HW assignment #8 (Due Friday, April 20)

Throughout this assignment, π, ρ, and $\pi_1, \ldots, \pi_k$ will denote finite-dimensional complex representations of a finite group G.

1. a. Read §18.1 of Dummit and Foote.

   b. Under the correspondence between representations and $\mathbf{C}G$-modules discussed in §18.1 of the text, if $\pi \leftrightarrow V$ and $\rho \leftrightarrow W$, prove that $\mathrm{Hom}_G(\pi, \rho)$ (as defined in class) is isomorphic to $\mathrm{Hom}_{\mathbf{C}G}(V, W)$ as vector spaces over $\mathbf{C}$.

2. a. If $M_1, \ldots, M_k$ and N are modules over a ring R, prove that

$$\mathrm{Hom}_R(\oplus_{i=1}^k M_i, N) \cong \oplus_{i=1}^k \mathrm{Hom}_R(M_i, N) \ .$$

   Deduce that

$$\mathrm{Hom}_G(\oplus_{i=1}^k \pi_i, \rho) \cong \oplus_{i=1}^k \mathrm{Hom}_G(\pi_i, \rho) \ .$$

   b. If ρ and $\pi_1, \ldots, \pi_k$ are irreducible, prove that

$$\dim_{\mathbf{C}} \mathrm{Hom}_G(\oplus_{i=1}^k \pi_i, \rho) = \#\{1 \le i \le k \ : \ \pi_i \cong \rho\} \ .$$

   Conclude that the irreducible representations which appear in any decomposition of π as a direct sum of irreducible representations, as well as their multiplicities, are uniquely determined up to isomorphism. They are called the *irreducible components* of π.

3. Let (λ_G, V) denote the left regular representation of G, and let (ρ, W) be any irreducible representation of G.

   a. For any nonzero vector $w \in W$, prove that $W = \{\rho(g)w \ : \ g \in G\}$.

   b. Choose a nonzero $w \in W$, and define $\theta_w : V \to W$ by

$$\theta_w(\sum_{g \in G} c_g e_g) = \sum_{g \in G} c_g \rho(g) w \ .$$

   Prove that θ_w is a nonzero element of $\mathrm{Hom}_G(\lambda_G, \rho)$.

c. Prove that ρ is an irreducible component of λ_G. In particular, conclude that there are only finitely many irreducible representations of G (up to isomorphism).

4. Find (up to isomorphism) all irreducible representations of S_3 by explicitly decomposing the left regular representation into its irreducible components.

5. Let V be a finite-dimensional vector space over a field F of characteristic different from 2, and let $e_1, \dots, e_n$ be a basis for V.

   a. Define a linear transformation $\theta : V \otimes V \to V \otimes V$ by $\theta(e_i \otimes e_j) = e_j \otimes e_i$. Prove that $\theta(x \otimes y) = y \otimes x$ for all $x, y \in V$.

   b. Define $\mathrm{Sym}^2(V) = \{z \in V \otimes V \ : \ \theta(z) = z\}$ and $\mathrm{Alt}^2(V) = \{z \in V \otimes V \ : \ \theta(z) = -z\}$. Prove that $\mathrm{Sym}^2(V)$ has dimension $\binom{n+1}{2}$, $\mathrm{Alt}^2(V)$ has dimension $\binom{n}{2}$, and that $V = \mathrm{Sym}^2(V) \oplus \mathrm{Alt}^2(V)$.

   c. If (π, V) is a representation of G, prove that $\mathrm{Sym}^2(V)$ and $\mathrm{Alt}^2(V)$ are π-invariant subspaces, and hence give rise to representations of G, called the *symmetric square* and *alternating square* of π.