Proof of Eisenstein's Irreducibility Criterion

Since the book's proof of Eisenstein's criterion on pp. 309-310 is incorrect, we give a corrected version here.

Theorem. *Let R be a UFD, and let $f(x) = x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 \in R[x]$ be a monic polynomial of degree $n \geq 1$. Let P be a prime ideal of R, and suppose that $c_0, c_1, \ldots, c_{n-1} \in P$ and $c_0 \notin P^2$. Then $f(x)$ is irreducible in $R[x]$ (and therefore irreducible in $K[x]$ as well by Gauss' lemma).*

Proof. Suppose to the contrary that $f(x) = a(x)b(x)$ with $a(x) = a_l x^l + \cdots + a_1 x + a_0 \in R[x]$, $b(x) = b_m x^m + \cdots + b_1 x + b_0 \in R[x]$, and $1 \leq l \leq n-1$. Since $a_0 b_0 = c_0 \in P$, we must have either $a_0 \in P$ or $b_0 \in P$. Without loss of generality suppose that $a_0 \in P$. Since $c_0 \notin P^2$, we must have $b_0 \notin P$. As $a_l b_m = 1 \notin P$, it follows that $a_l \notin P$. Let j be the smallest integer $1 \leq j \leq l$ such that $a_j \notin P$, so that $a_j \notin P$ but $a_0, a_1, \ldots, a_{j-1} \in P$. Since $c_j = a_j b_0 + a_{j-1} b_1 + \cdots + a_1 b_{j-1} + a_0 b_j$, $a_j b_0 \notin P$, and $a_{j-1} b_1 + \cdots + a_1 b_{j-1} + a_0 b_j \in P$, it follows that $c_j \notin P$. But $j \leq l \leq n-1$, so this contradicts the assumption that $c_0, c_1, \ldots, c_{n-1} \in P$. $\square$