μ. The downside is that we still have to establish the ratio asymptotic (5.16) for the Christoffel functions/ reproducing kernels, and to date, these have only been established in the stronger form

$$\lim_{n\to\infty}\frac{1}{n}K_n(\xi,\xi)\,\mu'(\xi)=\nu'(\xi),$$

where ν' is the density of the equilibrium measure of the support of μ, and μ is assumed to be regular, together with some local condition. However, it seems likely that the ratio asymptotic (5.16) should hold more generally than this last limit. Note too that the continuity assumption on w can be replaced by a Lebesgue point type condition.

Initially, I believed that the hypotheses in Theorem 5.2 should give (5.16) automatically. However, I am inclined to doubt this now. One can show under the hypotheses of Theorem 5.2, that the limit of any subsequence of

$\left\{ \frac{K_n\left(\xi + \frac{a}{\widetilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\xi,\xi)}\right)}{K_n(\xi,\xi)} \right\}$ is the reproducing kernel of a de Branges space that is norm equaivalent to a classical Paley-Wiener space. Since there are such spaces with reproducing kernel other than the sinc kernel, this raises the possibility that there might be other subsequential limits than the sinc kernel. This has been used in [38] to show that for sequences of measures, one can get universality limits "in some sense in the bulk" that are different from the sinc kernel. However, it remains to establish this for a fixed measure.

We note that the method of this section has been applied to varying exponential weights [30], at the hard edge of the spectrum in [37], at the soft edge of the spectrum [32], and to Cantor sets with positive measure by Avila, Last and Simon [2].

References

[1] S. Albeverio, L. Pastur, M. Shcherbina, *On the* $\frac{1}{n}$ *expansion for some unitary invariant ensembles of random matrices*, Comm. Math. Phys. 224 (2001), 271–305.

[2] A. Avila, Y. Last, and B. Simon, *Bulk universality and clock spacing of zeros for ergodic Jacobi matrices with a.c. spectrum,* submitted.

[3] A.I. Aptekarev, P.M. Bleher, and A.B. Kuijlaars, *Large n limit of Gaussian random matrices with external source, part II,* Comm. Math. Phys. 259 (2005), 367-389.

[4] J. Baik, T.Kriecherbauer, K.McLaughlin and P.Miller, *Uniform asymptotics for polynomials orthogonal with respect to a general class of discrete weights and universality results for associated Ensembles,* Annals of Math. Studies No. 164, Princeton University Press, Princeton, 2007.

[5] J. Baik, L. Li, T. Kriecherbauer, K. McLaughlin, C. Tomei, *Proceedings of the Conference on Integrable Systems, Random Matrices and Applications,* Contemporary Mathematics, Vol. 458, American Mathematical Society, Providence, 2008.

[6] E.L. Basor, *Toeplitz determinants, Fisher-Hartwig symbols, and random matrices*, (in) Recent perspectives in random matrix theory and number theory, London Math. Soc. Lecture Note Ser., Vol. 322, Cambridge Univ. Press, Cambridge, 2005, pp. 309–336.

[7] E.L. Basor, H. Widom, *Determinants of Airy operators and applications to random matrices,* J. Statist. Phys. 96 (1999), 1–20.

[8] P. Bleher and A. Its, *Semiclassical Asymptotics of Orthogonal Polynomials, Riemann-Hilbert Problem, and Universality in the Matrix Model,* Annals of Math., 150(1999), 185-266.

[9] P. Bleher and A. Its, *Random matrix models and their applications,* Cambridge University Press, Cambridge, 2001.

[10] P. Bleher and A. Its, *Asymptotics of the partition function of a random matrix model,* Ann. Inst. Fourier (Grenoble) 55 (2005), 6, 1943–2000.

[11] P. Bleher, and A. Kuijlaars, *Large n limit of Gaussian random matrices with external source. III. Double scaling limit*, Comm. Math. Phys., 270 (2007), 481–517.

[12] T. Claeys and A. Kuijlaars, *Universality of the double scaling limit in random matrix models,* Comm. Pure Appl. Math., 59 (2006), 1573-1603.

[13] T. Claeys and A. Kuijlaars, *Universality in unitary random matrix ensembles when the soft edge meets the hard edge*, Contemporary Mathematics (60th Birthday of Percy Deift), 458(2008), 265-280.

[14] T. Claeys, A. Kuijlaars, and M. Vanlessen, *Multi-critical unitary random matrix ensembles and the general Painlevé II equation*, Annals of Mathematics, 167(2008), 601-642.

[15] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach,* Courant Institute Lecture Notes, Vol. 3, New York University Pres, New York, 1999.

[16] P. Deift, *Riemann-Hilbert Methods in the theory of orthogonal polynomials*, (in) Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday, Proc. Sympos. Pure Math., Vol. 76 Amer. Math. Soc., Providence, RI, 2007, pp. 715–740.

[17] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, *Uniform Asymptotics for Polynomials Orthogonal with respect to Varying Exponential Weights and Applications to Universality Questions in Random Matrix Theory,* Communications in Pure and Applied Mathematics, 52(1999), 1335-1425.

[18] P. Desrosiers, P.J. Forrester, *Hermite and Laguerre β-ensembles: asymptotic corrections to the eigenvalue density,* Nuclear Phys. B 743 (2006), no. 3, 307–332.

[19] E. Findley, *Universality for Regular Measures satisfying Szegő's Condition,* J. Approx. Theory, 155 (2008), 136–154.

[20] P. J. Forrester, *Spacing distributions in random matrix ensembles,* (in) Recent perspectives in random matrix theory and number theory, London Math. Soc. Lecture Note Ser., 322, Cambridge Univ. Press, Cambridge, 2005, pp. 279–307.

[21] P. J. Forrester, *Log-gases and Random matrices*, online book, http://www.ms.unimelb.edu.au/~matpjf/matpjf.html .

[22] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[23] T.M. Garoni, P.J. Forrester, N. E. Frankel, *Asymptotic corrections to the eigenvalue density of the GUE and LUE,* J. Math. Phys., 46 (2005), no. 10, 103301, 17 pp.

[24] A.R. Its, A.B. Kuijlaars, and J. Ostensson, *Critical edge behavior in unitary random matrix ensembles and the thirty fourth Painlevé transcendent,* International Maths. Research Notices, 2008, article ID rnn017, 67 pages.

[25] A.B. Kuijlaars, *Asymptotic analysis of the density of states in random matrix models associated with a slowly decaying weight*, J. Comput. Appl. Math. 133 (2001), 127-140.

[26] A.B. Kuijlaars and M. Vanlessen, *Universality for Eigenvalue Correlations from the Modified Jacobi Unitary Ensemble*, International Maths. Research Notices, 30(2002), 1575-1600.

[27] A.B. Kuijlaars and M. Vanlessen, *Universality for eigenvalue correlations at the origin of the spectrum*, Comm. Math. Phys. 243 (2003), 163-191.

[28] B. Ya. Levin, in collaboration with Yu. Lyubarskii, M. Sodin, V. Tkachenko, *Lectures on Entire Functions*, Translations of Mathematical Monographs, Vol. 150, American Mathematical Society, Providence, 1996.

[29] Eli Levin and D.S. Lubinsky, *Universality Limits Involving Orthogonal Polynomials on the Unit Circle*, Computational Methods and Function Theory, 7(2007), 543-561.

[30] Eli Levin and D.S. Lubinsky, *Universality Limits in the Bulk for Varying Measures,* Advances in Mathematics, 219 (2008), 743–779..

[31] Eli Levin and D.S. Lubinsky, *Universality Limits for Exponential Weights*, Constructive Approximation, 29(2009), 247-275.

[32] Eli Levin and D.S. Lubinsky, *Universality Limits at the Soft Edge of the Spectrum via Classical Complex Analysis,* manuscript.

[33] D.S. Lubinsky, *A New Approach to Universality Limits involving Orthogonal Polynomials*, to appear in Annals of Mathematics.

[34] D.S. Lubinsky, *A New Approach to Universality Limits at the Edge of the Spectrum*, Contemporary Mathematics (60th Birthday of Percy Deift), 458(2008), 281-290.

[35] D.S. Lubinsky, *Mutually Regular Measures have Similar Universality Limits*, (in) Proceedings of Twelfth Texas Conference on Approximation Theory (eds. M. Neamtu, L. Schumaker), Nashboro Press, Nashville 2008, pp. 256-269.

[36] D.S. Lubinsky, *Universality limits in the bulk for arbitrary measures on compact sets*, J. d"Analyse Mathematique, 106 (2008), 373–394.

[37] D.S. Lubinsky, *Universality Limits at the Hard Edge of the Spectrum for Measures with Compact Support,* International Maths. Research Notices, IMRN 2008, Art. ID rnn 099, 39 pp. .

[38] D.S. Lubinsky, *Universality Limits for Random Matrices and de Branges Spaces of Entire Functions*, Journal of Functional Analysis, 256(2009), 3688-3729.

[39] A. Mate, P. Nevai, V. Totik, *Szego's Extremum Problem on the Unit Circle*, Annals of Math., 134(1991), 433-453.

[40] K. T.-R. McLaughlin, *Asymptotic analysis of random matrices with external source and a family of algebraic curves*, Nonlinearity 20 (2007), 1547–1571.

[41] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights,* Intern. Math. Res. Papers, 2006, Article ID 48673, pages 1-77.

[42] K. T.-R. McLaughlin and P. Miller, *The $\bar{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights,* Int. Math. Res. Not. IMRN 2008, Art. ID rnn 075, 66 pp.

[43] M.L. Mehta, *Random Matrices*, 2nd edn., Academic Press, Boston, 1991.

[44] P. Nevai, *Orthogonal Polynomials*, Memoirs of the AMS no. 213 (1979).

[45] P. Nevai, *Geza Freud, Orthogonal Polynomials and Christoffel Functions: A Case Study*, J. Approx. Theory, 48(1986), 3-167.

[46] L. Pastur, *From random matrices to quasi-periodic Jacobi matrices via orthogonal polynomials,* J. Approx. Theory 139 (2006), 269–292.

[47] L. Pastur and M. Shcherbina, *Universality of the local eigenvalue statistics for a class of Unitary Invariant Random Matrix Ensembles*, J. Statistical Physics, 86(1997), 109-147.

[48] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.

[49] B. Simon, *Two Extensions of Lubinsky's Universality Theorem*, Journal d'Analyse de Mathematique, 105 (2008), 345-362.

[50] B. Simon, *The Christoffel-Darboux Kernel*, to appear in Perspectives in PDE, Harmonic Analysis and Applications, Proceedings of Symposia in Pure and Applied Mathematics.

[51] A. Soshnikov, *Universality at the Edge of the Spectrum in Wigner Random Matrices,* Comm. Math. Phys., 207(1999), 697-733.

[52] H. Stahl and V. Totik, *General Orthogonal Polynomials,* Cambridge University Press, Cambridge, 1992.

[53] F. Stenger, *Numerical Methods Based on Sinc and Analytic Function*s, Springer, New York, 1993.

[54] V. Totik, *Asymptotics for Christoffel Functions for General Measures on the Real Line*, J. d'Analyse Math., 81(2000), 283-303.

[55] V. Totik, *Universality and fine zero spacing on general sets*, to appear in Arkiv for Matematik.

[56] C. Tracy, H.L. Widom, *Universality of the distribution functions of random matrix theory,* (in) Statistical physics on the eve of the 21st century, Ser. Adv. Statist. Mech., 14, World Sci. Publ., River Edge, NJ, 1999, pp. 230–239.

[57] C. Tracy, H.L. Widom, *Universality of the distribution functions of random matrix theory*, (in) Integrable systems: from classical to quantum (Montréal, QC, 1999), CRM Proc. Lecture Notes, 26, Amer. Math. Soc., Providence, RI, 2000, pp.251–264.

[58] C. Tracy, H.L. Widom, *The Pearcey process*, Communications in Mathematical Physics, 263 (2006), no. 2, 381–400.

[59] M. Vanlessen, *Strong Asymptotics of Lageurre-type Orthogonal Polynomials and Applications in Random Matrix Theory,* Constr. Approx., 25(2007), 125-175.

[60] H. Widom, *Toeplitz determinants, random matrices and random permutations*, (in) Toeplitz matrices and singular integral equations (Pobershau, 2001), Birkhäuser, Basel, 2002, pp.317–328,

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu