Calculus Problems

1. Let $a_1, \ldots, a_k$ be nonnegative real numbers. Evaluate

$$\lim_{n \to \infty} (a_1^n + \cdots + a_k^n)^{1/n} .$$

2. a. Find all differentiable functions $f : (0, \infty) \to \mathbf{R}$ such that $f(xy) = f(x) + f(y)$ for all $x, y > 0$.

   b. Find all continuous functions $f : \mathbf{R} \to \mathbf{R}$ such that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbf{R}$.

3. Let a be a positive real number, and define a sequence x_n by $x_0 = 0$ and $x_{n+1} = x_n^2 + a$ for $n \geq 0$. For which values of a does $\lim_{n \to \infty} x_n$ exist?

4. Show that for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

5. Let $f(x)$ be differentiable on $[0, 1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n, show that there exist distinct points $x_1, x_2, \ldots, x_n$ in $[0, 1]$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

6. For each $x > e^e$, define a sequence $S_x = u_0, u_1, u_2, \ldots$ recursively as follows: $u_0 = e$, while for $n \geq 0$,

$$u_{n+1} = \log_{u_n} x \ .$$

   Prove that S_x converges to a real number $g(x)$, and that the function g defined in this way is continuous for $x > e^e$.