Homework Assignment # 1

Due Wednesday, September 13

1. Prove that the following rings are not UFD's by explicitly finding two distinct factorizations of the same element.
   (a) $\mathbf{Z}[\sqrt{-13}]$
   (b) $\mathbf{Z}[\sqrt{10}]$ (**Hint:** Factor 6 in two different ways.)

2. Prove that the following rings are Euclidean domains (and hence UFD's).
   (a) $\mathbf{Z}[\sqrt{-2}]$ (**Hint:** For $x, y \in \mathbf{Z}[\sqrt{-2}]$ with $y \neq 0$, write $x/y = a + b\sqrt{-2}$ with $a, b \in \mathbf{Q}$, and choose $q = c + d\sqrt{-2} \in \mathbf{Z}[\sqrt{-2}]$ so that $|c - a| \leq 1/2, |d - b| \leq 1/2$.)
   (b) $\mathbf{Z}[\sqrt{2}]$ (**Hint:** Use the norm $\phi(a + b\sqrt{2}) = |a^2 - 2b^2|$.)

3. Find all integers x, y such that $x^3 - y^2 = 2$.

4. (a) Prove that every quadratic number field (a field of degree 2 over $\mathbf{Q}$) is of the form $\mathbf{Q}(\sqrt{d})$ for some square-free integer d.
   (b) Find an explicit example of a cubic number field which is not of the form $\mathbf{Q}(d^{1/3})$ for any integer d.

5. (a) Determine the ring of integers in $\mathbf{Q}(\sqrt{d})$ for all square-free integers d.
   (b) Determine the unit group of the ring of integers in $\mathbf{Q}(\sqrt{d})$ for all square-free integers $d < 0$.