UNIVERSALITY IN MEASURE IN THE BULK FOR VARYING WEIGHTS

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ABSTRACT. We prove that universality holds in measure for sequences $\left\{g_n e^{-2nQ}\right\}$ of varying weights, where e^{-Q} is an exponential weight and the functions $\{g_n\}$ admit suitable polynomial approximations.

1. INTRODUCTION[1]

In the theory of random Hermitian matrices, one considers a probability distribution $\mathcal{P}^{(n)}$ on the eigenvalues $x_1 \leq x_2 \leq ... \leq x_n$ of $n \times n$ Hermitian matrices,

$$\mathcal{P}^{(n)}\left(x_1, x_2, ..., x_n\right) = ce^{-\sum_{j=1}^{n} 2nQ_n(x_j)} \prod_{i<j}\left(x_i - x_j\right)^2.$$

See [5, p. 106 ff.]. Here, c is a normalizing constant, often called the partition function, and Q_n is a given function. In the Gaussian Unitary Ensemble, $Q_n(x) = \frac{1}{2}x^2$.

Orthogonal polynomials play a crucial role in analyzing these. For $n \geq 1$, let μ_n be a finite positive Borel measure with support $\operatorname{supp}[\mu_n]$ containing infinitely many points. We may define orthonormal polynomials

$$p_m\left(\mu_n, x\right) = \gamma_m\left(\mu_n\right) x^m + ..., \gamma_m\left(\mu_n\right) > 0,$$

$m = 0, 1, 2, ...$, satisfying the orthonormality conditions

$$\int p_k\left(\mu_n, x\right) p_\ell\left(\mu_n, x\right) d\mu_n(x) = \delta_{k\ell}.$$

Throughout we use μ'_n to denote the Radon-Nikodym derivative of μ_n. The nth reproducing kernel for μ_n is

$$(1.1) \qquad K_n\left(\mu_n, x, y\right) = \sum_{k=0}^{n-1} p_k\left(\mu_n, x\right) p_k\left(\mu_n, y\right),$$

and the normalized kernel is

$$(1.2) \qquad \widetilde{K}_n\left(\mu_n, x, y\right) = \mu'_n(x)^{1/2} \mu'_n(y)^{1/2} K_n\left(\mu_n, x, y\right).$$

The nth Christoffel function is

$$\lambda_n\left(\mu_n, x\right) = K_n\left(\mu_n, x, x\right)^{-1}.$$

When

$$d\mu_n(x) = e^{-2nQ_n(x)}dx,$$

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there is the basic formula for the probability distribution $\mathcal{P}^{(n)}$ [5, p.112]:

$$\mathcal{P}^{(n)}(x_1, x_2, ..., x_n) = \frac{1}{n!} \det\left(\tilde{K}_n(\mu_n, x_i, x_j)\right)_{1\leq i,j\leq n}.$$

Note that this is the standard definition, which includes the factor $\mu'_n(x_1)\,\mu'_n(x_2)\,...\mu'_n(x_n)$ in the right-hand side. One particularly important quantity is the $m-$point correlation function,

$$R_m(\mu_n, x_1, x_2, ..., x_m) = \frac{n!}{(n-m)!} \int ... \int \mathcal{P}^{(n)}(x_1, x_2 ..., x_n)\, dx_{m+1}\ dx_{m+2}\ ...dx_n.$$

R_m admits the remarkable identity [5, p. 112], first proved by Freeman Dyson,

$$R_m(\mu_n, x_1, x_2, ..., x_m) = \det\left(\tilde{K}_n(\mu_n, x_i, x_j)\right)_{1\leq i,j\leq m}.$$

The *universality limit in the bulk* asserts that for fixed $m \geq 2$, ξ in a suitable subset of the (common) supports of $\{\mu_n\}$, and real $a_1, a_2, ..., a_m$, we have

$$\lim_{n\to\infty} \frac{1}{\tilde{K}_n(\mu_n, \xi, \xi)^m} R_m\left(\mu_n, \xi + \frac{a_1}{\tilde{K}_n(\mu_n, \xi, \xi)}, \xi + \frac{a_2}{\tilde{K}_n(\mu_n, \xi, \xi)}, ..., \xi + \frac{a_m}{\tilde{K}_n(\mu_n, \xi, \xi)}\right)$$
$$= \det\left(\mathbb{S}(a_i - a_j)\right)_{1\leq i,j\leq m},$$

where

$$\mathbb{S}(t) = \frac{\sin \pi t}{\pi t}$$

is the sinc kernel. Because m is fixed in this limit, this reduces to the case $m = 2$, namely

$$(1.3) \qquad \lim_{n\to\infty} \frac{\tilde{K}_n\left(\mu_n, \xi + \frac{a}{\tilde{K}_n(\mu_n, \xi, \xi)}, \xi + \frac{b}{\tilde{K}_n(\mu_n, \xi, \xi)}\right)}{\tilde{K}_n(\mu_n, \xi, \xi)} = \mathbb{S}(a - b).$$

Typically, this is established uniformly for a, b in compact subsets of the real line. Thus, an assertion about the distribution of eigenvalues of random matrices has been reduced to a technical limit involving orthogonal polynomials. See [1], [2], [5], [6], [9], [13], [15], [16], [17], [18], [19], [24], [27], [28] for some references to the extensive literature on this topic.

We shall need some concepts from potential theory for external fields [22]. Let Σ be a closed set on the real line, and $W(x) = \exp(-Q(x))$ be an upper semi-continuous function on Σ that is positive on a set of positive linear Lebesgue measure. If Σ is unbounded, we assume that

$$(1.4) \qquad\qquad \lim_{|x|\to\infty, x\in\Sigma} W(x)|x| = 0.$$

We say, following Saff and Totik, [22], that W is *admissible*. Associated with Σ and Q, we may consider the extremal problem

$$\inf_{\nu}\left(\int\int \log \frac{1}{|x-t|} d\nu(x)\, d\nu(t) + 2\int Q\, d\nu\right),$$

where the inf is taken over all positive Borel measures ν with support in Σ and $\nu(\Sigma) = 1$. The inf is attained by a unique equilibrium measure ω_Q, characterized by the following conditions: let

$$U^{\omega_Q}(z) = \int \log \frac{1}{|z-t|} d\omega_Q(t)$$

denote the logarithmic potential for ω_Q. Let

$$\mathcal{S}_Q = \operatorname{supp}[\omega_Q]$$

denote the (compact) support of the equilibrium measure. Then [22, Thm. I.1.3, p. 27]

(1.5) $$U^{\omega_Q} + Q \geq F_Q \text{ q.e. on } \Sigma;$$

(1.6) $$U^{\omega_Q} + Q = F_Q \text{ q.e. in } \mathcal{S}_Q.$$

Here the number F_Q is a constant, and q.e. stands for quasi everywhere, that is, except on a set of capacity 0. Notice that we are using ω_Q for the equilibrium measure, rather than the more standard μ_W or ν_W, to avoid confusion with μ_n or ν_n.

Our first result in [15, p. 747, Thm. 1.1] was the following. It's proof depended heavily on asymptotics of Christoffel functions that had been established by Vili Totik [26]:

Theorem A
Let $W = e^{-Q}$ be a continuous non-negative function on the set Σ, which is assumed to consist of at most finitely many intervals. If Σ is unbounded, we assume also (1.4). Let h be a bounded positive continuous function on Σ, and for $n \geq 1$, let

$$d\mu_n(x) = \left(hW^{2n}\right)(x)\,dx.$$

Let J be a closed interval lying in the interior of $\mathcal{S}_Q = supp[\omega_Q]$, where ω_Q denotes the equilibrium measure for W. Assume that ω_Q is absolutely continuous in a neighborhood of J, and that ω_Q' and Q' are continuous in that neighborhood, while $\omega_Q' > 0$ there. Then uniformly for $\xi \in J$, and a, b in compact subsets of the real line, we have

(1.7) $$\lim_{n \to \infty} \frac{\widetilde{K}_n\left(\mu_n, \xi + \frac{a}{\widetilde{K}_n(\mu_n,\xi,\xi)}, \xi + \frac{b}{\widetilde{K}_n(\mu_n,\xi,\xi)}\right)}{\widetilde{K}_n(\mu_n, \xi, \xi)} = \mathbb{S}(a - b).$$

In particular, when Q' satisfies a Lipschitz condition of some positive order in a neighborhood of J, then [22, p. 199] ω_Q', is continuous there, and hence we obtain universality except near zeros of ω_Q'. Note too that when Q is convex in Σ, or $xQ'(x)$ is increasing there, then the support of ω_Q consists of at most finitely many intervals, with at most one interval per component of Σ [22, p. 199].

Theorem A is a special case of a more general result in [15, p. 748, Thm. 1.2], which allowed fairly general sequences of weights, but involved hypotheses on asymptotics of ratios of their Christoffel functions. In [15], applications were also made to universality for fixed exponential weights on the real line.

Our goal in this paper is to prove that universality holds in measure for varying weights, under more general hypotheses than those in Theorem A. Our first result is an equiconvergence result. Throughout, meas denotes linear Lebesgue measure. We also say that a sequence of real valued functions $\{f_n\}$ defined on a compact set Σ converges in measure to a function g, if for each $\varepsilon > 0$,

$$meas\{x \in \Sigma : |f_n(x) - g(x)| > \varepsilon\} \to 0 \text{ as } n \to \infty.$$

Theorem 1.1

Let Σ be a closed set and $\{\nu_n\}$ and $\{\mu_n\}$ be sequences of positive measures on Σ. Assume that for $n \geq 1$,

$$(1.8) \qquad \mu'_n(x) = h(x) e^{-2nQ(x)}, x \in \Sigma,$$

where h is bounded, positive and continuous on Σ, and e^{-Q} is admissible, while also as $n \to \infty$,

$$(1.9) \qquad \frac{1}{n} K_n(\mu_n, x, x) \mu'_n(x) \to \omega'_Q \text{ in measure in } S_Q.$$

Assume also that for $n \geq 1$,

$$(1.10) \qquad \nu'_n(x) = g_n(x) \mu'_n(x), x \in \Sigma,$$

and there exists for $n \geq 1$, a polynomial S_n of degree ℓ_n, that is positive in Σ, with

$$(1.11) \qquad \ell_n = o(n), n \to \infty$$

and

$$(1.12) \qquad S_n^2(x) g_n(x) \leq 1, \text{ a.e. } x \in \Sigma,$$

while as $n \to \infty$,

$$(1.13) \qquad S_n^2 g_n \to 1 \text{ in measure in } \mathcal{S}_Q.$$

Let $R, \varepsilon > 0$. Then for large enough n, there exists a set $\mathcal{E}_n$ of linear Lebesgue measure $< \varepsilon$, such that

$$(1.14)$$
$$\sup_{\substack{|a|,|b| \leq R, \\ \xi \in \mathcal{S}_Q \setminus \mathcal{E}_n}} \frac{1}{n} \left| K_n\left(\nu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) - S_n\left(\xi + \frac{a}{n}\right) S_n\left(\xi + \frac{b}{n}\right) K_n\left(\mu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n}\right) \right| \nu'_n(\xi) = o(1).$$

The set $\mathcal{E}_n$ does not depend on R.

Remarks

(a) Note that in (1.14), a, b are complex numbers.

(b) In (1.9), we could equivalently assume that the convergence in measure takes place in Σ.

(c) Note that (1.9) holds under the conditions of Theorem A, see [26].

(d) A more explicit form of Theorem 1.1 is given in Theorem 4.3.

(e) The conclusion of Theorem 1.2 is really almost uniform convergence, as used in Egorov's Theorem, and so is a little stronger than convergence in measure.

One case where we can find suitable approximating polynomials $\{S_n\}$ is in the following result, where Σ is assumed to be a union of finitely many compact intervals. Let $\alpha \in (0, \infty)$, and for $n \geq 1$, let $\varphi_n : \Sigma \to \mathbb{R}$. We say that $\{\varphi_n\}$ is *uniformly smooth of order α*, if

(i) when $\alpha = k$, where k is a positive integer, $\varphi_n^{(k-1)}$ is absolutely continuous for each $n \geq 1$, with

$$\sup_n \left\| \varphi_n^{(k)} \right\|_{L_\infty(\Sigma)} < \infty.$$

(ii) when $\alpha = k + \Delta$, where k is a nonnegative integer, and $0 < \Delta < 1$, $\varphi_n^{(k)}$ satisfies a uniform Lipschitz condition of order Δ :

$$\sup_n \sup_{x,y \in \Sigma} \frac{\left| \varphi_n^{(k)}(x) - \varphi_n^{(k)}(y) \right|}{|x - y|^\Delta} < \infty.$$

Theorem 1.2
Let Σ consist of finitely many compact intervals and let

$$(1.15) \qquad \nu'_n(x) = h(x) e^{\tau_n \varphi_n(x)} \left(\prod_{j=1}^{N} |x - b_{nj}|^{\beta_{nj}} \right)^n e^{-2nQ(x)}, x \in \Sigma,$$

where h is a positive continuous function on Σ, and where Q is continuous, while Q' is continuous in Σ, except perhaps at finitely many points. Assume moreover, that the equilibrium measure ω_Q is absolutely continuous, and that ω'_Q is positive and continuous in $\mathcal{S}^o_Q$, except perhaps at finitely many points. Assume also that $N \geq 1$, and all $\{b_{nj}\}_{n,j}$ lie in some compact subset of the real line, while

$$(1.16) \qquad \max_{1 \leq j \leq N} \beta_{nj} \to 0 \text{ as } n \to \infty;$$

In addition, assume that $\alpha \in (0, \infty)$, and $\{\varphi_n\}$ is a sequence of real valued functions on Σ that is uniformly smooth of order α, while for $n \geq 1$,

$$(1.17) \qquad \|\varphi_n\|_{L_\infty(\Sigma)} \leq 1$$

and

$$(1.18) \qquad \tau_n = o\left(n^{\frac{\alpha}{1+\alpha}}\right), n \to \infty.$$

Then, given $\varepsilon > 0$,

$$meas \left\{ x \in \mathcal{S}_Q : \sup_{|a|,|b| \leq R} \left| \frac{\widetilde{K}_n\left(\nu_n, \xi + \frac{a}{n\omega'_Q(\xi)}, \xi + \frac{b}{n\omega'_Q(\xi)}\right)}{n\omega'_Q(\xi)} - \mathbb{S}(a-b) \right| > \varepsilon \right\}$$
$$\to \quad 0 \text{ as } n \to \infty.$$

(1.19)

Remarks
(a) Note that in (1.19), we restrict x to lie in $\mathcal{S}_Q$ as we cannot have universality (at least in the sense above) outside this set. Moreover, a, b are real as $\widetilde{K}_n$ is only defined in Σ.
(b) The requirement that Σ be compact can be removed, if we assume that the measures decay exponentially at ∞, allowing one to apply restricted range inequalities from, for example, [14], [22].
(c) We could allow more general rates of growth of $\{\tau_n\}$ than (1.18). Moreover, if all $\{\varphi_n\}$ are analytic and uniformly bounded in an open set containing Σ that is independent of n, we could replace (1.18) by

$$\tau_n = o\left(\frac{n}{\log n}\right), n \to \infty.$$

(d) One could multiply by other factors, such as $e^{o(n)U^{\rho_n}}$, where $\{\rho_n\}$ are a sequence of measures supported on $\Sigma = [0, 1]$ that admit polynomial approximation of the type described in [25, Theorem 8.1, p. 94].

2. AN INTEGRAL ESTIMATE

We start with the following simple comparison inequality:

Lemma 2.1
Let $m, n \geq 1$, and μ_n and ν_n be positive measures on the real line. Assume that S_n is a polynomial of degree $\ell < n$, and

$$(2.1) \qquad n = m + \ell.$$

Assume also that on $\mathbb{R}$,

$$(2.2) \qquad S_n^2 d\nu_n \leq d\mu_n.$$

Then

$$
\Gamma_n = \frac{1}{n} \int \int \left(K_n(\nu_n, x, y) - S_n(x) S_n(y) K_n(\mu_n, x, y) \right)^2 d\nu_n(x) \, d\nu_n(y)
$$

$$(2.3) \qquad \leq 4 \left(1 + \frac{\ell}{n} - \frac{1}{n} \int S_n^2(y) K_n(\mu_n, y, y) \, d\nu_n(y) \right).$$

Proof
Now using the reproducing kernel property and (2.1), followed by (2.2),

$$
\Delta_n := \int \int \left(K_n(\nu_n, x, y) - S_n(x) S_n(y) K_m(\mu_n, x, y) \right)^2 d\nu_n(x) \, d\nu_n(y)
$$

$$
= \int \int K_n(\nu_n, x, y)^2 \, d\nu_n(x) \, d\nu_n(y)
$$

$$
- 2 \int S_n(y) \left[\int S_n(x) K_m(\mu_n, x, y) K_n(\nu_n, x, y) \, d\nu_n(x) \right] d\nu_n(y)
$$

$$
+ \int S_n(y)^2 \left[\int S_n(x)^2 K_m(\mu_n, x, y)^2 \, d\nu_n(x) \right] d\nu_n(y)
$$

$$
= \int K_n(\nu_n, y, y) \, d\nu_n(y) - 2 \int S_n^2(y) K_m(\mu_n, y, y) \, d\nu_n(y)
$$

$$
+ \int S_n(y)^2 \left[\int S_n(x)^2 K_m(\mu_n, x, y)^2 \, d\nu_n(x) \right] d\nu_n(y)
$$

$$
\leq n - 2 \int S_n^2(y) K_m(\mu_n, y, y) \, d\nu_n(y)
$$

$$
+ \int \int K_m(\mu_n, x, y)^2 \, d\mu_n(x) \, d\mu_n(y)
$$

$$
= n + m - 2 \int S_n^2(y) K_m(\mu_n, y, y) \, d\nu_n(y).
$$

$$(2.4)$$

Next,

$$
\int S_n^2(y) \left(K_n - K_m \right)(\mu_n, y, y) \, d\nu_n(y)
$$

$$
\leq \int \left(K_n - K_m \right)(\mu_n, y, y) \, d\mu_n(y) = n - m.
$$

Substituting into (2.4), and using $n - m = \ell$, gives

$$\Delta_n \leq n + m - 2 \int S_n^2 (y) K_n (\mu_n, y, y) \, d\nu_n (y) + 2 (n - m)$$

$$(2.5) \qquad = \quad 2n + \ell - 2 \int S_n^2 (y) K_n (\mu_n, y, y) \, d\nu_n (y).$$

Using the inequality $(x + y)^2 \leq 2 (x^2 + y^2)$, we see that

$$n\Gamma_n \quad \leq \quad 2\Delta_n + 2 \int \int (K_n (\mu_n, x, y) - K_m (\mu_n, x, y))^2 S_n^2 (x) S_n^2 (y) \, d\nu_n (x) \, d\nu_n (y)$$

$$\leq \quad 2\Delta_n + 2 \int \int (K_n (\mu_n, x, y) - K_m (\mu_n, x, y))^2 \, d\mu_n (x) \, d\mu_n (y)$$

$$= \quad 2\Delta_n + 2 (n - m) = 2\Delta_n + 2\ell.$$

Now use (2.5). ∎

Under the hypotheses of Theorem 1.1, we can estimate Γ_n defined by (2.3).

Lemma 2.2
Assume the hypotheses of Theorem 1.1. Then

$$(2.6) \qquad\qquad \lim_{n \to \infty} \Gamma_n = 0.$$

Proof
By our hypotheses,

$$\frac{1}{n} K_n (\mu_n, x, x) \nu_n' (x) S_n^2 (x)$$

$$= \quad \frac{1}{n} K_n (\mu_n, x, x) \mu_n' (x) \left[S_n^2 (x) g_n (x) \right]$$

$$\to \quad \omega_Q'$$

in measure in $\mathcal{S}_Q$, as $n \to \infty$. Let $\varepsilon > 0$ and

$$\mathcal{E}_n = \left\{ x \in \mathcal{S}_Q : \left| \frac{1}{n} K_n (\mu_n, x, x) \nu_n' (x) S_n^2 (x) - \omega_Q' (x) \right| > \varepsilon \right\},$$

so that $\text{meas}(\mathcal{E}_n) \to 0$ as $n \to \infty$. Then as $n \to \infty$,

$$\frac{1}{n} \int S_n^2 (x) K_n (\mu_n, x, x) \, d\nu_n (x)$$

$$\geq \quad \int_{\mathcal{S}_Q \setminus \mathcal{E}_n} \left(\omega_Q' (x) - \varepsilon \right) dx$$

$$\to \quad \int_{\mathcal{S}_Q} \left(\omega_Q' (x) - \varepsilon \right) dx = 1 - \varepsilon.$$

As $\varepsilon > 0$ is arbitrary,

$$\liminf_{n \to \infty} \frac{1}{n} \int S_n^2 (x) K_n (\mu_n, x, x) \, d\nu_n (x) \geq 1,$$

because of integrability of ω_Q'. Together with our hypothesis (1.11), and (2.3), this gives (2.6). ∎

3. POTENTIAL THEORETIC ESTIMATES

Next, we turn to some potential theoretic estimates. Given a positive measure ρ with compact support in the plane, recall that

$$U^{\rho}(z) = \int \log \frac{1}{|z - t|} d\rho(t)$$

denotes the associated potential. We need a general growth estimate:

Lemma 3.1
Let ρ be a measure with compact support in the real line, and with total mass ≤ 1. Let $\varepsilon > 0$. There is a set $\mathcal{F} \subset \mathbb{R}$ with $\mathrm{meas}(\mathcal{F}) < \varepsilon$, such that for $\xi \in \mathbb{R}\backslash\mathcal{F}$, and all $u \in \mathbb{C}$,

(3.1) $$U^{\rho}(\xi) - U^{\rho}(\xi + u) \leq C_0 |u| / \varepsilon.$$

Here C_0 is independent of $\rho, \xi, \varepsilon, u$.
Proof
This is essentially the same as that of Lemma 4.1 and Corollary 4.2, [18, pp. 232-233]. There estimates were made for the Green's function, but on replacing the equilibrium measure by the measure ρ, one obtains essentially what we need. Nevertheless, we provide the details. Recall that the maximal function of the measure ρ is

$$\mathcal{M}[\rho](x) = \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} d\rho.$$

Moreover, $\mathcal{H}^{*}$ denotes the maximal Hilbert transform of ρ, defined by

$$\mathcal{H}^{*}[\rho](x) = \sup_{\varepsilon>0} \left| \int_{|t-x|\geq\varepsilon} \frac{1}{t - x} d\rho(t) \right|.$$

Step 1: A growth estimate
We first show that

(3.2) $$U^{\rho}(\xi) - U^{\rho}(\xi + u) \leq 26 |u| \mathcal{M}[\rho](\xi) + |\mathrm{Re}\, u| \mathcal{H}^{*}[\rho](\xi).$$

Write $u = x + iy$. Now for real ξ,

$$U^{\rho}(\xi) - U^{\rho}(\xi + u)$$
$$= \frac{1}{2} \int \log \left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2} \right] d\rho(t).$$

Let $\mathcal{S}_1$ denote the set of t for which

$$\left| \frac{2x}{\xi - t} \right| \leq \frac{2 |u|^2}{(\xi - t)^2} \iff |\xi - t| \leq \frac{|u|^2}{|x|}.$$

Let $\mathcal{S}_2$ denote the complementary range. In the case where $x = 0$, of course $\mathcal{S}_2$ is empty. Let us assume $x \neq 0$, the case $x = 0$ is easier. We see that

$$\int_{\mathcal{S}_1} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t)$$

$$\leq \int_{|\xi - t| \leq \frac{|u|^2}{|x|}} \log\left[1 + \frac{3|u|^2}{(\xi - t)^2}\right] d\rho(t)$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{-k-1}\frac{|u|^2}{|x|} \leq |\xi - t| \leq 2^{-k}\frac{|u|^2}{|x|}} \log\left[1 + \frac{12x^2}{|u|^2} 2^{2k}\right] d\rho(t)$$

$$\leq \sum_{k=0}^{\infty} \log\left[1 + \frac{12x^2}{|u|^2} 2^{2k}\right] 2^{-k+1} \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi)$$

$$\leq \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi) 4 \int_0^{\infty} \log\left[1 + \frac{12x^2}{|u|^2 t^2}\right] dt$$

$$(3.3) \qquad = |u| \mathcal{M}[\rho](\xi) 8\sqrt{3}\pi,$$

cf. [12, p. 525, no. 4.222.1]. Next, in $\mathcal{S}_2$, we have $|\xi - t| \geq |u|^2 / |x|$, so using the inequality $\log(1 + t) \leq t$, $t \geq -1$, we obtain

$$\int_{\mathcal{S}_2} \log\left[1 + \frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t)$$

$$\leq \int_{|\xi - t| \geq |u|^2/|x|} \left[\frac{2x}{\xi - t} + \frac{|u|^2}{(\xi - t)^2}\right] d\rho(t)$$

$$(3.4) \qquad \leq 2|x| |\mathcal{H}^*[\rho](\xi)| + |u|^2 \int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2} d\rho(t).$$

Here,

$$\int_{|\xi - t| \geq |u|^2/|x|} \frac{1}{(\xi - t)^2} d\rho(t)$$

$$\leq \sum_{k=0}^{\infty} \int_{2^{k+1}|u|^2/|x| \geq |\xi - t| \geq 2^k |u|^2/|x|} \frac{1}{\left(2^k |u|^2 / |x|\right)^2} d\rho(t)$$

$$\leq \sum_{k=0}^{\infty} \frac{x^2}{|u|^4} 2^{-2k} \cdot 2^{k+2} \frac{|u|^2}{|x|} \mathcal{M}[\rho](\xi)$$

$$= \frac{|x|}{|u|^2} 8\mathcal{M}[\rho](\xi).$$

Combining this with (3.3) and (3.4) gives

$$U^\rho(\xi) - U^\rho(\xi + u)$$

$$\leq 4\sqrt{3}\pi |u| \mathcal{M}[\rho](\xi) + |x| |\mathcal{H}^*[\rho](\xi)| + |x| 4\mathcal{M}[\rho](\xi).$$

Estimating the constants gives (3.2).

Step 2: Estimate the maximal function and Hilbert transform

Next, we use the fact that both the maximal function and the maximal Hilbert transform are weak type (1,1). That is, [21, p. 137, Thm. 1.4]

$$meas\{\xi : \mathcal{M}[\rho](\xi) > \lambda\} \leq \frac{3}{\lambda} \int d\rho \leq \frac{3}{\lambda};$$

and [3, p. 139], [11, p. 128 ff.]

$$meas\{\xi : \mathcal{H}^*[\rho](\xi) > \lambda\} \leq \frac{C_1}{\lambda} \int d\rho \leq \frac{C_1}{\lambda}.$$

Here C_1 is independent of ρ and λ. Note that in these references the bound for $\mathcal{H}^*[\rho]$ is proved assuming that ρ is absolutely continuous. However, it is true without this restriction. Indeed, John Garnett emailed to the authors about [11]: "The proof of Theorem 2.1 of Chapter III about the maximal conjugate function goes through with the L_1 function replaced by a finite measure. Also on the line it is easy to estimate the difference between the truncated Hilbert transform of $|x| > \varepsilon$ and the conjugate function at i/ε bythe ordinary maximal function". Choosing $\lambda = \frac{2}{\varepsilon} \max\{3, C_1\}$, we obtain a set $\mathcal{F}$ of measure $\leq \varepsilon$ such that for $\xi \in \mathbb{R} \backslash \mathcal{F}$, and all complex u, (3.1) holds with a constant C_0 independent of $\rho, \xi, u, \varepsilon$. ∎

Lemma 3.2
Let Σ be a closed set, let $Q : \Sigma \rightarrow [0, \infty)$ be such that e^{-Q} is admissible, and let ω_Q denote the equilibrium measure for e^{-Q}, with support $\mathcal{S}_Q$. Let L be a compact subset of Σ, and assume L is the closure of its (non-empty) interior. Let $n > \ell \geq 1$. Let S_n be a polynomial of degree ℓ that has no zeros on Σ. Let $\varepsilon > 0$ and $R : \mathbb{C} \rightarrow \mathbb{C}$ be a function such that $\log|R|$ is subharmonic in $\mathbb{C}$. Assume also that

$$(3.5) \qquad \lim_{|z| \rightarrow \infty} (\log|R(z)| - (n + \ell) \log|z|)$$

exists and is finite, and that

$$(3.6) \qquad |R(x)| e^{-nQ(x)} / |S_n(x)| \leq 1 \text{ for } x \in L.$$

Then there exists a set $\mathcal{F}_n \subset \mathbb{R}$ with $\text{meas}(\mathcal{F}_n) < \varepsilon$, such that for $\xi \in (L \cap \mathcal{S}_Q) \backslash \mathcal{F}_n$, and all $u \in \mathbb{C}$,

$$(3.7) \qquad |R(\xi + u)| e^{-nQ(\xi)} / |S_n(\xi)| \leq e^{C_0 n |u|/\varepsilon}, u \in \mathbb{C}.$$

The set $\mathcal{F}_n$ depends on $Q, \varepsilon, n, \ell, S_n$ but not on the particular R. Also, C_0 is independent of $Q, R, n, \xi, \varepsilon, u$.
Proof
Recall that

$$U^{\omega_Q}(x) + Q = F_Q \text{ q.e. in } \mathcal{S}_Q.$$

Now balayage that part of ω_Q with support in $\mathcal{S}_Q \backslash L^o$ onto L. More precisely, we apply Theorem II.4.7 in [22, p. 116] with the domain $G = \mathbb{C} \backslash L$, giving the balayage measure

$$\nu = \widehat{\omega_{Q|G}}.$$

It has the property that for some constant C_0,

$$U^{\nu}(x) = U^{\omega_{Q|G}}(x) + C_0, \text{ q.e. } x \in L.$$

If we let

$$\tilde{\omega}_Q = \omega_{Q|L} + \nu,$$

(the sum of ν and the restriction of ω_Q to L), then for q.e. $x \in L \cap \mathcal{S}_Q$,

$$(3.8) \qquad U^{\tilde{\omega}_Q}(x) = U^{\omega_Q}(x) + C_0 = -Q(x) + F_Q + C_0,$$

recall (1.6). Note that $\tilde{\omega}_Q$ has support in L. Next, we may assume that S_n is monic (a constant can be absorbed into R). Then we can write

$$S_n(x) = \prod_{j=1}^{\ell}(x - b_j),$$

and if

$$\rho = \sum_{j=1}^{\ell}\delta_{b_j},$$

a measure with compact support in $\mathbb{C}\backslash\Sigma$, we have

$$-\log|S_n(z)| = U^{\rho}(z).$$

Let $\tilde{\rho}$ denote the balayage measure of ρ onto $L \cap \mathcal{S}_Q$, so that [22, Thm. II.4.7, p. 116] it has total mass ℓ, support in $L \cap \mathcal{S}_Q$, and for some constant C_1,

$$(3.9) \qquad U^{\tilde{\rho}}(x) = U^{\rho}(x) + C_1 \text{ q.e. in } L \cap \mathcal{S}_Q.$$

Let

$$C_2 = n(F_Q + C_0) + C_1.$$

Then for q.e. $x \in L \cap \mathcal{S}_Q$, we have from (3.8) and (3.9),

$$(3.10) \qquad e^{nU^{\tilde{\omega}_Q}(x)+U^{\tilde{\rho}}(x)-C_2} = e^{-nQ(x)}/|S_n(x)|$$

so we can recast (3.6) as

$$|R(x)| e^{nU^{\tilde{\omega}_Q}(x)+U^{\tilde{\rho}}(x)-C_2} \leq 1, \text{ for q.e. } x \in L \cap \mathcal{S}_Q.$$

Moreover, since $\tilde{\omega}_Q$ and $\tilde{\rho}$ have support in $L \cap \mathcal{S}_Q$, their potentials are harmonic outside this set, so $\log|R(z)|+nU^{\tilde{\omega}_Q}(z)+U^{\tilde{\rho}}(z)-C_2$ is subharmonic in $\mathbb{C}\backslash(L \cap \mathcal{S}_Q)$, with a finite limit at ∞, in view of (3.5). By the maximum principle for subharmonic functions, we have

$$(3.11) \qquad |R(z)| e^{nU^{\tilde{\omega}_Q}(z)+U^{\tilde{\rho}}(z)-C_2} \leq 1, z \in \mathbb{C}.$$

Now let

$$\omega := \frac{1}{n+\ell}[n\tilde{\omega}_Q + \tilde{\rho}]$$

a measure of total mass 1, with support in $L \cap \mathcal{S}_Q$. We have from (3.11),

$$|R(z)| e^{(n+\ell)U^{\omega}(z)-C_2} \leq 1, z \in \mathbb{C}.$$

Then for $\xi \in \mathbb{R}$, and complex u, we have

$$|R(\xi + u)| e^{(n+\ell)U^{\omega}(\xi)-C_2} \leq e^{(n+\ell)[U^{\omega}(\xi)-U^{\omega}(\xi+u)]}$$

and now we can apply Lemma 3.1. Note that the set $\mathcal{F}_n$ in Lemma 3.1 does not depend on the particular R, as ω does not depend on R. Finally, by (3.10), in $L \cap \mathcal{S}_Q$, we have

$$e^{(n+\ell)U^{\omega}(\xi)-C_2} = e^{-nQ(\xi)}/|S_n(\xi)|.$$

∎

4. Pointwise Estimates and Proof of Theorem 1.1

Throughout, we assume the hypotheses of Theorem 1.1 and the notation of the previous section. We begin with

Lemma 4.1
Let $\varepsilon > 0$. For $n \geq n_0(\varepsilon)$, there exists a set $\mathcal{G}_n$ of measure $\leq \Gamma_n^{1/2} + \varepsilon$ such that for $\xi \in \mathcal{S}_Q \backslash \mathcal{G}_n$, and for all $u, z \in \mathbb{C}$, we have

$$\frac{1}{n} \left| K_n(\nu_n, \xi + u, z) - S_n(\xi + u) S_n(z) K_n(\mu_n, \xi + u, z) \right| \nu_n'(\xi)$$

$$(4.1) \quad \leq \quad 4 \left[\frac{1}{n} K_{n+\ell}(\nu_n, z, \bar{z}) \nu_n'(\xi) \right]^{1/2} \Gamma_n^{1/4} e^{C_0 n |u|/\varepsilon}.$$

Proof
Step 1: An integral estimate and exceptional set
Let $\mathcal{E}_n$ denote the set of $x \in \Sigma$ for which

$$\frac{1}{n} \int \left(K_n(\nu_n, x, y) - S_n(x) S_n(y) K_n(\mu_n, x, y) \right)^2 d\nu_n(y) \nu_n'(x) > \sqrt{\Gamma_n}.$$

Then

$$\text{meas}(\mathcal{E}_n) \sqrt{\Gamma_n}$$

$$\leq \quad \frac{1}{n} \int \int \left(K_n(\nu_n, x, y) - S_n(x) S_n(y) K_n(\mu_n, x, y) \right)^2 d\nu_n(y) \, d\nu_n(x) = \Gamma_n$$

$$\Rightarrow \quad \text{meas}(\mathcal{E}_n) \leq \sqrt{\Gamma_n}.$$

(4.2)

Step 2: A Christoffel function estimate
We use the inequality

$$|P(z)|^2 \leq K_{n+\ell}(\nu_n, z, \bar{z}) \int |P|^2 \, d\nu_n,$$

valid for all polynomials P of degree $< n + \ell$, and all $z \in \mathbb{C}$, applied to

$$P(z) = \frac{1}{n} \left(K_n(\nu_n, x, z) - S_n(x) S_n(z) K_n(\mu_n, x, z) \right),$$

with $x \in \Sigma \backslash \mathcal{E}_n$ fixed. We then obtain from Step 1, for $x \in \Sigma \backslash \mathcal{E}_n$, and all $z \in \mathbb{C}$,

$$\frac{1}{n^2} \left| K_n(\nu_n, x, z) - S_n(x) S_n(z) K_n(\mu_n, x, z) \right|^2 \nu_n'(x)$$

$$= \quad |P(z)|^2 \nu_n'(x)$$

$$\leq \quad K_{n+\ell}(\nu_n, z, \bar{z}) \left(\int |P|^2 \, d\nu_n \right) \nu_n'(x) \leq \frac{1}{n} K_{n+\ell}(\nu_n, z, \bar{z}) \Gamma_n^{1/2}.$$

Now take square roots: for $x \in \Sigma \backslash \mathcal{E}_n$, and all $z \in \mathbb{C}$,

$$\frac{1}{n} \left| K_n(\nu_n, x, z) - S_n(x) S_n(z) K_n(\mu_n, x, z) \right| \nu_n'(x)^{1/2}$$

$$(4.3) \quad \leq \quad \left(\frac{1}{n} K_{n+\ell}(\nu_n, z, \bar{z}) \right)^{1/2} \Gamma_n^{1/4}.$$

Step 3: Fix z and apply Lemma 3.2

Fix $n \geq 1$, fix $z \in \mathbb{C}$, and let $L_n = \mathcal{S}_Q \backslash \mathcal{E}_n$ (which is independent of z), and

$$A_n = \left[\frac{1}{n} K_{n+\ell} \left(\nu_n, z, \bar{z} \right) \right]^{1/2} \Gamma_n^{1/4};$$

$$R\left(t \right) = \frac{1}{nA_n} \left(K_n \left(\nu_n, t, z \right) - S_n \left(t \right) S_n \left(z \right) K_n \left(\mu_n, t, z \right) \right).$$

Then $\log \left| R \left(t \right) \right|$ is subharmonic in the plane, and (assuming that S_n has full degree $\ell \geq 1$, as we may) $\log \left| R \left(t \right) \right| - \left(n + \ell - 1 \right) \log \left| t \right|$ has a finite limit as $\left| t \right| \to \infty$. Moreover, for $x \in L_n$, we can recast (4.3) as

$$\left| R \left(x \right) \right| h \left(x \right)^{1/2} e^{-nQ(x)} g_n \left(x \right)^{1/2} \leq 1.$$

Since h is bounded above and below by positive constants in Σ, and does not depend on n, we can simply assume that $h \equiv 1$. Next, by our hypothesis that $g_n S_n^2 \to 1$ in measure in $\mathcal{S}_Q$, we may for large enough n replace $g_n^{1/2}$ in this last inequality by $\frac{1}{2} \left| S_n \right|^{-1}$ except on a set of small measure. Absorbing that set of small measure into $\mathcal{E}_n$, (perhaps by replacing ε by $\varepsilon/2$), we obtain for large enough n,

$$\frac{1}{2} \left| R \left(x \right) \right| e^{-nQ(x)} \left| S_n \left(x \right) \right|^{-1} \leq 1, x \in L_n.$$

Since the left-hand side is continuous, we may assume that L_n is closed, and (by removing isolated points), is even the closure of its interior. Moreover, the new L_n is still independent of z above. Then by Lemma 3.2, there exists a set $\mathcal{F}_n \subset \mathbb{R}$ with $\mathrm{meas}(\mathcal{F}_n) < \varepsilon$, such that for $\xi \in L_n \backslash \mathcal{F}_n = \mathcal{S}_Q \backslash \left(\mathcal{E}_n \cup \mathcal{F}_n \right)$ and all $u \in \mathbb{C}$,

$$\frac{1}{2nA_n} \left| K_n \left(\nu_n, \xi + u, z \right) - S_n \left(\xi + u \right) S_n \left(z \right) K_n \left(\mu_n, \xi + u, z \right) \right| e^{-nQ(\xi)} / \left| S_n \left(\xi \right) \right| \leq e^{C_0 n |u|/\varepsilon}.$$

Note that the set $\mathcal{F}_n$ did not depend on the particular R, so it works for all complex z. We may replace $e^{-nQ(\xi)} / \left| S_n \left(\xi \right) \right|$ by $\nu_n' \left(\xi \right)^{1/2}$ in this last inequality, at the cost of adding a set of small measure to $\mathcal{E}_n \cup \mathcal{F}_n$. Finally, we set $\mathcal{G}_n = \mathcal{E}_n \cup \mathcal{F}_n$, multiply by $\nu_n' \left(\xi \right)^{1/2}$, and take account of the definition of A_n to obtain the result. ∎

Next, we estimate $K_{n+\ell}$. We shall assume that in μ_n', $h \equiv 1$. The general case involves trivial modifications.

Lemma 4.2

Let $\varepsilon > 0$. There exists a set $\mathcal{H}_n$ of measure at most 2ε such that for $\xi \in \mathcal{S}_Q \backslash \mathcal{H}_n$, and all complex v,

(4.4) $$\frac{1}{n} \left| K_{n+\ell} \left(\nu_n, \xi + v, \xi + \bar{v} \right) \right| \nu_n' \left(\xi \right) \leq \frac{4}{\varepsilon} e^{nC_1 |v|/\varepsilon}.$$

C_1 is independent of n, ξ, a, ε.

Proof

We shall abbreviate $p_k \left(\nu_n, z \right)$ as $p_k \left(z \right)$. First, we note that for $n \geq 1$,

$$\log K_n \left(\nu_n, z, \bar{z} \right) = \log \left(\sum_{k=0}^{n-1} \left| p_k \left(z \right) \right|^2 \right)$$

is subharmonic in the plane. Indeed, if we fix z, and $r > 0$, we can choose unimodular constants $\{\alpha_k\}$ such that

$$
\begin{aligned}
\log K_n\left(\nu_n, z, \bar{z}\right) &= \log \left|\sum_{k=0}^{n-1} \alpha_k p_k\left(z\right)^2\right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left|\sum_{k=0}^{n-1} \alpha_k p_k\left(z + re^{i\theta}\right)^2\right| d\theta \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left(\sum_{k=0}^{n-1} \left|p_k\left(z + re^{i\theta}\right)\right|^2\right) d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log K_n\left(\nu_n, z + re^{i\theta}, \overline{z + re^{i\theta}}\right) d\theta.
\end{aligned}
$$

In the second last line, we used subharmonicity of absolute values of analytic functions. Moreover, we see that as $|z| \to \infty$,

$$
\log K_n\left(\nu_n, z, \bar{z}\right) = (2n - 2)\log|z| + 2\log \gamma_{n-1} + o\left(1\right).
$$

Now for small $\varepsilon > 0$, let

$$
\mathcal{I}_n = \left\{t \in \mathcal{S}_Q : \frac{1}{n} K_{n+\ell}\left(\nu_n, t, t\right) \nu_n'\left(t\right) \geq \frac{2}{\varepsilon}\right\}.
$$

Observe that

$$
\operatorname{meas}\left(\mathcal{I}_n\right) \frac{2}{\varepsilon} \leq \int \frac{1}{n} K_{n+\ell}\left(\nu_n, t, t\right) d\nu_n\left(t\right) < 2,
$$

as $n > \ell$, so

$$
(4.5) \qquad\qquad\qquad \operatorname{meas}\left(\mathcal{I}_n\right) \leq \varepsilon.
$$

Since $S_n^2 g_n \to 1$ in measure, we may absorb a set of small measure into $\mathcal{I}_n$ and obtain

$$
\frac{1}{2n} K_{n+\ell}\left(\nu_n, t, t\right) e^{-2nQ(t)}/S_n^2\left(t\right) \leq \frac{2}{\varepsilon}, t \in \mathcal{S}_Q \backslash \mathcal{I}_n.
$$

We now let $L_n = \mathcal{S}_Q \backslash \mathcal{I}_n$. Since the left-hand side in this last inequality is continuous, we may assume that L_n is closed and (by removing isolated points) is also the closure of its interior. Next, by Lemma 3.2, with $R\left(t\right) = K_{n+\ell}\left(\nu_n, t, \bar{t}\right)$, with L taken as L_n, Q replaced by $2Q$, and S_n by S_n^2, there is a set $\mathcal{J}_n$ of measure at most ε such that for $\xi \in L_n \backslash \mathcal{J}_n = \mathcal{S}_Q \backslash \left(\mathcal{I}_n \cup \mathcal{J}_n\right)$, and all complex v,

$$
(4.6) \qquad\qquad \frac{\varepsilon}{4n} K_{n+\ell}\left(\nu_n, \xi + v, \overline{\xi + v}\right) e^{-2nQ(\xi)}/S_n^2\left(\xi\right) \leq e^{nC_1|v|/\varepsilon}.
$$

Here C_1 is independent of n, v, ε, ξ. Setting $\mathcal{H}_n = \mathcal{I}_n \cup \mathcal{J}_n$, we see that $\operatorname{meas}(\mathcal{H}_n) < 2\varepsilon$, and for $\xi \in \mathcal{S}_Q \backslash \mathcal{H}_n$, and all complex v, we have (4.4), except that ν_n' is replaced by $e^{-2nQ(\xi)}/S_n^2\left(\xi\right)$. Again, we can add a small set to $\mathcal{H}_n$ and replace $e^{-2nQ(\xi)}/S_n^2\left(\xi\right)$ by $\nu_n'\left(\xi\right)$. ∎

Now we put it all together:

Proof of Theorem 1.1

We apply Lemmas 4.1 and 4.2 above: let $\mathcal{E}_n = \mathcal{G}_n \cup \mathcal{H}_n$ so that $\mathrm{meas}(\mathcal{E}_n) < 3\varepsilon + 2\Gamma_n^{1/2}$, and such that for $\xi \in \mathcal{S}_Q \backslash \mathcal{E}_n$ and all $u, v \in \mathbb{C}$,

$$\frac{1}{n} \left| K_n \left(\nu_n, \xi + u, \xi + v \right) - S_n \left(\xi + u \right) S_n \left(\xi + v \right) K_n \left(\mu_n, \xi + u, \xi + v \right) \right| \nu_n' \left(\xi \right)$$

$$\leq \quad \frac{8}{\sqrt{\varepsilon}} \Gamma_n^{1/4} e^{C_0 n (|u| + |v|)/\varepsilon}.$$

Finally, we know that (2.6) holds, so can just choose $u = a/n, v = b/n$, and replace ε by $\varepsilon/4$. $\blacksquare$

For future use, we record this last inequality as:

Theorem 4.3

Assume the hypotheses of Theorem 1.1. Let $\varepsilon > 0$. Then there exists for $n \geq 1$, a set $\mathcal{E}_n$ of measure $\leq \varepsilon$, such that for $\xi \in S_Q \backslash \mathcal{E}_n$ and all complex a, b,

$$\frac{1}{n} \left| K_n \left(\nu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n} \right) - S_n \left(\xi + \frac{a}{n} \right) S_n \left(\xi + \frac{b}{n} \right) K_n \left(\mu_n, \xi + \frac{a}{n}, \xi + \frac{b}{n} \right) \right| \nu_n' \left(\xi \right)$$

$$\leq \quad \frac{8}{\sqrt{\varepsilon}} \Gamma_n^{1/4} e^{C_0 (|a| + |b|)/\varepsilon}.$$

(4.7)

Here C_0 is independent of $n, \xi, a, b, \varepsilon$. We may replace $\nu_n' \left(\xi \right)$ by $\mu_n' \left(\xi \right) / S_n^2 \left(\xi \right)$ in this last left-hand side.

5. PROOF OF THEOREM 1.2

Our main task is to find approximating polynomials $\{S_n\}$ satisfying the hypotheses (1.11-1.13) of Theorem 1.1. We begin with:

Lemma 5.1

Let $\{\delta_n\} \subset (0, \infty)$ with

(5.1)
$$\lim_{n \to \infty} \delta_n = 0.$$

Let $r > 0$. There exist nonnegative polynomials $\{P_n\}$ with

(5.2)
$$\deg \left(P_n \right) = o \left(n \right), \ n \to \infty$$

and as $n \to \infty$, uniformly in compact subsets of $(-r, r) \backslash \{0\}$,

(5.3)
$$|x|^{n \delta_n} P_n \left(x \right) \to 1,$$

while for $n \geq 1$,

(5.4)
$$\sup_{x \in [-r, r]} |x|^{n \delta_n} P_n \left(x \right) \leq 1.$$

Proof

First, note that by dilating the variable, we may assume $r = 1$. Next, it suffices to find nonnegative polynomials $\{P_n\}$ satisfying (5.2), and also as $n \to \infty$,

(5.5)
$$x^{n \delta_n} P_n \left(x \right) \to 1 \text{ uniformly in compact subsets of } (0, 1),$$

while for $n \geq 1$,

(5.6)
$$0 \leq x^{n \delta_n} P_n \left(x \right) \leq 1 \text{ in } [0, 1].$$

Indeed, we can always replace $P_n(x)$ satisfying these last two conditions by $P_n(x^2)$, after also replacing δ_n by $\delta_n/2$.

Next, we recall classical results from the theory of incomplete polynomials. Let $\theta \in \left(0, \frac{1}{4}\right)$ and $f_\theta : [0,1] \to \mathbb{R}$ be continuous with $f_\theta = 0$ in $\left[0, \theta^2\right] \cup \{1\}$. Then [22, p. 283] there exist polynomials $R_{m,\theta}$ of degree $\leq m$, such that

(5.7) $$\lim_{m \to \infty} x^{m\theta/(1-\theta)} R_{m,\theta}(x) = f_\theta(x)$$

uniformly in $[0,1]$. We may also assume that $R_{m,\theta} \geq 0$ in $[0,1]$. (If not consider $R_{m,\theta}^2$ after modifying the value of θ). By dividing by $R_{m,\theta}$ by $1 + \eta_m$, where η_m decreases to 0 sufficiently slowly, we can also assume that

(5.8) $$0 \leq x^{m\theta/(1-\theta)} R_{m,\theta}(x) < 1 \text{ in } [0,1].$$

We shall choose our f_θ so that $0 \leq f_\theta \leq 1$ in $[0,1]$ and $f_\theta = 1$ in $[\theta, 1-\theta]$.

Next, fix $\varepsilon \in \left(0, \frac{1}{2}\right)$ and set $\theta = \frac{\varepsilon}{2}$. For large enough n, write

$$\frac{1-\theta}{\theta} \delta_n n = m_n + \Delta_n,$$

where m_n is a positive integer, and $\Delta_n \in [0,1)$. We shall set, for some suitable polynomial V_n,

$$P_n(x) = R_{m_n,\theta}(x) V_n(x),$$

so that

$$x^{\delta_n n} P_n(x) = \left(x^{m_n \theta/(1-\theta)} R_{m_n,\theta}(x)\right) \left(x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x)\right).$$

Assume now that $\deg(V_n) \leq \log n$ and

(5.9) $$0 \leq x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) \leq 1 \text{ in } [0,1],$$

while uniformly in compact subsets of $(0,1]$,

(5.10) $$\lim_{n \to \infty} x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) = 1.$$

Then P_n has degree $\leq m_n + \log n \leq \frac{1-\theta}{\theta}\delta_n n + \log n$, and from (5.6) and (5.8),

$$0 \leq x^{\delta_n n} P_n(x) \leq 1$$

while uniformly in $\left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right]$, from (5.7), (5.10), and our choice of f_θ,

$$\lim_{n \to \infty} x^{\delta_n n} P_n(x) = 1.$$

(Recall that $f_\theta = 1$ in $\left[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}\right] = [\theta, 1-\theta]$.) The conclusion of the lemma then follows by choosing a sufficently slowly decreasing sequence of values for ε, and corresponding $\{P_n\}$ (with "long" subsequences of $\{P_n\}$ arising from each of the values of ε).

It remains to prove the existence of $\{V_n\}$ satisfying (5.9) and (5.10). Let

$$A_\ell(t) = -\sum_{j=1}^{\ell} \frac{t^j}{j}$$

denote the ℓth partial sum of the Maclaurin series of $\log(1-t)$, so that

(5.11) $$A_\ell(t) \geq \log(1-t), t \in [0,1)$$

and

$$\text{(5.12)} \qquad\qquad \sup_{t\in[0,1]} |A_\ell(t)| \leq 1 + \log \ell.$$

Also, let B_ℓ denote the ℓth partial sum of the Maclaurin series of $\exp(-t)$, so that, by straightforward calculations,

$$\text{(5.13)} \qquad\qquad \sup_{|t|\leq \ell/10} |B_\ell(t)\exp(t) - 1| = o(1) \text{ for } \ell \to \infty.$$

We let $\ell = \left[\sqrt{\log n}\right]$ (where $[x]$ denotes the greatest integer $\leq x$) and

$$V_n(x) = B_{\left[\sqrt{\log n}\right]}\left(-\frac{\theta}{1-\theta}\Delta_n A_{\left[\sqrt{\log n}\right]}(1-x)\right)$$

so that, in view of (5.11-5.13), uniformly for $x \in [0,1]$,

$$
\begin{aligned}
0 &\leq & x^{\frac{\theta}{1-\theta}\Delta_n} V_n(x) \\
&=& (1+o(1))\exp\left(\frac{\theta}{1-\theta}\Delta_n\left\{\log(1-(1-x)) - A_{\left[\sqrt{\log n}\right]}(1-x)\right\}\right) \\
&\leq& 1 + o(1).
\end{aligned}
$$

Moreover, the desired uniform convergence in compact subsets of $(0,1]$ in (5.10) follows from the convergence of the partial sums $\{A_\ell\}$ and $\{B_\ell\}$. The factor $1+o(1)$ may be replaced by 1 by dividing by $1+\eta_n$ for some slowly decreasing sequence $\{\eta_n\}$ with limit 0. ∎

Next, we handle the factors $e^{\tau_n \varphi_n}$:

Lemma 5.2

Let $\alpha, \{\tau_n\}, \{\varphi_n\}$ be as in Theorem 1.2. Then there exist polynomials $\{R_n\}$ with

$$\text{(5.14)} \qquad\qquad \deg R_n = o(n), n \to \infty;$$

$$\text{(5.15)} \qquad\qquad 0 \leq R_n(x)e^{\tau_n \varphi_n(x)} \leq 1; x \in \Sigma;$$

$$\text{(5.16)} \qquad\qquad \lim_{n\to\infty} R_n(x)e^{\tau_n \varphi_n(x)} = 1 \text{ uniformly in } \Sigma.$$

Proof

For the purposes of this lemma, we can assume that Σ is a single closed interval, as we can extend the domain of definition of the functions $\{\varphi_n\}$ without increasing the sup norms of φ_n and their derivatives or Lipschitz norms. Let

$$\text{(5.17)} \qquad\qquad \ell_n = \left[n^{\frac{1}{1+\alpha}}\right], n \geq 1.$$

so that by (1.18)

$$\text{(5.18)} \qquad\qquad \lim_{n\to\infty} \tau_n/\ell_n^\alpha = 0.$$

By Jackson's Theorem [7, Thm. 6.2, p. 219], and our hypothesis of uniform smoothness of order α, we can find a polynomial A_n of degree $\leq \ell_n$, such that

$$\|\varphi_n - A_n\|_{L_\infty(\Sigma)} \leq C/\ell_n^\alpha.$$

Here C is independent of n. Then

$$\text{(5.19)} \qquad\qquad \|\tau_n\varphi_n - \tau_n A_n\|_{L_\infty(\Sigma)} \leq C\tau_n/\ell_n^\alpha \to 0, n \to \infty.$$

By adding a sequence of positive numbers with limit 0, we may assume that

(5.20) $$\tau_n \varphi_n \leq \tau_n A_n \text{ in } \Sigma.$$

Next, let $B_n(t)$ denote the nth partial sum of the Maclaurin series of e^t. We use (5.13) above, and choose

$$R_n(x) = B_{[\tau_n]}(-\tau_n A_n(x)).$$

Note that this has degree at most $[\tau_n] \ell_n = o\left(\ell_n^{1+\alpha}\right) = o(n)$, by (5.17) and (5.18). Also,

$$\|\tau_n A_n\|_{L_\infty(\Sigma)} \leq \tau_n \left(1 + O\left(1/\ell_n^\alpha\right)\right) = \tau_n + o(1),$$

so (5.13), followed by (5.20), show that uniformly for $x \in \Sigma$,

$$R_n(x) e^{\tau_n \varphi_n(x)} = \exp\left(\tau_n\left(\varphi_n(x) - A_n(x)\right)\right)(1 + o(1)) \leq 1 + o(1),$$

and we also have uniformly for $x \in \Sigma$, by (5.19),

$$R_n(x) e^{\tau_n \varphi_n(x)} = \exp\left(\tau_n O\left(1/\ell_n^\alpha\right)\right) = 1 + o(1).$$

Now multiply R_n by a sequence that is $1 + o(1)$ to ensure the inequality (5.15). ∎
    We need one more lemma on local growth of ν_n' :

Lemma 5.3
Let $R, \varepsilon > 0$. There exists $C > 1$ and for $n \geq 1$, a set $\mathcal{F}_n$ of measure $< \varepsilon$ such that

(5.21) $$C^{-1} \leq \sup_{\xi \in S_Q \setminus \mathcal{E}_n, |a| \leq R} \frac{\nu_n'(\xi)}{\nu_n'\left(\xi + \frac{a}{n\omega_Q'(\xi)}\right)} \leq C.$$

Proof
In view of the form (1.15) of ν_n', it suffices to show the following three estimates:

(5.22) $$C^{-1} \leq \sup_{\xi \in S_Q \setminus \mathcal{E}_{n,1}, |a| \leq R} e^{\tau_n\left[\varphi_n(\xi) - \varphi_n\left(\xi + \frac{a}{n\omega_Q'(\xi)}\right)\right]} \leq C;$$

(5.23) $$C^{-1} \leq \sup_{\xi \in S_Q \setminus \mathcal{E}_{n,2}, |a| \leq R} \left|\frac{\xi - b_n}{\xi - b_n + \frac{a}{n\omega_Q'(\xi)}}\right|^{n\beta_n} \leq C;$$

(5.24) $$C^{-1} \leq \sup_{\xi \in S_Q \setminus \mathcal{E}_{n,3}, |a| \leq R} e^{2n\left[Q(\xi) - Q\left(\xi + \frac{a}{n\omega_Q'(\xi)}\right)\right]} \leq C,$$

where $\mathcal{E}_{n,j}, j = 1, 2, 3$ are sets of small measure, and where $\lim_{n \to \infty} \beta_n = 0$, while $\{b_n\}$ is a sequence of real numbers. The proof of (5.22) is easy, and follows from the Mean Value Theorem, or our assumed Lipschitz condition (if $\alpha \leq 1$). Note that since Σ consists of finitely many intervals, ξ and $\xi + \frac{a}{n\omega_Q'(\xi)}$ belong to the same interval, for large enough n, and ξ outside a set $\mathcal{E}_{n,1}$ of small measure (outside

which ω'_Q is bounded above and below by specified constants). Then

$$\sup_{\xi \in S_Q \setminus \mathcal{E}_{n,1}, |a| \leq R} \tau_n \left| \varphi_n(\xi) - \varphi_n \left(\xi + \frac{a}{n\omega'_Q(\xi)} \right) \right|$$

$$\leq \tau_n C \left(\frac{R}{n \inf_{\xi \in S_Q \setminus \mathcal{E}_{n,1}} \omega'_Q(\xi)} \right)^{\min\{1,\alpha\}}$$

$$= o \left(n^{\frac{\alpha}{1+\alpha} - \min\{1,\alpha\}} \right) \to 0, n \to \infty,$$

recall (1.18). So we have (5.22).

Next, for $|a| \leq R$,

$$\left| \frac{\xi - b_n}{\xi - b + \frac{a}{n\omega'_Q(\xi)}} \right|^{n\beta_n}$$

$$= \exp \left(-n\beta_n \log \left| 1 + \frac{a}{n\omega'_Q(\xi)(\xi - b_n)} \right| \right)$$

$$= \exp \left(n\beta_n O \left(\frac{R}{n\omega'_Q(\xi)(\xi - b_n)} \right) \right) = \exp(o(1)),$$

uniformly for for $|a| \leq R$, and $\xi \in S_Q \setminus \mathcal{E}_{n,2}$, where again $\mathcal{E}_{n,2}$ is a set of small measure, chosen so that $\omega'_Q(\xi)$ is bounded above and below outside $\mathcal{E}_{n,2}$, while a small interval around b_n is excluded. Then (5.23) follows.

Finally, recall that Q' is assumed to be continuous except at finitely many points in Σ, while ω'_Q is positive and continuous except at finitely many points. By taking $\mathcal{E}_{n,3}$ to consist of small intervals centered on these points, we obtain a small set $\mathcal{E}_{n,3}$ such that

$$\sup_{\xi \in S_Q \setminus \mathcal{E}_{n,3}, |a| \leq R} 2n \left| Q(\xi) - Q \left(\xi + \frac{a}{n\omega'_Q(\xi)} \right) \right|$$

$$\leq 2n \sup_{t \in S_Q \setminus \mathcal{E}_{n,3}} |Q'(t)| \frac{R}{n\omega'_Q(\xi)} \leq C < \infty.$$

So we have (5.24). ∎

Proof of Theorem 1.2
We apply Theorem 1.1 with

$$\mu'_n(x) = h(x) e^{-2nQ(x)}$$

and

$$\nu'_n(x) = g_n(x) \mu'_n(x),$$

where

$$g_n(x) = e^{\tau_n h_n(x)} \left(\prod_{j=1}^{N} |x - \alpha_{nj}|^{\beta_{nj}} \right)^n.$$

By a result of Totik [26, Thm. 1.2, p. 326], uniformly for x in compact subsets of S_Q omitting discontinuities of ω'_Q, we have

$$(5.25) \qquad \lim_{n \to \infty} \frac{1}{n} K_n (\mu_n, x, x) \, \mu'_n (x) = \omega'_Q (x),$$

so (1.9) is certainly true, so we can apply Lemma 2.2. Note too, that by the aforementioned Theorem A of the authors, [15, Thm. 1.1, p. 747],

$$\lim_{n \to \infty} \frac{\widetilde{K}_n \left(\mu_n, \xi + \frac{a}{\widetilde{K}_n(\mu_n, \xi, \xi)}, \xi + \frac{b}{\widetilde{K}_n(\mu_n, \xi, \xi)} \right)}{\widetilde{K}_n (\mu_n, \xi, \xi)} = \mathbb{S} (a - b),$$

uniformly for ξ in compact subset of S_Q omitting zeros of Q' and zeros or discontinuities of ω'_Q. We can replace $\widetilde{K}_n (\nu_n, \xi, \xi)$ in this limit by $n\omega'_Q (\xi)$ in view of Totik's asymptotics for Christoffel functions and the uniform convergence in a, b, giving, for the same range of ξ, a, b,

$$(5.26) \qquad \lim_{n \to \infty} \frac{\widetilde{K}_n \left(\mu_n, \xi + \frac{a}{n\omega'_Q(\xi)}, \xi + \frac{b}{n\omega'_Q(\xi)} \right)}{n\omega'_Q (\xi)} = \mathbb{S} (a - b).$$

Now let us turn to the construction of the polynomials $\{S_n\}$ in Theorem 1.1. We apply Lemmas 5.1 and 5.2 above. In applying Lemma 5.1, we choose r sufficently large, and form a product of N terms of the form $P_n (x - b_{n,j})$, each with appropriately chosen δ_j. We multiply the polynomials from Lemmas 5.1 and 5.2, to obtain $\{S_n\}$ satisfying (1.11)-(1.13). In fact, Lemma 5.1 and 5.2 give much more than (1.11) - (1.13). Indeed, let $0 < \eta < \frac{1}{2N}$, and

$$\mathcal{U}_n (\eta) = \bigcup_{j=1}^{N} [b_{n,j} - \eta, b_{n,j} + \eta],$$

a set of measure $\leq 2N\eta$. Lemmas 5.1 and 5.2 give polynomials S_n of degree $o(n)$ such that for each small enough η,

$$\sup_{\xi \in \Sigma \backslash \mathcal{U}_n(\eta)} |(g_n S_n) (\xi) - 1| \to 0, n \to \infty.$$

Note that also, given $R > 0$, and $\eta > 0$,

$$(5.27) \qquad \sup_{\xi \in \Sigma \backslash \mathcal{U}_n(\eta), |a| \leq R} \left| (g_n S_n) \left(\xi + \frac{a}{n} \right) - 1 \right| \to 0, n \to \infty.$$

Next, let us set

$$\xi_{n,a} = \xi + \frac{a}{n\omega'_Q (\xi)}$$

with a similar notation for $\xi_{n,b}$. Also, let us replace a, b in (4.7) by $a/\omega'_Q (\xi)$ and $b/\omega'_Q (\xi)$. We add to the set $\mathcal{E}_n$ in Theorem 4.3, small intervals centered on the zeros and discontinuities of ω'_Q. We obtain for $\xi \in S_Q \backslash \mathcal{E}_n$ and all complex a, b,

$$\frac{1}{n\omega'_Q (\xi)} \left| \tilde{K}_n \left(\nu_n, \xi_{n,a}, \xi_{n,b} \right) - \left[(g_n S_n^2) (\xi_{n,a}) (g_n S_n^2) (\xi_{n,b}) \right]^{1/2} \tilde{K}_n \left(\mu_n, \xi_{n,a}, \xi_{n,b} \right) \right|$$

$$\times \frac{\nu'_n (\xi)}{\left[\nu'_n (\xi_{n,a}) \, \nu'_n (\xi_{n,b}) \right]^{1/2}}$$

$$\leq \frac{8}{\sqrt{\varepsilon} \omega'_Q (\xi)} \Gamma_n^{1/4} e^{C_0 (|a| + |b|)/\varepsilon}.$$

Applying (5.26) and (5.27), and Lemma 5.3, we obtain from some suitable set $\mathcal{H}_n$ of small measure, and for each $R > 0$,

$$
\sup_{\xi \in S_Q \setminus \mathcal{H}_n, |a|, |b| \leq R} \left| \frac{1}{n\omega_Q'(\xi)} \tilde{K}_n\left(\nu_n, \xi_{n,a}, \xi_{n,b}\right) - \mathbb{S}\left(a - b\right) \right|
$$
$$
\leq \frac{C}{\sqrt{\varepsilon}} \sup_{\xi \in \Sigma \setminus \mathcal{H}_n, |a|, |b| \leq R} \Gamma_n^{1/4} \frac{e^{C_0 R/\varepsilon}}{\omega_Q'(\xi)} \to 0, n \to \infty.
$$

∎

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