On Boundedness of Lagrange Interpolation in $L_p, p < 1$

D.S. Lubinsky
Mathematics Department,
Witwatersrand University,
Wits 2050, South Africa.
e-mail: 036dsl@cosmos.wits.ac.za

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Abstract

We estimate the distribution function of a Lagrange interpolation polynomial and deduce mean boundedness in L_p, $p < 1$.

1 The Result

There is a vast literature on mean convergence of Lagrange interpolation, see [4–8] for recent references. In this note, we use distribution functions to investigate mean convergence. We believe the simplicity of the approach merits attention.

Recall that if $g : \mathbb{R} \to R$, and m denotes Lebesgue measure, then the *distribution function* m_g of g is

$$m_g(\lambda) := m\left(\{x : |g(x)| > \lambda\}\right),\ \lambda \geq 0. \tag{1}$$

One of the uses of m_g is in the identity [1,p.43]

$$\| g \|^p_{L_p(\mathbb{R})} = \int_0^\infty pt^{p-1} m_g(t)dt,\ 0 < p < \infty. \tag{2}$$

Moreover, the weak L_1 norm of g may be defined by

$$\| g \|_{weak(L_1)} = \sup_{\lambda > 0} \lambda m_g(\lambda). \tag{3}$$

If

$$\| g \|_{L_p(\mathbb{R})} < \infty,$$

then for $p<\infty$, it is easily seen that

$$m_g(\lambda) \leq \lambda^{-p} \parallel g \parallel^p_{L_p(\mathbb{R})},\ \lambda>0. \tag{4}$$

and if $p=\infty$,

$$m_g(\lambda)=0,\ \lambda > \parallel g \parallel_{L_\infty(\mathbb{R})}.$$

Our result is:

Theorem 1
Let $w,\nu:\mathbb{R}\to R$ be measurable and let ν have compact support. Let $n\geq 1$ and let π_n be a polynomial of degree n with n real simple zeros $\{t_{jn}\}_{j=1}^n$. Let

$$\Omega_n := \sum_{j=1}^n \frac{1}{|\pi_n' w|(t_{jn})}. \tag{5}$$

(a) *Let $0<r<\infty$ and assume there exists $A>0$ such that*

$$m_{\pi_n\nu}(\lambda) \leq A\lambda^{-r},\ \lambda>0. \tag{6}$$

Then if $L_n[f]$ denotes the Lagrange interpolation polynomial to f at the zeros $\{t_{jn}\}$ of π_n, we have

$$m_{L_n[f]\nu}(\lambda) \leq 2A^{\frac{1}{r+1}} \left(8 \parallel fw \parallel_{L_\infty(\mathbb{R})} \Omega_n/\lambda\right)^{\frac{r}{r+1}},\ \lambda>0; \tag{7}$$

(b) *Assume that*

$$m_{\pi_n\nu}(\lambda)=0,\ \lambda>A. \tag{8}$$

Then

$$m_{L_n[f]\nu}(\lambda) \leq A \parallel fw \parallel_{L_\infty(\mathbb{R})} \Omega_n/\lambda,\ \lambda>0. \tag{9}$$

Corollary 2
Let w,ν be as in Theorem 1 and assume that we are given $\pi_n,\{t_{jn}\}_{j=1}^n$ for each $n\geq 1$ and

$$\Omega := \sup_{n\geq 1} \sum_{j=1}^n \frac{1}{|\pi_n' w|(t_{jn})} < \infty. \tag{10}$$

(a) *If $r<\infty$ and (6) holds for $n\geq 1$, then for $0<p<\frac{r}{1+r}$, we have for some C_1 independent of f,n*

$$\parallel L_n[f]\nu \parallel_{L_p(\mathbb{R})} \leq C_1 \parallel fw \parallel_{L_\infty(\mathbb{R})}. \tag{11}$$

(b) *If (8) holds for $n \geq 1$, then we have (11) for $0 < p < 1$, as well as*

$$\| L_n[f]\nu \|_{weak(L_1)} \leq C_1 \| fw \|_{L_\infty(\mathbb{R})} . \tag{12}$$

Remarks

(a) Note that (6) holds if

$$\| \pi_n \nu \|^r_{L_r(\mathbb{R})} \leq A, \ n \geq 1$$

and (8) holds if

$$\| \pi_n \nu \|_{L_\infty(\mathbb{R})} \leq A.$$

Of course (6) is a weak L_r condition.

(b) Under mild additional conditions on w and ν that guarantee density of the polynomials in the relevant spaces, the projection property $L_n[P] = P$, $\deg(P) \leq n-1$, allows us to deduce mean convergence of $L_n[f]$ to f.

(c) Orthogonal polynomials $\{p_n(u,x)\}_{n=0}^{\infty}$ such as those for generalized Jacobi weights u [4] or the exponential weights u in [2] admit the bound

$$|p_n(u,x)|\, u^{1/2}(x) \leq C \left| 1 - \frac{|x|}{a_n} \right|^{-1/4}, \ x \in [-1,1]$$

for a C independent of n and a suitable choice of a_n. Thus these polynomials admit the bound (6) with $r = 4$. Moreover, if $\{t_{jn}\}$ are the zeros of p_n, then a great deal is known about $p_n'(t_{jn})$, and in particular (10) holds with an appropriate choice of w. More generally, for extended Lagrange interpolation, involving interpolation at the zeros of $S_n p_n$, where S_n is a polynomial of fixed degree, it is easy to verify (10) under mild conditions on S_n.

(d) A result of Shi [7] implies that if (11) holds with C_1 independent of f and n, and if π_n is normalized by the condition

$$\| \pi_n \nu \|_{L_p(\mathbb{R})} = 1,$$

while the $\{t_{jn}\}$ are all contained in a bounded interval, then (10) holds. Thus in this case (10) is necessary for (11). However, our normalisation (6) or (8) of π_n involves a condition with $r > p$, so there is a gap.

(e) Of course (10) requires $w(t_{jn}) \neq 0 \forall j, n$. We may weaken (10) to

$$\sup_{n \geq 1} \sum_{j: w(t_{jn}) \neq 0} \frac{1}{|\pi_n' w|\,(t_{jn})} < \infty$$

if we restrict f by the condition $w(t_{jn}) = 0 \Rightarrow f(t_{jn}) = 0$. In particular this allows us to consider w with compact support even when $\{t_{jn}\}_{j,n}$ is not contained in a bounded interval.

Our proofs rely on a lemma of Loomis [1,p. 129].

Lemma 3
Let $n \geq 1$ *and* $\{x_j\}_{j=1}^n, \{c_j\}_{j=1}^n \subset \mathbb{R}$. *Then for* $\lambda > 0$,

$$m\left(\left\{x : \left|\sum_{j=1}^n \frac{c_j}{x - x_j}\right| > \lambda\right\}\right) \leq \frac{8}{\lambda} \sum_{j=1}^n |c_j| . \tag{13}$$

Proof
When all $c_j \geq 0$, we have equality in (13) with 8 replaced by 2 [1,p.129]. The general case follows by writing

$$c_j = c_j^+ - c_j^-$$

where $c_j^+ = \max\{0, c_j\}$, $c_j^- = -\min\{0, c_j\}$ and noting that

$$\left|\sum_{j=1}^n \frac{c_j}{x - x_j}\right| > \lambda \Rightarrow \left|\sum_{j=1}^n \frac{c_j^+}{x - x_j}\right| > \frac{\lambda}{2} \text{ or } \left|\sum_{j=1}^n \frac{c_j^-}{x - x_j}\right| > \frac{\lambda}{2} \text{ or both.} \quad \square$$

Proof of Theorem 1

(a) Assume that $r < \infty$ and let $a \in \mathbb{R}, \lambda > 0$. We may assume that

$$\| fw \|_{L_\infty(\mathbb{R})} = 1. \tag{14}$$

(The general case follows from the identity $m_{bg}(\lambda) = m_g(\lambda/b)$ for $b, \lambda > 0$). Now

$$(L_n[f]\nu)(x) = (\pi_n \nu)(x) \sum_{j=1}^n \frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x - t_{jn})}$$

so

$$|L_n[f]\nu|(x) > \lambda$$

implies

$$|\pi_n \nu|(x) > \lambda^a \tag{15}$$

or

$$\left|\sum_{j=1}^n \frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x - t_{jn})}\right| > \lambda^{1-a} \tag{16}$$

or both. The set of x satisfying (15) has, by (6), measure at most $A\lambda^{-ar}$. The set of x satisfying (16) has by Loomis' Lemma, measure at most

$$\frac{8}{\lambda^{1-a}}\sum_{j=1}^{n}\left|\frac{fw}{\pi_n' w}\right|(t_{jn}) \leq 8\lambda^{a-1}\Omega_n.$$

Now, if $\lambda \neq 1$, we choose a so that

$$A\lambda^{-ar} = 8\lambda^{a-1}\Omega_n \Leftrightarrow a = \frac{1}{r+1}\left[1 - \frac{\log\left[8\Omega_n/A\right]}{\log\lambda}\right].$$

Then we obtain

$$m_{L_n[f]v}(\lambda) \leq 2A^{\frac{1}{r+1}}\left(8\Omega_n/\lambda\right)^{\frac{r}{r+1}},$$

that is (7) holds. The case $\lambda = 1$ follows from continuity properties of Lebesgue measure.

(b) Here we have instead

$$\left|L_n[f]\nu\right|(x) > \lambda \Rightarrow \left|\sum_{j=1}^{n}\frac{(fw)(t_{jn})}{(\pi_n' w)(t_{jn})(x-t_{jn})}\right| > \frac{\lambda}{A}$$

and again (9) follows from Loomis' Lemma. □

Proof of Corollary 2

(a) We may assume (14). Now by hypothesis, there exists $b > 0$ such that ν vanishes outside $[-b, b]$. Thus in addition to (7), we have the estimate

$$m_{L_n[f]\nu}(\lambda) \leq 2b, \ \lambda > 0.$$

Then from (2), if $0 < p < \frac{r}{r+1}$, we have

$$\| L_n[f]\nu \|_{L_p(\mathbb{R})}^p \leq p\left(\int_0^1 t^{p-1}(2b)dt + 2A^{\frac{1}{r+1}}\left(8\Omega\right)^{\frac{r}{r+1}}\int_1^\infty t^{p-1-\frac{r}{r+1}}dt\right) =: C_1 < \infty.$$

(b) Here trivial modifications of this last estimate allow us to treat $0 < p < 1$, while (9) gives

$$\| L_n[f]\nu \|_{weak(L_1)} = \sup_{\lambda>0}\lambda m_{L_n[f]\nu}(\lambda) \leq C\Omega. \quad \square$$

We make two final remarks: The proof of Theorem 1 also gives a weak converse Marcinkiewicz–Zygmund inequality. For a given f, define

$$\Omega_n(f) := \sum_{j=1}^{n} \frac{|fw|(t_{jn})}{|\pi_n' w|(t_{jn})}.$$

Then (7) holds with Ω_n replaced by $\Omega_n(f)$. Moreover, (7) can be reformulated in the following way: If P is a polynomial of degree $\leq n-1$ satisfying

$$|Pw|(t_{jn}) \leq 1, \ 1 \leq j \leq n,$$

then

$$m_{P\nu}(\lambda) \leq 2A^{\frac{1}{r+1}} \left(8\Omega_n/\lambda\right)^{\frac{r}{r+1}}, \ \lambda > 0.$$

It would be useful to have more sophisticated estimates for $m_{P\nu}$. For special weights w, ν and points $\{t_{jn}\}$, converse quadrature sum inequalities imply these [4].

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